ANNEX B

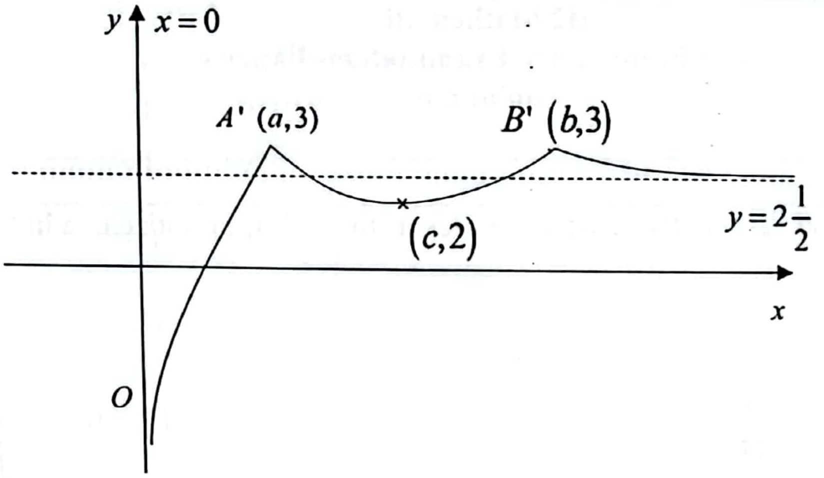
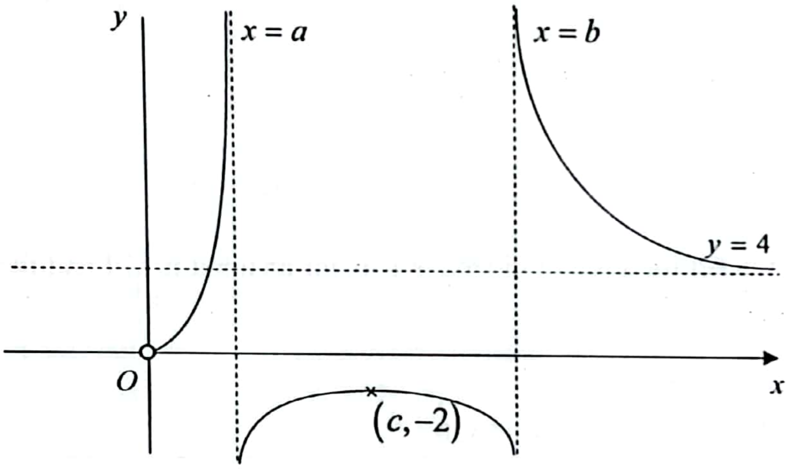
IJC H2 Preliminary Examination (Paper 1)

Qn/No	Topic Set	Answers
1	System of linear equations	\$570
2	Further Curve Sketching	------
3	Small angle approximation, Binomial expansion	$AC \approx 3+\frac{2}{3}\theta^2$, where $a=3$ and $b=\frac{2}{3}$
4	Application of differentiation (Stationary value)	(i) $h=\frac{2}{x^2}$ (iii) 0.89
5	Vectors (scalar and cross-product)	(i) $\lvert \mathbf{b}-\mathbf{a}\rvert=\sqrt{7}\lvert \mathbf{b}\rvert$ (ii) $\sqrt{\frac{3}{7}}\lvert \mathbf{b}\rvert$
6	Application of Integration (Volume of revolution)	(ii) $a=0,\ b=\frac{\pi}{2}$ $\frac{2}{3}\pi$ units^3
7	Application of differentiation (Tangent & Normal), Maclaurin Series	(i) $y=-\frac{1}{2}x+\frac{2}{5}$ (ii) $y=\frac{2}{3}-\frac{3}{2}x+\frac{27}{20}x^2+\ldots$ (iii) $y=\frac{2}{3}-\frac{3}{2}x$
8	Complex numbers	(a) $k=\sqrt{3}\tan\left(-\frac{\pi}{5}\right)$ or $k=\sqrt{3}\tan\left(-\frac{2\pi}{5}\right)$ (b)(ii) $2\sin\theta$; $\theta-\frac{\pi}{2}$
9	Functions	(ii) $\{x\in\mathbb{R}:\ x\geq 0.5,\ x\neq 3\}$ (iii) $\mathrm{gf}(x)=-\frac{1}{x^2-x-6},\quad x\leq\frac{1}{2};\ -1$
10	Mathematical Induction, Sequence & Series (M.O.D.)	(ii) $\frac{1}{2}-\frac{N+1}{(N+2)!}$ (iii) $\sum_{n=1}^{\infty}\frac{n^2+n-1}{(n+2)!}\to\frac{1}{2}$ which is a constant, hence it is a convergent series. $S_\infty=\frac{1}{2}$

