Section A: Pure Mathematics

Question 1

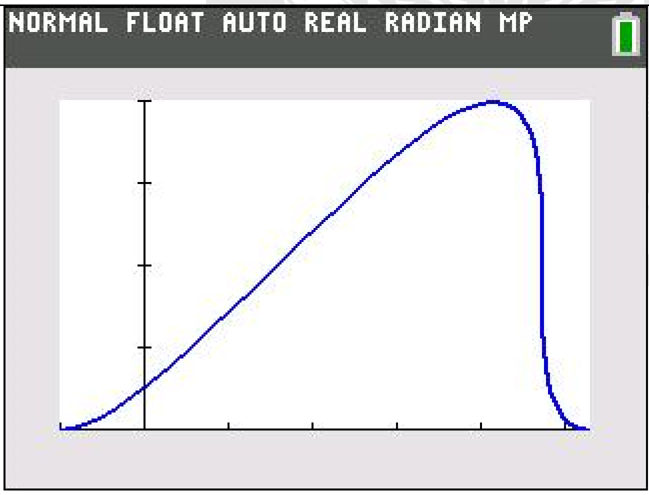
No.	Suggested Solution	Remarks for Student
	$\frac{\mathrm{d}V}{\mathrm{d}t}=0.1$ $\tan\alpha=\frac{r}{h}\Rightarrow r=h\tan\alpha=\frac{1}{2}h$ where $h=$ depth of water $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h=\frac{1}{12}\pi h^3$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{4}\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t}$ When $V=3,\ V=\frac{1}{12}\pi h^3\Rightarrow h=\left(\frac{36}{\pi}\right)^{\frac{1}{3}}$ $0.1=\frac{1}{4}\pi\left(\frac{36}{\pi}\right)^{\frac{2}{3}}\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=0.0251$ Rate of increase of depth of water is 0.0251 m per minute.	

Question 2

No.	Suggested Solution		Remarks for Student
(a)(i)	$\int x^2 \cos nx \, \mathrm{d}x$ $= \frac{1}{n}x^2 \sin nx - \frac{2}{n}\int x \sin nx \, \mathrm{d}x$ $= \frac{1}{n}x^2 \sin nx - \frac{2}{n}\left(-\frac{1}{n}x\cos nx + \frac{1}{n}\int \cos nx \, \mathrm{d}x\right)$ $= \frac{1}{n}x^2 \sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx + c$	$u = x^2, \frac{\mathrm{d}v}{\mathrm{d}x} = \cos nx$ $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x, v = \frac{1}{n}\sin nx$ $u = x, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin nx$ $\frac{\mathrm{d}u}{\mathrm{d}x} = 1, v = -\frac{1}{n}\cos nx$	
(ii)	$\int_{\pi}^{2\pi} x^2 \cos nx \, \mathrm{d}x = \left[\frac{1}{n}x^2 \sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_{\pi}^{2\pi}$ $= \frac{2}{n^2}(2\pi)\cos 2n\pi - \frac{2}{n^2}(\pi)\cos n\pi$ $= \frac{\pi}{n^2}(4\cos 2n\pi - 2\cos n\pi)$ If n is even, $\int_{\pi}^{2\pi} x^2 \cos nx \, \mathrm{d}x = \frac{\pi}{n^2}(4-2) = \frac{2\pi}{n^2}$ If n is odd, $\int_{\pi}^{2\pi} x^2 \cos nx \, \mathrm{d}x = \frac{\pi}{n^2}(4+2) = \frac{6\pi}{n^2}$	 $a = 2$ $a = 6$	

(b)	Volume $= \pi \int_0^2 y^2 \, \mathrm{d}x$ $u = 9 - x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2x \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{2x}$ $= \pi \int_0^2 \left(\frac{x\sqrt{x}}{9 - x^2} \right)^2 \mathrm{d}x$ $x = 0 \Rightarrow u = 9$ $= \pi \int_0^2 \frac{x^2(x)}{(9 - x^2)^2} \mathrm{d}x$ $x = 2 \Rightarrow u = 5$ $= -\frac{1}{2}\pi \int_9^5 \frac{9 - u}{u^2} \mathrm{d}u$ $= \frac{1}{2}\pi \int_5^9 \left(\frac{9}{u^2} - \frac{1}{u} \right) \mathrm{d}u$ $= \frac{1}{2}\pi \left[-\frac{9}{u} - \ln u \right]_5^9$ $= \frac{1}{2}\pi \left[-1 - \ln 9 + \frac{9}{5} + \ln 5 \right]$ $= \frac{1}{2}\pi \left(\frac{4}{5} + \ln \frac{5}{9} \right)$	

