HWA CHONG INSTITUTION
2023 JC1 BLOCK TEST

FURTHER MATHEMATICS
Higher 2

9649/01

Paper 1

Wednesday **28 June 2023** **3 hours**

Additional materials: 12-page Answer Booklet
List of Formula (MF26)
4-page Additional Answer Booklet (upon request)

READ THESE INSTRUCTIONS FIRST

Write your name and class on the 12-page Answer Booklet and any other additional 4-page Answer Booklets you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.
Do not write anything on the List of Formula (MF26).

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, slot any additional 4-page Answer Booklets used in your 12-page Answer Booklet and indicate on the 12-page Answer Booklet the number of additional 4-page Answer Booklets used (if any)

1. ~~Prove by mathematical induction that~~ $\cancel{\left(\frac{a+b}{2}\right)^n \le \frac{a^n+b^n}{2}}$ ~~for~~ $\cancel{n \in \mathbb{Z}^+}$~~, where~~ $\cancel{a}$ ~~and~~ $\cancel{b}$ ~~are positive.~~ ~~[5]~~

2. A point P lies on the curve C with polar equation $r = 2a\cos\theta$, $-\frac{\pi}{2} < \theta \le \frac{\pi}{2}$, where a is a positive constant. The point N is the foot of perpendicular from the pole to the tangent of C at P.

(a) Sketch C, P and N on the same diagram. [2]

(b) By considering the polar coordinates of N, show that, as P varies, the locus of N is given by the polar equation $r = a(1+\cos\theta), -\pi < \theta \le \pi$. [4]

3. The sequence $\{u_n\}$ is given by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n\,, \quad n \in \mathbb{Z}^+,$$

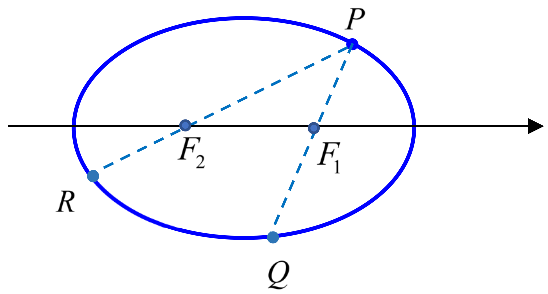
together with terms $u_1 = a$ and $u_2 = b$.

(a) Find the expression of u_n in terms of a and b. [4]

(b) Find algebraically the possible limits of $\left(\frac{u_n}{u_{n-1}}\right)$. [3]

4. **(a)** ~~A straight line l passes through the focus F of a conic with polar equation $r = \dfrac{k}{1+e\cos\theta}$ and meets the conic at 2 points A and B. Show that $\dfrac{1}{AF} + \dfrac{1}{BF} = \dfrac{2}{k}$.~~ ~~[4]~~

~~(b)~~



~~The diagram above shows an ellipse with foci F_1 and F_2. The points P, Q and R lies on the ellipse with PQ and PR passing through F_1 and F_2 respectively. By using the result in~~ **~~part (a)~~**~~, or otherwise, show that $\dfrac{PF_1}{F_1Q} + \dfrac{PF_2}{F_2R}$ is a constant.~~ ~~[4]~~

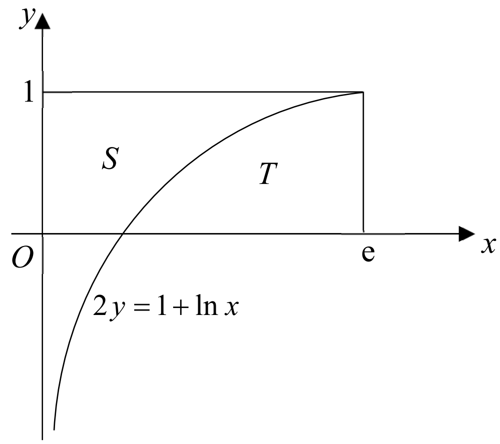
5. The points A and B have Cartesian coordinates $(a,0)$ and $(-a,0)$ respectively, where a is a positive constant. The point P is such that $AP \cdot BP = a^2$. The curve C describes the locus of P.

(a) Show that C has polar equation $r^2 = 2a^2 \cos 2\theta$, $0 \le \theta \le 2\pi$. [4]

(b) Sketch the graph of C, indicating all key features and symmetries of the curve. [2]

(c) Find the exact area of the region enclosed by the curve C. [3]

6. In the diagram, the curve with equation $2y = 1 + \ln x$ for $x > 0$ divides the rectangle bounded by the axes, the lines $y = 1$ and $x = \mathrm{e}$ into two regions, S and T.