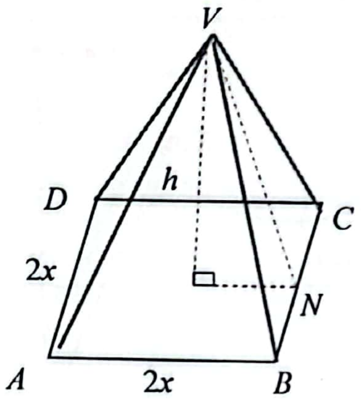
11	Application of differentiation (Stationary point), Curve Sketching, Application of Integration (Area)	(iii) $\int_{\sqrt{e}-1}^{\sqrt{e}} \frac{\sqrt{1-\left(x-\sqrt{e}\right)^2}}{2e}\,dx - \int_{1}^{\sqrt{e}} \frac{\ln x}{x^2}\,dx$; 0.0543
12	Complex numbers (including Loci)	(a)(i) $2^{\frac{1}{6}}e^{i\frac{-11\pi}{12}}, 2^{\frac{1}{6}}e^{i\frac{-7\pi}{12}}, 2^{\frac{1}{6}}e^{i\frac{-\pi}{4}}, 2^{\frac{1}{6}}e^{i\frac{\pi}{12}}, 2^{\frac{1}{6}}e^{i\frac{5\pi}{12}}, 2^{\frac{1}{6}}e^{i\frac{3\pi}{4}}$ (a)(iii) 6.735 (b)(ii) Maximum $\lvert w+10\rvert = 26$; Minimum $\lvert w+10\rvert = 12$

Innova Junior College
H2 Mathematics
JC2 Preliminary Examinations Paper 1
Solutions

1	**Solution**
	Let $\$x$, $\$y$ and $\$z$ be the cost of a ticket for a senior citizen, adult and child respectively. $2x+19y+9z=1982$ $10y+3z=908$ $x+7y+4z=778$ Using GC, $x=36$ $y=74$ $z=56$ Thus, the cost of a ticket for a senior citizen is \$36, for an adult is \$74 and for a child is \$56. 4(36) + 5(74) + 1(56) = 570 Therefore, the total cost for Group D = \$570

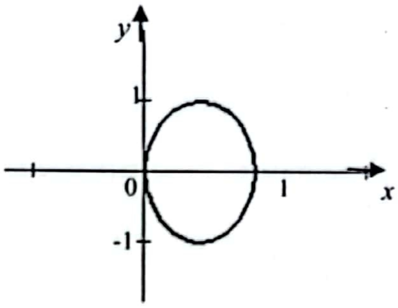
2	Solution
(a)	
(b)	

3	Solution
	Using cosine rule, $$AC^2 = 1^2 + 4^2 - 2(1)(4)\cos\theta$$ $$= 1 + 16 - 8\cos\theta$$ $$= 17 - 8\cos\theta$$ $$\approx 17 - 8\left(1 - \frac{\theta^2}{2}\right)$$ $$= 9 + 4\theta^2$$ $$AC \approx \left(9 + 4\theta^2\right)^{\frac{1}{2}} \quad (\because AC > 0)$$ $$AC \approx \left(9 + 4\theta^2\right)^{\frac{1}{2}}$$ $$= 9^{\frac{1}{2}}\left(1 + \frac{4}{9}\theta^2\right)^{\frac{1}{2}}$$ $$= 3\left(1 + \frac{1}{2}\left(\frac{4}{9}\theta^2\right) + \ldots\right)$$ $$\approx 3\left(1 + \frac{2}{9}\theta^2\right)$$ $$= 3 + \frac{2}{3}\theta^2$$ Therefore, $a = 3$ and $b = \frac{2}{3}$