Question 3

No.	Suggested Solution	Remarks for Student
	$x = t - \cos t, \quad y = 1 - \cos t, \quad$ for $0 \le t \le 2\pi$	
(i)		

$y = 0$

$\Rightarrow 1 - \cos t = 0$

$\Rightarrow t = 0$ or 2π

$\Rightarrow x = 0 - \cos 0 = -1$ or $x = 2\pi - \cos 2\pi = 2\pi - 1$

$(-1, 0)$ and $(2\pi - 1, 0)$

$x = t - \cos t, \qquad y = 1 - \cos t, \qquad$ for $0 \le t \le 2\pi$

$\frac{dx}{dt} = 1 + \sin t, \; \frac{dy}{dt} = \sin t$

$\frac{dy}{dx} = \frac{\sin t}{1 + \sin t} = 0 \quad \Rightarrow \sin t = 0$

$\Rightarrow t = 0, \pi, 2\pi$

From above points where D meets the x-axis, max point is when $t = \pi$

$\therefore x = \pi - \cos \pi = \pi + 1, \; y = 1 - \cos \pi = 2$

$(\pi + 1, 2)$

(ii)

$$\begin{aligned}
\text{Area} &= \int_{-1}^{a - \cos a} y \, dx \\
&= \int_0^a (1 - \cos t)(1 + \sin t) \, dt \\
&= \int_0^a (1 - \cos t + \sin t - \sin t \cos t) \, dt \\
&= \int_0^a \left(1 - \cos t + \sin t - \frac{1}{2}\sin 2t\right) dt \\
&= \left[t - \sin t - \cos t + \frac{1}{4}\cos 2t \right]_0^a \\
&= a - \sin a - \cos a + \frac{1}{4}\cos 2a + \frac{3}{4}
\end{aligned}$$

(iii)