(a) Show that the volume of the solid generated when S is rotated completely about the x-axis is given by $2\pi\int_0^1 \mathrm{F}(y)\,\mathrm{d}y$, where $\mathrm{F}(y)$ is a function to be determined. [1]

(b) Find the exact value of $\int_0^1 \mathrm{F}(y)\,\mathrm{d}y$. [2]

(c) By using the result in **part (b)**, find the exact value of $\int_{\mathrm{e}^{-1}}^{\mathrm{e}} (\ln x + 1)^2\,\mathrm{d}x$. [4]

(d) The arc of the curve between the x-intercept and $x = \mathrm{e}$ is rotated through 2π radians about the x-axis. Find the area of the surface generated.

Without further calculation, deduce the area of the surface generated when the arc of a curve with equation $y = \mathrm{e}^{2x-1}$ between the y-intercept and $y = \mathrm{e}$ is rotated through 2π radians about the y-axis, justifying your answer. [4]

7. It is given that the volume of a water reservoir between two cross-sectional areas can be calculated using $\int_a^b A(x)\,\mathrm{d}x$, where $A(x)$ is the cross sectional area at depth x metres from the water surface.

In real life, various numerical methods are used to estmiate the value of the integral $\int_a^b A(x)\,\mathrm{d}x$. One of the methods that can be used to estimate the integral $\int_a^b A(x)\,\mathrm{d}x$ is Simpson's rule, which is given by

$$\int_a^b A(x)\,\mathrm{d}x \approx \frac{L}{3}\left(A_a + 4A_M + A_b\right),$$

where L is the perpendicular distance between the two cross-sections. A_a and A_b are the cross-sectional areas of the end faces and A_M is the cross-sectional area of the face in the middle as shown in Figure 1 below.

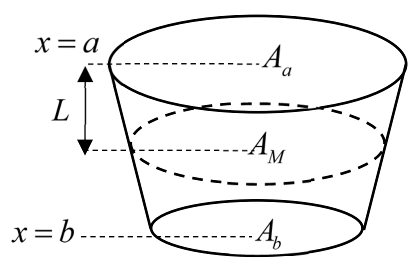


Figure 1

(a) State a condition on the number of given cross-sectional areas for Simpson's rule to be used to estimate the volume of a water reservoir. [1]

The table below shows the cross-sectional areas of the water reservoir in City D at different depths.

Depth (in metres)	Cross-Sectional Area (in ten thousand square metres)
0	190
4	127
8	100
12	76
16	20
19 (max depth)	14

When the condition to use Simpson's rule is not met, it is a usual practice to calculate the volume between two cross-sections using Trapezium Rule.

~~**(b)** Using Simpson's rule and Trapezium rule together, find an estimated volume of the reservoir.~~ ~~[3]~~

~~A frustum is a truncated cone where a cone is cut off from the top parallel to its base, as shown in Figure 2. Alternatively, the volume of an inverted frustum can be used to estimate the volume of the water reservoir.~~

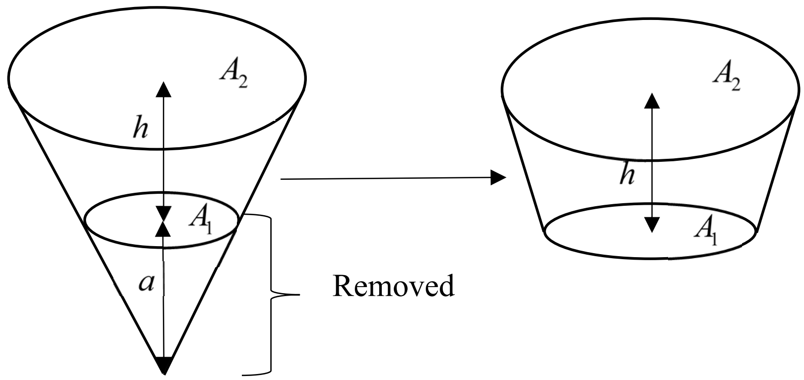


Figure 2

~~[The volume of a cone of base radius r and height H is given by $\frac{1}{3}\pi r^2 H$]~~

~~**(c)** Show that the volume V of an inverted frustum is given by~~

$$V = \frac{1}{3}h\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)$$

~~where h is the distance between the two cross sectional areas A_1 and A_2.~~ ~~[3]~~