4	**Solution**
(i)	 Volume of the pyramid $= \dfrac{8}{3}$ $\Rightarrow \frac{1}{3}(2x)^2 h = \frac{8}{3}$ $\Rightarrow h = \frac{2}{x^2}$
(ii)	In triangle VBC, height $= VN = \sqrt{h^2 + x^2}$ Area of the triangle $VBC = \frac{1}{2}(2x)\sqrt{h^2 + x^2}$ $= x\sqrt{\frac{4}{x^4} + x^2} = x\sqrt{x^2\left(\frac{4}{x^6} + 1\right)}$ $= x^2\sqrt{1 + 4x^{-6}}$ Hence total surface area of the pyramid, $S =$ base area $+ 4 \times$ area of triangle VBC $= (2x)^2 + 4\left(x^2\sqrt{1 + 4x^{-6}}\right)$ $S = 4x^2\left[1 + \sqrt{\left(1 + \frac{4}{x^6}\right)}\right]$ (shown)
(iii)	$S = 4x^2\left[1 + \sqrt{(1 + 4x^{-6})}\right]$ $\frac{dS}{dx} = 4x^2\left(\frac{1}{2}\left(1 + 4x^{-6}\right)^{-\frac{1}{2}}\left(-24x^{-7}\right)\right) + (8x)\left[1 + \sqrt{(1 + 4x^{-6})}\right]$ $= -48x^{-5}\left(1 + 4x^{-6}\right)^{-\frac{1}{2}} + 8x\left[1 + \sqrt{(1 + 4x^{-6})}\right]$ $= -8x\left[6x^{-6}\left(1 + 4x^{-6}\right)^{-\frac{1}{2}} - 1 - \sqrt{(1 + 4x^{-6})}\right]$ At the stationary value of S, $\frac{dS}{dx} = 0$.

	$\therefore\quad -8x\left[6x^{-6}\left(1+4x^{-6}\right)^{-\frac{1}{2}}-1-\sqrt{(1+4x^{-6})}\right]=0$ By G.C., $x=0.89090=0.89$ (to 2dp)

5	Solution
(i)	$(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \lvert\mathbf{b}\rvert^2 + \lvert\mathbf{a}\rvert^2 - 2\mathbf{a}.\mathbf{b}$ $= \lvert\mathbf{b}\rvert^2 + 9\lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60°$ $\lvert\mathbf{b}-\mathbf{a}\rvert^2 = 10\lvert\mathbf{b}\rvert^2 - 2(3\lvert\mathbf{b}\rvert)\lvert\mathbf{b}\rvert\frac{1}{2}$ $\lvert\mathbf{b}-\mathbf{a}\rvert = \sqrt{7}\lvert\mathbf{b}\rvert$ Therefore, $k = \sqrt{7}$.
(ii)	$\mathbf{c} = \frac{1}{3}\mathbf{a}$ $\overrightarrow{CA} = \frac{2}{3}\mathbf{a}$ Shortest distance of C to l = $\dfrac{\left\lvert\frac{2}{3}\mathbf{a}\times(\mathbf{b}-\mathbf{a})\right\rvert}{\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \dfrac{\left\lvert\frac{2}{3}\mathbf{a}\times\mathbf{b} - \frac{2}{3}\mathbf{a}\times\mathbf{a}\right\rvert}{\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \dfrac{2\lvert\mathbf{a}\times\mathbf{b}\rvert}{3\lvert\mathbf{b}-\mathbf{a}\rvert} \quad \because \mathbf{a}\times\mathbf{a} = \mathbf{0}$ $= \dfrac{2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 60°}{3\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \dfrac{6\lvert\mathbf{b}\rvert^2\frac{\sqrt{3}}{2}}{3\sqrt{7}\lvert\mathbf{b}\rvert}$ $= \dfrac{\sqrt{3}\lvert\mathbf{b}\rvert}{\sqrt{7}}$ $= \sqrt{\dfrac{3}{7}}\lvert\mathbf{b}\rvert$