$\left.\frac{dy}{dx}\right|_{t = \frac{1}{2}\pi} = \frac{1}{2}$

$t = \frac{1}{2}\pi \Rightarrow x = \frac{1}{2}\pi, \; y = 1$

Thus, equation of normal is

$y - 1 = -2\left(x - \frac{\pi}{2}\right)$

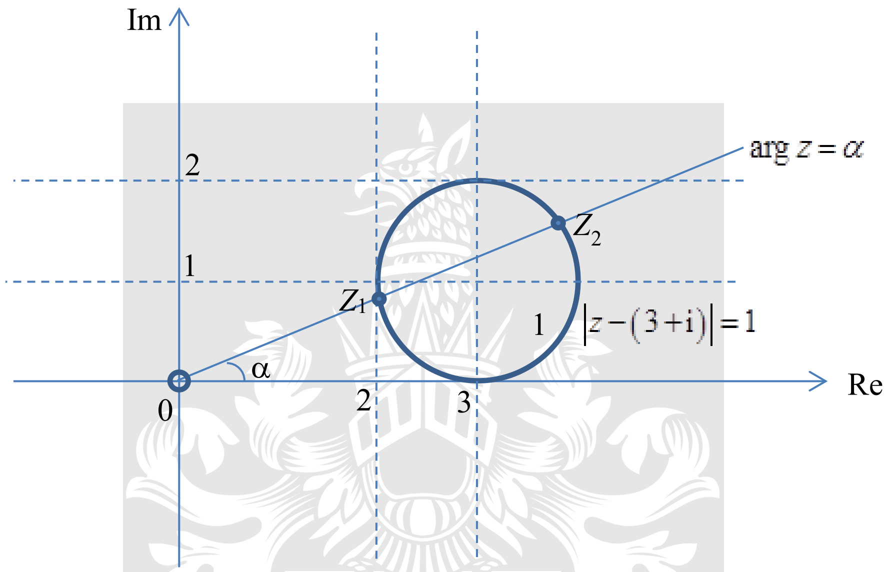
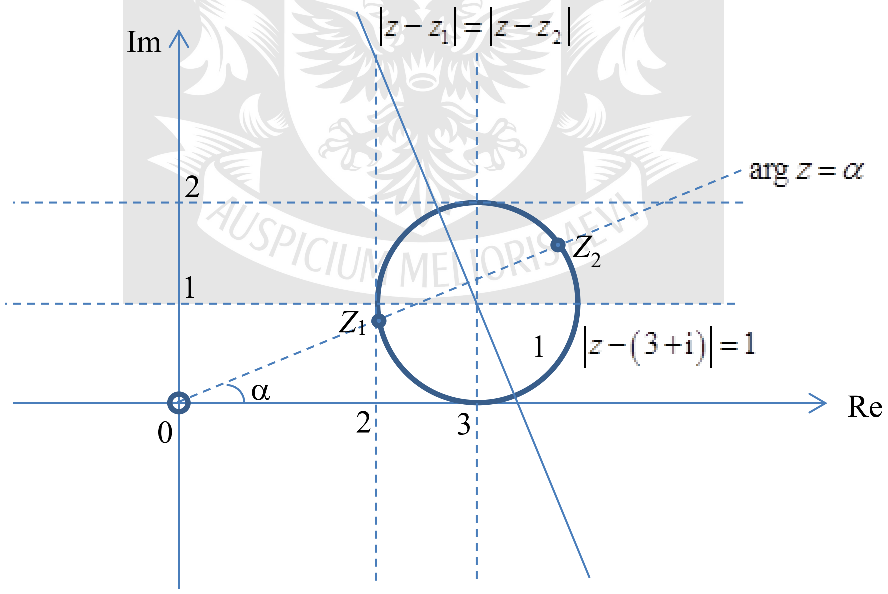
$y = -2x + \pi + 1$

E is $\left(\frac{\pi + 1}{2}, 0\right)$

F is $(0, \pi + 1)$

$\therefore$ area of triangle $OEF = \frac{1}{2}(\pi + 1)\left(\frac{\pi + 1}{2}\right) = \frac{1}{4}(\pi + 1)^2$

Question 4

No.	Suggested Solution	Remarks for Student
(a)	$\lvert z-3-\mathrm{i}\rvert=1 \Rightarrow \lvert z-(3+\mathrm{i})\rvert=1$ $\arg z=\alpha$ where $\tan\alpha=\frac{2}{5}$	
(a)(i)		
(ii)		
	Equation of circle: $(x-3)^2+(y-1)^2=1$ ...(1)	

	Equation of line: $y=-\frac{5}{2}x+c$ Sub (3, 1) into line, we have $1=-\frac{15}{2}+c \Rightarrow c=\frac{17}{2}$ $y=-\frac{5}{2}x+\frac{17}{2}$ ...(2) Solving (1) and (2), the two values are 3.37 + 0.0715i, 2.62 + 1.93i	
(b)(i)	$w=2-2\mathrm{i}=\sqrt{8}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}$ $z=w^{\frac{1}{3}}$ $z^3=w=\sqrt{8}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}=2^{\frac{3}{2}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}+2k\pi\right)}=2^{\frac{3}{2}}\mathrm{e}^{\mathrm{i}\pi\left(\frac{8k-1}{4}\right)}$ $z=2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\pi\left(\frac{8k-1}{12}\right)},\ k=-1,0,1$ $z=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}, z=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{12}\right)}, z=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{12}\right)}$	
(ii)	$\arg\left(w^*w^n\right)=\frac{\pi}{2}$ $\arg\left(w^*ww^{n-1}\right)=\frac{\pi}{2}$ $\arg\left(w^*w\right)+\arg\left(w^{n-1}\right)=\frac{\pi}{2}$ $\arg\left(w^{n-1}\right)=\frac{\pi}{2}$ since $w^*w=\lvert w\rvert^2$ is a positive real number $(n-1)\arg w=\frac{\pi}{2}$ $(n-1)\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}$ or $-\frac{3\pi}{2}$ or $-\frac{7\pi}{2}$... $-\frac{n\pi}{4}=\frac{\pi}{4}$ or $-\frac{7\pi}{4}$ or $-\frac{15\pi}{4}$... Required $n=7$	

