Mastery Questions

11. [N2007/1/2]

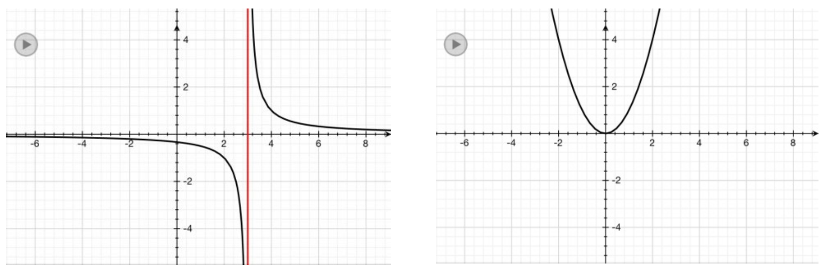
Functions f and g are defined by

$$\text{f}: x \mapsto \frac{1}{x-3}, \text{for } x \in \mathbb{R}, x \neq 3,$$
$$\text{g}: x \mapsto x^2, \text{for } x \in \mathbb{R}$$

(i) Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist.
(ii) Find $\text{f}^{-1}(x)$ and state the domain of f^{-1}.

[Ans: **(i)** $\text{gf}: x \mapsto \frac{1}{(x-3)^2},\ x \in \mathbb{R},\ x \neq 3$; **(ii)** $\text{f}^{-1}: x \mapsto 3 + \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0$]

[Solution]



(i) $R_\text{f} = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R} = D_\text{g}$ $\therefore$ gf exists.

$\text{gf}: x \mapsto \frac{1}{(x-3)^2},\ x \in \mathbb{R},\ x \neq 3$

$R_g = [0, \infty)$ is not a subset of $\mathbb{R} \setminus \{3\} = D_f$ $\therefore$ fg does not exist.

(ii) Let $y = \frac{1}{x-3}$

$x - 3 = \frac{1}{y}$

$x = \frac{1}{y} + 3$

$\therefore \text{f}^{-1}: x \mapsto 3 + \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0$

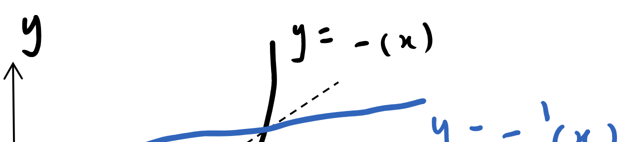
12. [N2008/2/4]

The function f is defined by $\text{f}: x \mapsto (x-4)^2 + 1$ for $x \in \mathbb{R}, x > 4$.

(i) Sketch the graph of $y = \text{f}(x)$. Your sketch should indicate the position of the graph in relation to the origin.
(ii) Find $\text{f}^{-1}(x)$, stating the domain of f^{-1}.
(iii) On the same diagram as in part **(i)**, sketch the graph of $y = \text{f}^{-1}(x)$.
(iv) Write down the equation of the line in which the graph of $y = \text{f}(x)$ must be reflected in order to obtain the graph of $y = \text{f}^{-1}(x)$, and hence find the exact solution of the equation $\text{f}(x) = \text{f}^{-1}(x)$.

[Ans: **(ii)** $\text{f}^{-1}(x) = 4 + \sqrt{x-1}, x \in \mathbb{R}, x > 1$; **(iv)** $y = x$; $x = \frac{9+\sqrt{13}}{2}$]

[Solution]



(i) and **(iii)**

(4, 1)

(ii) $y = (x-4)^2 + 1$

$(x-4)^2 = y - 1$

$x = 4 \pm \sqrt{y-1}$

Since $x > 4$, choose $x = 4 + \sqrt{y-1}$

$\mathrm{f}^{-1}(x) = 4 + \sqrt{x-1}, x \in \mathbb{R}, x > 1.$

(iv) It must be reflected about the line $y = x$

In this case, solving $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ is equivalent to solving $\mathrm{f}(x) = x$.

$(x-4)^2 + 1 = x$

$x^2 - 8x + 17 = x$

$x^2 - 9x + 17 = 0$

$x = \frac{9 \pm \sqrt{9^2 - 4(17)}}{2} = \frac{9 \pm \sqrt{13}}{2}$

Since $x > 4$, $x = \frac{9+\sqrt{13}}{2}$.

13 [N2014/1/1]

The function f is defined by $\mathrm{f} : x \mapsto \dfrac{1}{1-x}, x \in \mathbb{R}, x \neq 1, x \neq 0.$

(i) Show that $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$. [4]

(ii) Find $\mathrm{f}^3(x)$ in simplified form. [1]

[Solution]

(i)

$$\begin{aligned}
\mathrm{f}^2(x) &= \mathrm{f}(\mathrm{f}(x)) \\
&= \mathrm{f}\left(\frac{1}{1-x}\right) \\
&= \frac{1}{1-\left(\frac{1}{1-x}\right)} \\
&= \frac{1-x}{-x} \\
&= \frac{x-1}{x} \\
&= 1-\frac{1}{x}
\end{aligned}$$

$\therefore \mathrm{f}^2 : x \mapsto 1-\frac{1}{x}, x \in \mathbb{R}, x \neq 1, x \neq 0.$ (Since $\mathrm{D}_{\mathrm{f}^2} = \mathrm{D}_{\mathrm{f}}$)

$\mathrm{R_f} = \mathrm{D}_{\mathrm{f}^{-1}} = (-\infty, 0) \cup (0,1) \cup (1, \infty)$

Let $\mathrm{f}(x) = y$.

$$y = \frac{1}{1-x}$$

$$y(1-x) = 1$$

$$y - xy = 1$$

$$xy = y - 1$$

$$\begin{aligned}
x &= \frac{y-1}{y} \\
&= 1-\frac{1}{y}
\end{aligned}$$

$\therefore\ \mathrm{f}^{-1} : x \mapsto 1-\frac{1}{x}, x \in \mathbb{R}, x \neq 1, x \neq 0.$

Hence $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$.

(ii)

Since $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$,

$$\begin{aligned}
\mathrm{f}^3(x) &= \mathrm{f}(\mathrm{f}^{-1}(x)) \\
&= x
\end{aligned}$$