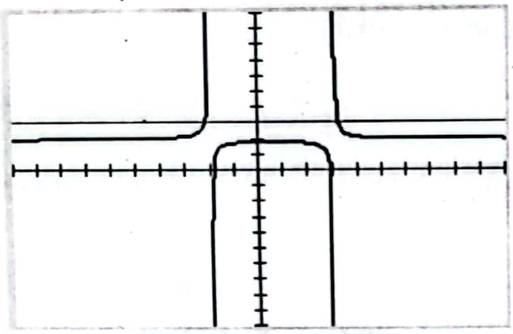
6	Solution
(i)	$x = \cos^2 \theta, \quad y = \sin 2\theta, \quad \text{for } -\frac{\pi}{2} < \theta \le \frac{\pi}{2}.$ 
(ii)	$x = \cos^2 \theta$ $dx = -2\cos\theta \sin\theta \, d\theta$ When $y = 0$, $\sin 2\theta = 0$ $2\theta = 0, \pi$ $\theta = 0, \frac{\pi}{2}$ When $\theta = 0$, $x = 1$; When $\theta = \frac{\pi}{2}$, $x = 0$ Volume of the solid formed $= \pi \int_0^1 y^2 \, dx$ $= \pi \int_{\frac{\pi}{2}}^{0} (\sin 2\theta)^2 (-2\cos\theta \sin\theta \, d\theta)$ $= \pi \int_{\frac{\pi}{2}}^{0} (\sin 2\theta)^2 (-\sin 2\theta \, d\theta)$ $= \pi \int_0^{\frac{\pi}{2}} \sin^3 2\theta \, d\theta$ (shown) $\therefore a = 0, \ b = \frac{\pi}{2}.$ Let $u = \cos 2\theta$. $\therefore du = -2\sin 2\theta \, d\theta$ When $\theta = 0$, $u = 1$; When $\theta = \frac{\pi}{2}$, $u = -1$ Volume of the solid formed $= \pi \int_0^{\frac{\pi}{2}} \sin^3 2\theta \, d\theta$

	$=\frac{1}{2}\pi\int_0^{\frac{\pi}{2}}\sin^2 2\theta(2\sin 2\theta)\mathrm{d}\theta$ $=\frac{1}{2}\pi\int_0^{\frac{\pi}{2}}\left(1-\cos^2 2\theta\right)(2\sin 2\theta\,\mathrm{d}\theta)$ $=\frac{1}{2}\pi\int_1^{-1}\left(1-u^2\right)(-\mathrm{d}u)$ $=-\frac{1}{2}\pi\left[u-\frac{u^3}{3}\right]_1^{-1}$ $=-\frac{1}{2}\pi\left[\left(-1+\frac{1}{3}\right)-\left(1-\frac{1}{3}\right)\right]$ $=-\frac{1}{2}\pi\left(-\frac{4}{3}\right)$ $=\frac{2}{3}\pi$ units^3

7	Solution
(i)	$3y^3 - 8y^2 + 10y = 4 - 5x$ Differentiate wrt x, $(9y^2 - 16y + 10)\frac{\mathrm{d}y}{\mathrm{d}x} = -5$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-5}{9y^2 - 16y + 10}$ When $x = \frac{4}{5}$, $3y^3 - 8y^2 + 10y = 0$. $\therefore y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}$. Eqn of tangent : $y - 0 = -\frac{1}{2}\left(x - \frac{4}{5}\right)$, ie $y = -\frac{1}{2}x + \frac{2}{5}$
(ii)	$(9y^2 - 16y + 10)\frac{\mathrm{d}y}{\mathrm{d}x} = -5$ Differentiate wrt x, $(9y^2 - 16y + 10)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (18y - 16)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0$. When $x = 0$, $y = \frac{2}{3}, \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{2}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{27}{10}$. $\therefore \quad y = \frac{2}{3} - \frac{3}{2}x + \frac{27}{20}x^2 + \ldots$
(iii)	$y = \frac{2}{3} - \frac{3}{2}x$

