

TEMASEK JUNIOR COLLEGE

JC2 Preliminary Examination



9649/02

Higher 2

FURTHER MATHEMATICS

Paper 2

12 September 2024 3 hours

Additional Materials: Answer Booklet List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Write your Civics Group and name on all the work that you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [] at the end of each question or part question.

Section A: Pure Mathematics [50 marks]

- 1 (a) Solve the equation $z^3 = 32 32\sqrt{3}i$, giving your answers z_1, z_2 and z_3 in exact form $re^{i\theta}$, where r > 0, $-\pi < \theta \le \pi$ and $\arg z_1 < \arg z_2 < \arg z_3$. [3]
 - (b) The roots z_1 , z_2 and z_3 in part (a) are represented by points P_1 , P_2 and P_3 . Find the exact perimeter of triangle $P_1P_2P_3$. [2]
- 2 The Fibonacci numbers F_n are defined by the conditions $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1}$$
 for all $n \ge 1$.

- (a) Compute F_2, F_3, F_4 and F_5 . [1]
- **(b)** Compute $F_{n+1}F_{n-1} F_n^2$ for n = 1, 2, 3, 4. [2]
- (c) Conjecture and prove by induction, for all $n \ge 1$, an expression for $F_{n+1}F_{n-1} F_n^2$. [6]
- 3 At the start of the month in January 2023, Sandy started her career as a lifestyle vlog content creator with 60 subscribers for her channel. By the end of each month, she lost 5% of her existing subscribers but gained 100 new subscribers each month through her publicity efforts. Let u_n be the number of subscribers at the end of the nth month taking January 2023 as the first month.
 - (a) Write down a recurrence relation to model the number of subscribers at the end of the n^{th} month. [2]
 - (b) Find the general formula of u_n in terms of *n*. Hence show that the number of subscribers at the end of December 2023 is 952, correct to the nearest integer. [4]
 - (c) Based on the model in (a), comment on the long-term prospect of Sandy's career. [2]

At the start of January 2024, Sandy decided to collaborate with another well-known content creator to gain more exposure for her own channel. By the end of each month, she gained k% of the number of existing subscribers but lost 100 existing subscribers.

Let w_n be the number of subscribers at the end of the n^{th} month taking January 2024 as the first month.

(d) By considering the general formula of w_n , find an inequality in terms of k that will lead to an increase in the number of subscribers in the long run. Hence find the least integer value of k. [4]

4 A car travels over a rough surface. The vertical motion of the front suspension is modelled by the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 25 \, y = 30 \cos 5t$$

where y is the vertical displacement of the top of the suspension and t is time.

(a) Find the general solution of the above differential equation. [6]

It is given that y = 1 and $\frac{dy}{dt} = 0$ initially.

(b) Find the solution subject to these conditions.

A second model of the motion of the suspension is given by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 25y = 30\cos 5t$$

- (c) Verify that $y = 3\sin 5t$ is a particular integral for this differential equation. Hence find the general solution. [3]
- (d) Compare the behaviour of the suspension predicted by the two models. [1]

5 The ellipse *E* is given by the equation $\frac{x^2}{9} + \frac{y^2}{8} = 1$. $P(x_0, y_0)$ is a point on *E* where $x_0 > 0$ and $y_0 > 0$.

The tangent to *E* at *P* intersects the *x*-axis at a point *Q* and *O* is the origin.

(a) Show that
$$(\tan \angle POQ) \cdot (\tan \angle PQO) = \frac{8}{9}$$
. [4]

(b) (i) Find the eccentricity of E. [1]

 $F_1(-c, 0)$ and $F_2(c, 0)$ are the foci of *E*, where c > 0.

Write down the polar equation of E if the pole is located at

- (ii) F_1 , [1]
- (iii) F_2 . [1]

It is given that $\angle PF_2Q = \theta$ and $\angle F_2PQ = \alpha$.

(iv) Using the reflective property of ellipse and both polar equations above, show that $\frac{4}{3 + \cos \theta} + \frac{4}{3 + \cos (\theta + 2\alpha)} = 3.$ [5]

[Turn over

[2]

Section B: Probability and Statistics [50 marks]

- 6 A random sample of 100 university students were taken from the faculty of Medicine, Dentistry and Health Sciences. The sample consists of 32 Medicine students, 16 Dentistry students and 52 Health Sciences students. Out of the 100 students, 65 were females. A test, at the 1% level of significance was carried out on this data and it was found that there is association between the types of courses enrolled and gender. Given that there were 11 female dentistry students and *n* male medicine students, find the set of possible values of *n*. [5]
- 7 The Weibull distribution is a versatile distribution characterised by its ability to model a wide range of data types. It has important applications in meteorological studies such as wind speeds and thunderstorms. The probability density function is given by

$$f(x) = k\lambda^k x^{k-1} e^{-(\lambda x)^k} , \quad x > 0$$

where k is the shape parameter and λ is a non-negative scale parameter.

(a) For the case k = 2, show that f(x) is a probability density function for any value of λ .