Section B: Statistics

Question 5

No.	Suggested Solution	Remarks for Student
(i)	$\mathrm{P}(R*)+\mathrm{P}(B*)+\mathrm{P}(Y*)=\left(\frac{1}{7}\right)\left(\frac{1}{2}\right)+\left(\frac{2}{7}\right)\left(\frac{1}{3}\right)+\left(\frac{4}{7}\right)\left(\frac{1}{6}\right)=\frac{11}{42}$	
(ii)	$\mathrm{P}(B\mid W)=\frac{\mathrm{P}(B*)}{\mathrm{P}(W)}=\frac{\left(\frac{2}{7}\right)\left(\frac{1}{3}\right)}{\frac{11}{42}}=\frac{4}{11}$	
(iii)	$\mathrm{P}(R*)\times\mathrm{P}(B*)\times\mathrm{P}(Y*)3!=\left(\frac{1}{7}\right)\left(\frac{1}{2}\right)\left(\frac{2}{7}\right)\left(\frac{1}{3}\right)\left(\frac{4}{7}\right)\left(\frac{1}{6}\right)3!=\frac{4}{1029}$	

Question 6

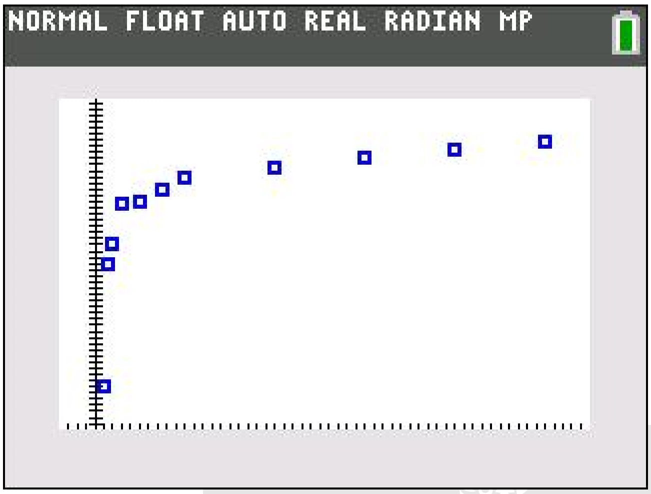
No.	Suggested Solution	Remarks for Student
	<table><tr><td></td><td>P</td><td>D</td><td>A</td><td>F</td><td>Total</td></tr><tr><td>Male</td><td>2345</td><td>1013</td><td>237</td><td>344</td><td>3939</td></tr><tr><td>Female</td><td>867</td><td>679</td><td>591</td><td>523</td><td>2660</td></tr><tr><td>Total</td><td>3212</td><td>1692</td><td>828</td><td>867</td><td>6599</td></tr></table>	
(i)(a)	$\frac{3939}{6599}=59.69\approx 60$	
(b)	$\frac{679}{6599}=10.29\approx 10$	
(ii)	Part (i) only takes into account gender and dept NOT age. Age group may not be evenly spread across all dept and gender.	
(iii)	Null hypothesis, $H_0:\mu=37$ Alternative hypothesis, $H_1:\mu<37$ Perform an one-tailed test at 5% significance level. Under H_0, $\bar{X}\sim N\left(37,\frac{140}{80}\right)$ approximately by Central Limit Theorem since $n=80$ is large Managing director's belief should be accepted $\Rightarrow H_0$ is rejected $p\text{-value}=P\left(\bar{X}<\bar{x}\right)\le 0.05$ $\Rightarrow 0<\bar{x}\le 34.824$ $\therefore$ Set of values of x is $(0,34.8]$	
(iv)	Now, perform an one-tailed test at α % significance level. Managing director's belief should not be accepted $\Rightarrow H_0$ is not rejected	

	p-value $= \mathrm{P}\left(\bar{X} < 35.2\right) > \dfrac{\alpha}{100}$ $\Rightarrow 0.086809 > \dfrac{\alpha}{100}$ $\Rightarrow 0 < \alpha < 8.6809$ $\Rightarrow 0 < \alpha \le 8.68$ $\therefore$ Set of values of α is $\left(0, 8.68\right]$	