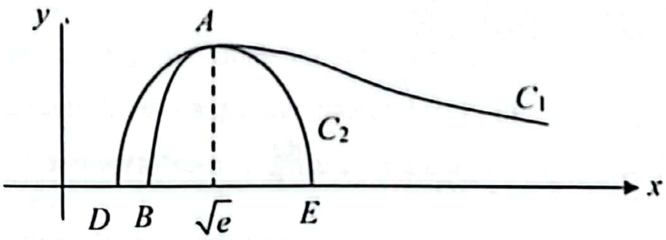
8	Solution
(a)	$\arg(w^5)=5\arg(w)=0,\pm\pi,\pm2\pi\ldots$ $\arg(w)=0,\frac{\pi}{5},-\frac{\pi}{5},\frac{2\pi}{5},\frac{-2\pi}{5},\ldots$ Since $k<0$, $\arg(w)=-\frac{\pi}{5}$ or $-\frac{2\pi}{5}$. $\frac{k}{\sqrt{3}}=\tan\left(-\frac{\pi}{5}\right)$ or $\frac{k}{\sqrt{3}}=\tan\left(-\frac{2\pi}{5}\right)$ $k=\sqrt{3}\tan\left(-\frac{\pi}{5}\right)$ or $k=\sqrt{3}\tan\left(-\frac{2\pi}{5}\right)$ $n=-\frac{1}{5}$ or $-\frac{2}{5}$
(bi)	**Method 1** $1-z^2=1-(\cos\theta+\mathrm{i}\sin\theta)^2$ $=1-\left(\cos^2\theta+2\mathrm{i}\cos\theta\sin\theta+(\mathrm{i}\sin\theta)^2\right)$ $=1-\left(1-\sin^2\theta+2\mathrm{i}\sin\theta\cos\theta-\sin^2\theta\right)$ $=1-1+2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta(\sin\theta-\mathrm{i}\cos\theta)$ **Method 2** $1-z^2=1-(\cos\theta+\mathrm{i}\sin\theta)^2$ $=1-(\cos2\theta+\mathrm{i}\sin2\theta)$ $=1-\cos2\theta-\mathrm{i}\sin2\theta$ $=1-(1-2\sin^2\theta)-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta(\sin\theta-\mathrm{i}\cos\theta)$
(bii)	**Method 1** $\lvert 1-z^2\rvert=\lvert 2\sin\theta(\sin\theta-\mathrm{i}\cos\theta)\rvert$ $=2\sin\theta\sqrt{\sin^2\theta+\cos^2\theta}$ $=2\sin\theta$ Given that $0\le\theta\le\frac{\pi}{2}$