[2]

At a weather station, the wind speed is measured at noon each day. The wind speed, X m/s, is modelled by the Weibull distribution.

(b) For the case k = 2 and $\lambda = 0.25$, explain what the shape of the density function indicates about the wind speed at noon at the weather station. Find the most likely wind speed. [2]

The average power output of wind turbines near the weather station is related to the wind speed, *x*. A model is proposed in which the power, *Y* megawatts, is given by $Y = CX^3$, where *C* is a constant.

(c) For the case k = 2 and $\lambda = 0.25$, find E(Y) in terms of C. [2]

- 8 The Health Promotion Board recommends that adults should engage in at least 150 minutes of moderate-intensity aerobic activity per week. The results of a survey from a random sample of 2105 adults in Singapore show that 1577 adults follow the recommendation.
 - (a) Calculate a 95% confidence interval for the true proportion *p*, of adults in Singapore who engage in at least 150 minutes of moderate-intensity aerobic activity per week. Give the end points of the interval correct to 4 significant figures. [2]
 - (b) Give two reasons why this interval is an approximation. [2]
 - (c) Suppose that a 90% confidence interval is required, and that the width of the interval is no more than 0.04. Determine the smallest sample size that will satisfy the requirement regardless the value of p. [3]
- 9 Alvin and Bernard take alternate turns at kicking a football at a goal, and their probabilities of scoring a goal on each kick are p_1 and p_2 respectively, independently of previous outcomes. The first person to score allows the other person one more kick. If the other then scores, the game is drawn. If the other then misses, the first has won the game. Alvin begins a game.
 - (a) Show that the probability that Alvin scores first, on his *n*th kick, is $p_1(q_1q_2)^{n-1}$, where $q_1 = 1 p_1$ and $q_2 = 1 p_2$. Hence find a simplified expression for the probability that Alvin wins the game. [3]
 - (b) Find a simplified expression for the probability that the game is drawn. [3]

It is now given that $p_1 = p_2 = \frac{1}{3}$.

Find the expected total number of kicks in the game.

[2]

Studies have shown that food rich in a substance known as flavonoid help to lower blood pressure. Flavonoid is found naturally in dark chocolate and is absent in white chocolate. A team of researchers conducted a study to investigate whether there is a greater reduction in blood pressure for people who consume dark chocolate than people who consume white chocolate. A random sample of 19 healthy adults in a particular age group was chosen to participate in the study. Of the 19 healthy adults, 10 were randomly assigned to the dark chocolate group and the rest to the white chocolate group. At the beginning of the study, all participants had their blood pressure recorded in mmHg (millimetres of mercury) before adding a fixed amount of chocolate to their daily diet. At the end of 30 days, their blood pressure was recorded again. The sample mean and standard deviation of the difference of the blood pressures (before diet – after diet) for each group were as follow.

	Sample size	Mean (mmHg)	Standard deviation
			(mmHg)
Dark	10	6.6	2.9
White	9	4.5	2.6

- (a) A researcher concluded that since the difference in the sample means of 2.1 mmHg is greater than 0, there is convincing statistical evidence to conclude that the population mean reduction in blood pressure for those who consume dark chocolate is greater than for those who consume white chocolate. Comment on the researcher's conclusion. [2]
- (b) Carry out an appropriate hypothesis test, using a 5% significance level, and state the conclusion that the researcher should reach.State the assumptions you have made in carrying out the test. [7]
- (c) State one non-statistical assumption that is necessary for the validity of the conclusion made in part (b).
- (d) One of the values for standard deviation in the table was wrongly computed and the correct value should be greater than shown. Explain whether the use of the correct value would change the conclusion made in part (b). [2]

11 A dietician is studying the possible effect of a diet plan in lowering cholesterol levels. A random sample of 12 patients is selected and she measures their cholesterol levels before and after they have followed the diet plan for 12 weeks. Their decrease in cholesterol levels measured in millimoles per litre (mmol/L) after following the diet plan for 12 weeks are recorded as follows:

-0.1, 1.7, -1.2, 1.1, 1.4, 0.5, 0.9, 2.2, -1.0, 2, 0.7, 0.3.

- (a) (i) Explain why the dietician has used a paired design. [1]
 - (ii) Carry out a suitable Wilcoxon matched-pairs signed rank test to examine at the 5% level of significance whether, on the whole, the diet plan is effective in lowering cholesterol level. State any assumption(s) required for the test to be valid.
 - (iii) The dietician originally planned to carry out a paired sample t test. However, she decided that the distributional assumption required for this test might not have been satisfied. State this assumption and explain why in general, if this assumption is satisfied, it is preferable to carry out a t test rather than a Wilcoxon test.
 [2]
- (b) Suppose that *k* of the 12 volunteers reported weight loss while the rest of the volunteers reported weight gain after following the diet plan for 12 weeks. Determine the possible values of *k* if a sign test at the 5% level of significance shows that the diet plan had also helped in weight loss. [4]