Question 7

No.	Suggested Solution	Remarks for Student
(i)	Number of ways is ${}^{4}P_{3} = 24$	
(ii)	Number of ways is ${}^{10}P_{3} - {}^{6}P_{3} - {}^{4}P_{3} = 720 - 120 - 24 = 576$	Applying complement principle
(iii)	Required probability $= \dfrac{(8-1)!3!}{(10-1)!} = \dfrac{1}{12}$	Grouping method
(iv)	Required probability $= \dfrac{(7-1)!\,{}^{7}P_{3}}{(10-1)!} = \dfrac{5}{12}$	Slotting method

Question 8

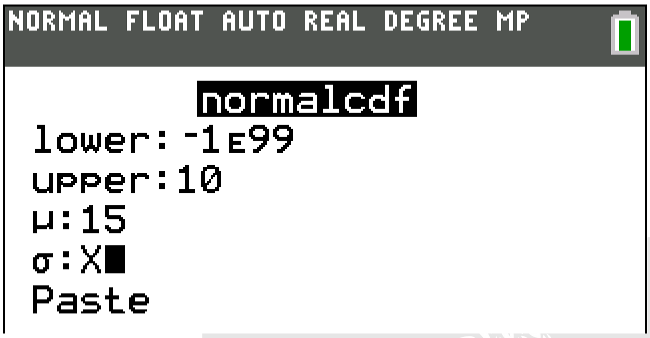
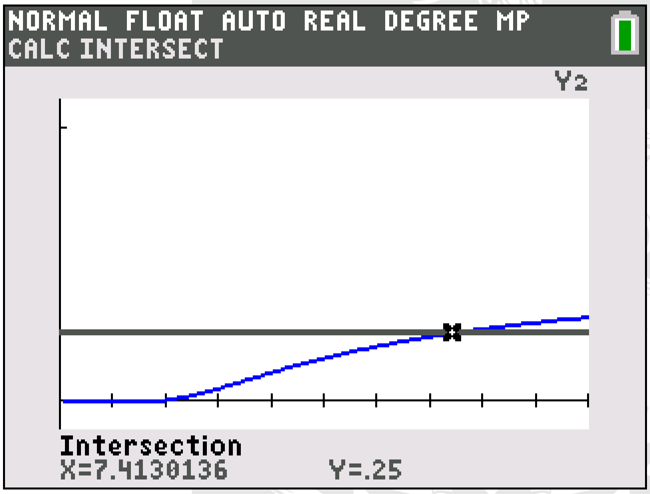
<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(a)(i)</td><td>


(3, 87.4) should be excluded.
</td><td>Read question carefully. Question requires students to use GRAPH paper, with given scale.

Is (1,72.5) acceptable?</td></tr>
<tr><td>(ii)</td><td>Scatter diagram does not display linear relationship, since as x increases, y increases at a decreasing rate. Thus it should not be modelled as an equation of the form
$y = ax + b$</td><td></td></tr>
<tr><td>(iii)</td><td>$y = \frac{c}{x} + d$
All x and y values are positive, thus $d > 0$ to account for large values of x when $\frac{c}{x} \to 0$.
Values of y increases as x increase, so $c < 0$</td><td></td></tr>
<tr><td>(iv)</td><td>
NORMAL FLOAT AUTO REAL RADIAN MP
<table>
<tr><th>L1</th><th>L2</th><th>L3</th><th>L4</th><th>L5</th><th>3</th></tr>
<tr><td>1</td><td>72.5</td><td>1</td><td>------</td><td>------</td><td></td></tr>
<tr><td>1.5</td><td>82.5</td><td>.66667</td><td></td><td></td><td></td></tr>
<tr><td>2</td><td>84</td><td>.5</td><td></td><td></td><td></td></tr>
<tr><td>5</td><td>87.5</td><td>.2</td><td></td><td></td><td></td></tr>
<tr><td>7.5</td><td>88.5</td><td>.13333</td><td></td><td></td><td></td></tr>
<tr><td>10</td><td>89.5</td><td>.1</td><td></td><td></td><td></td></tr>
<tr><td>20</td><td>90.2</td><td>.05</td><td></td><td></td><td></td></tr>
<tr><td>30</td><td>91</td><td>.03333</td><td></td><td></td><td></td></tr>
<tr><td>40</td><td>91.7</td><td>.025</td><td></td><td></td><td></td></tr>
<tr><td>50</td><td>92.4</td><td>.02</td><td></td><td></td><td></td></tr>
<tr><td>------</td><td>------</td><td>------</td><td></td><td></td><td></td></tr>
</table>
L3(1)=1
</td><td></td></tr>
</table>