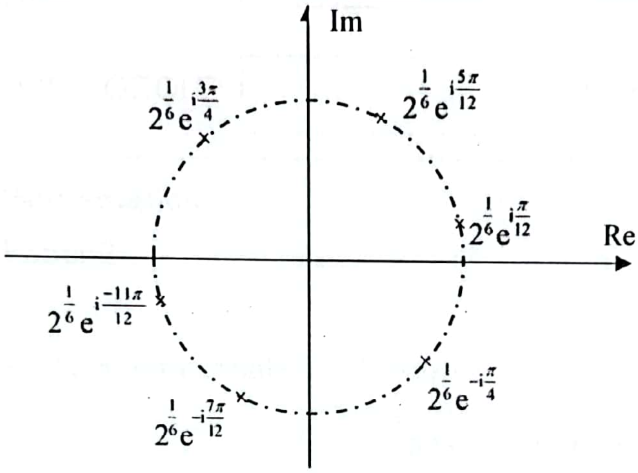
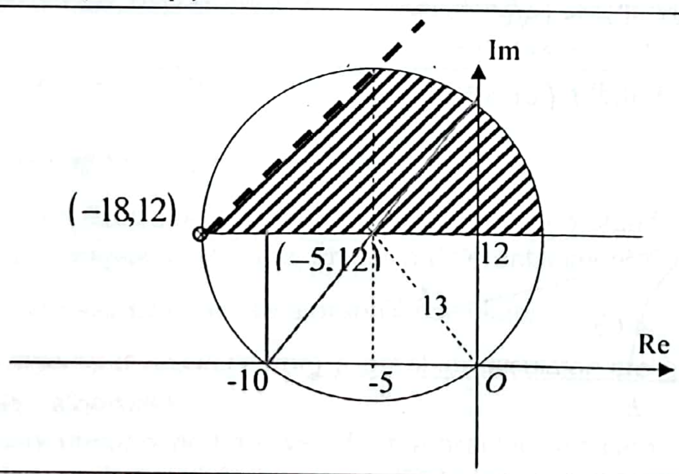
	$$\arg(1-z^2) = \arg[2\sin\theta(\sin\theta - \mathrm{i}\cos\theta)]$$ $$= \arg(2\sin\theta) + \arg(\sin\theta - \mathrm{i}\cos\theta)$$ $$= 0 - \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$ $$= -\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)$$ $$= -\left(\frac{\pi}{2}-\theta\right)$$ $$= \theta - \frac{\pi}{2}$$ **Method 2** $$1-z^2 = 2\sin\theta(\sin\theta - \mathrm{i}\cos\theta)$$ $$= 2\sin\theta(-\mathrm{i})(\cos\theta + \mathrm{i}\sin\theta)$$ $$= (-2\mathrm{i}\sin\theta)\mathrm{e}^{\mathrm{i}\theta}$$ $$\left|1-z^2\right| = \left|(-2\mathrm{i}\sin\theta)\mathrm{e}^{\mathrm{i}\theta}\right|$$ $$= 2\sin\theta$$ $$\arg(1-z^2) = \arg\left((-2\mathrm{i}\sin\theta)\mathrm{e}^{\mathrm{i}\theta}\right)$$ $$= \arg(-2\mathrm{i}\sin\theta) + \arg\left(\mathrm{e}^{\mathrm{i}\theta}\right)$$ $$= -\frac{\pi}{2} + \theta$$ **Method 3** $$1-z^2 = 2\sin\theta(\sin\theta - \mathrm{i}\cos\theta)$$ $$= 2\sin\theta\left(\cos\left(\frac{\pi}{2}-\theta\right) - \mathrm{i}\sin\left(\frac{\pi}{2}-\theta\right)\right)$$ $$= 2\sin\theta\left(\cos\left(\theta-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta-\frac{\pi}{2}\right)\right)$$ $$\left|1-z^2\right| = 2\sin\theta$$ $$\arg(1-z^2) = \theta - \frac{\pi}{2}$$

9	Solution
(i)	 From graph, the horizontal line $y=3$ cuts the graph at two points. Hence f is not a **one-one** function, hence f^{-1} does not exist.
(ii)	$f(x)=\frac{1}{x^2-x-6}+2$ $f'(x)=-(x^2-x-6)^{-2}(2x-1)$ $=\frac{1-2x}{(x^2-x-6)^2}$ For the function to be decreasing, $f'(x)\le 0$. $1-2x\le 0$ $1\le 2x$ $x\ge 0.5$ $\{x\in\mathbb{R}: x\ge 0.5, x\ne 3\}$
(iii)	$gf(x)=2-\frac{1}{x^2-x-6}-2$ $=-\frac{1}{x^2-x-6},\ x\le\frac{1}{2}$ $(gf)^{-1}\left(\frac{1}{4}\right)=x$ $gf(x)=\frac{1}{4}$ $-\frac{1}{x^2-x-6}=\frac{1}{4}$ $x^2-x-6=-4$ $x^2-x-2=0$ $(x-2)(x+1)=0$ $x=2$ (rejected) or $x=-1$ $x=-1$

10	**Solution**
(i)	Let P_n be the statement $u_n = \frac{n}{(n+1)!}$ for $n \in \mathbb{Z}^+$. P_1 is true since $u_1 = \frac{1}{2!} = \frac{1}{2}$. Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \frac{k}{(k+1)!}$ Consider P_{k+1}: i.e. $u_{k+1} = \frac{k+1}{(k+2)!}$ $u_{k+1} = \frac{k}{(k+1)!} - \frac{k^2+k-1}{(k+2)!}$ $= \frac{k(k+2)-(k^2+k-1)}{(k+2)!}$ $= \frac{k^2+2k-k^2-k+1}{(k+2)!}$ $= \frac{k+1}{(k+2)!}$ Thus, P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
(ii)	$\sum_{n=1}^{N} \frac{n^2+n-1}{(n+2)!} = \sum_{n=1}^{N} (u_n - u_{n+1})$ [$u_1 - u_2$ $+ u_2 - u_3$ $+ u_3 - u_4$ $+ \ldots$ $+ u_{N-1} - u_N$ $+ u_N - u_{N+1}$] $= u_1 - u_{N+1} = \frac{1}{2} - \frac{N+1}{(N+2)!}$

(iii)	As $N \to \infty$, $\dfrac{N+1}{(N+2)!} \to 0$ $\sum_{n=1}^{\infty} \dfrac{n^2+n-1}{(n+2)!} \to \dfrac{1}{2}$ which is a constant, hence it is a convergent series.

11	**Solution**
(i)	$y=\dfrac{\ln x}{x^2},\ x\geq 1$ $\dfrac{dy}{dx}=\dfrac{x^2\cdot\frac{1}{x}-\ln x\cdot 2x}{x^4}$ $=\dfrac{x-2x\ln x}{x^4}$ When $\dfrac{dy}{dx}=0$ and since $x\neq 0$, $1-2\ln x=0$ $2\ln x=1$ $\ln x=\dfrac{1}{2}$ $x=e^{\frac{1}{2}}=\sqrt{e}$ When $x=\sqrt{e}$, $y=\dfrac{\ln\sqrt{e}}{\left(\sqrt{e}\right)^2}$ $=\dfrac{1}{2e}$ Hence the coordinates of A is $\left(\sqrt{e},\dfrac{1}{2e}\right)$.
(ii)	$B(1,0)$, $D\left(\sqrt{e}-1,0\right)$ and $E\left(\sqrt{e}+1,0\right)$ 
(iii)	Area $=\displaystyle\int_{\sqrt{e}-1}^{\sqrt{e}}\frac{\sqrt{1-\left(x-\sqrt{e}\right)^2}}{2e}\,dx-\int_1^{\sqrt{e}}\frac{\ln x}{x^2}\,dx$ $=0.14446942-0.09020401$ $=0.05426541$ $=0.0543$ (correct to 3 s.f.)

12	**Solution**
(ai)	$z^6 - 2\text{i} = 0$ $z^6 = 2\text{i}$ $z^6 = 2\text{e}^{\text{i}\left(\frac{\pi}{2}+2k\pi\right)}, k = 0, \pm 1, \pm 2, -3$ $z = 2^{\frac{1}{6}}\text{e}^{\text{i}\frac{-11\pi}{12}}, 2^{\frac{1}{6}}\text{e}^{\text{i}\frac{-7\pi}{12}}, 2^{\frac{1}{6}}\text{e}^{\text{i}\frac{-\pi}{4}}, 2^{\frac{1}{6}}\text{e}^{\text{i}\frac{\pi}{12}}, 2^{\frac{1}{6}}\text{e}^{\text{i}\frac{5\pi}{12}}, 2^{\frac{1}{6}}\text{e}^{\text{i}\frac{3\pi}{4}}$
(aii)	
(aiii)	Since *ABCDEF* is a regular hexagon, the triangles *OAB*, *OBC* … are equilateral triangles. Perimeter of the polygon $= 6 \times 2^{\frac{1}{6}}$ $= 6.735$ (to 3 d.p.)
(bi)	
(bii)	Minimum $\lvert w+10 \rvert = 12$ Maximum $\lvert w+10 \rvert = 26$ (diameter of circle)