	 Product Moment Correlation Coefficient is –0.980 (3sf) $c = -17.5$ (3sf) and $d = 91.8$ (3sf)	
(v)	Value of y when $x = 3$ is 85.922 = 85.9 (3sf) $r = -0.97955$ is near -1 which suggest good fit of the model. Besides, the value 3 is within the given range of values of x, so we did not extrapolate the information.	

Question 9

No.	Suggested Solution	Remarks for Student
(a)	$X \sim \mathrm{N}\left(15, a^2\right)$ $\mathrm{P}(10 < X < 20) = 0.5 \quad \Rightarrow \mathrm{P}(X < 10) = 0.25$ $\therefore a = 7.41$ (3 s.f.)  	Note that (10,20) is symmetrical about 15. Alternatively, you may standardize X
(b)	$Y \sim \mathrm{B}(4, p)$ $\mathrm{P}(Y=1) + \mathrm{P}(Y=2) = 0.5$ ${}^4C_1 p(1-p)^3 + {}^4C_2 p^2 (1-p)^2 = \frac{1}{2}$ $4p\left(1-3p+3p^2-p^3\right) + 6p^2\left(1-2p+p^2\right) = \frac{1}{2}$ $4p - 12p^2 + 12p^3 - 4p^4 + 6p^2 - 12p^3 + 6p^4 = \frac{1}{2}$ $4p - 6p^2 + 2p^4 = \frac{1}{2}$ $4p^4 - 12p^2 + 8p = 1$ (shown) For $0 < p < 1$, $p = 0.166,\ 0.599$	
(c)	Let V be the number correct answers out of 100.	

	$V \sim \mathrm{B}\left(100, \frac{1}{3}\right)$ Since $n = 100$ is large, and $np = \frac{100}{3} > 5$ and $n(1-p) = \frac{200}{3} > 5$, $V \sim \mathrm{N}\left(\frac{100}{3}, \frac{200}{9}\right)$ approximately. $\mathrm{P}(V \geq 30) = \mathrm{P}(Y \geq 29.5)$ by continuity corrections $= 0.792$ (3 s.f.)	

Question 10

No.	Suggested Solution	Remarks for Student
(i)	Number of weeds are assumed to be uniformly distributed across the field. The existence of weeds are independently of each other.	
	Let X be the number of dandelion plants per m^2. $X \sim \text{Po}(1.5)$	
(ii)	$\text{P}(X \geq 2) = 1 - \text{P}(X \leq 1) = 1 - 0.55783 \approx 0.442$	
(iii)	$Y = X_1 + X_2 + X_3 + X_4 \sim \text{Po}(6)$ $\text{P}(Y \leq 3) = 0.15120 \approx 0.151$	
(iv)	$W = X_1 + X_2 + ... + X_{80} \sim \text{Po}(120)$ Since 120 is large, $W \sim \text{N}(120, 120)$ approximately. $\text{P}(110 \leq W \leq 140) = \text{P}(109.5 < W < 140.5)$ by continuity corrections $= 0.80045$ $= 0.800$ (3 s.f.)	
(v)	Let D be the number of daisies per m^2. $D \sim \text{Po}(\lambda)$ Let E be the number of daisies in 2m^2. $E \sim \text{Po}(2\lambda)$ $\text{P}(D \leq 2) = \text{P}(E > 2)$ $\text{e}^{-\lambda} + \frac{\text{e}^{-\lambda}\lambda}{1!} + \frac{\text{e}^{-\lambda}\lambda^2}{2!} = 1 - \left(\text{e}^{-2\lambda} + \frac{\text{e}^{-2\lambda}(2\lambda)}{1!} + \frac{\text{e}^{-2\lambda}(2\lambda)^2}{2!}\right)$ $\text{e}^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right) + \text{e}^{-2\lambda}\left(1 + 2\lambda + 2\lambda^2\right) = 1$ $\lambda = 1.8543 \approx 1.85$ (3 s.f.)	