~~Hence find an estimated volume of the water reservoir in City D using volume of the frustum of a cone.~~ ~~[2]~~

~~**(d)** Find the difference between the estimated volume found in **part (b)** and **(c)**.~~ ~~[1]~~

~~**(e)** Due to climate change, City D is experiencing dry spell when there is very little rainfall. The water usage of the city is 700 thousand m^3 per day. Find, to the nearest number of days, the water in the reservoir can last before it reaches 20% of its estimated volume found in **part (b)**. Without further calculation, justify whether there would be a difference in the number of days if the estimated volume found in **part (c)** is used instead.~~ ~~[2]~~

8. The function g is given by $g(x)=x^{\frac{3}{2}}-2\sqrt{x}-1$ for $x \geq 0$. Show that the equation $g(x)=0$ has only one real root, $x=\alpha$. State an integer n such that $n<\alpha<n+1$. [3]

(a) To find an approximate value for α, the following rearrangements of $g(x)=0$ are suggested as a basis for the iteration method of the form $x_{n+1}=f(x_n)$.

(1) $x=\frac{1}{4}\left(x^{\frac{3}{2}}-1\right)^2$

(2) $x=2+\frac{1}{\sqrt{x}}$

(3) $x=x^{\frac{5}{2}}-x^{\frac{3}{2}}$

(i) By considering $f'(x)$, identify the iteration method which converges to α. Use a graph to explain why the chosen iteration method converges to α. [4]

(ii) Using the appropriate iteration method found in **part (a)(i)**, and $x_0=n$, find the value of α, correct to 4 decimal places, and demonstrate how to verify its correctness. [3]

(b) **(i)** Show that a Newton Raphson iteration method for the root α is given by

$$x_{n+1}=\frac{x_n^{\frac{3}{2}}+2x_n^{\frac{1}{2}}+2}{3x_n^{\frac{1}{2}}-2x_n^{-\frac{1}{2}}}.$$ [2]

(ii) Explain why the Newton-Raphson iteration method fails when the initial value x_0 is less than or equal to $\frac{2}{3}$. [1]

9. A point P resides on the circumference of a circular gear with centre C and radius a, which rolls in an anticlockwise direction externally without slipping on the circumference of a fixed circular axle with centre O and radius a. Figure 1 shows the initial position of P at $(a, 0)$ and Figure 2 shows its position $P(x, y)$ where OC makes an angle θ with the positive x-axis.

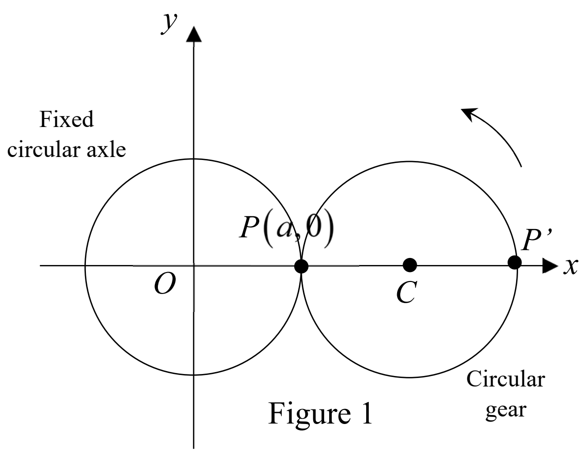


Figure 1

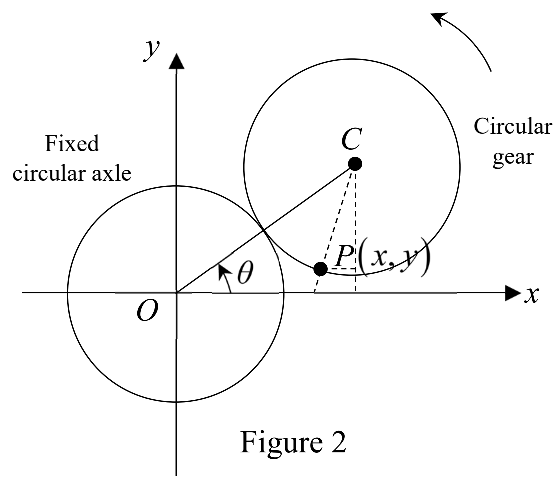


Figure 2

(a) Show that the equation of one full revolution of the path of P can be represented by

$$x = 2a\cos\theta - a\cos 2\theta$$
$$y = 2a\sin\theta - a\sin 2\theta$$

[3]

(b) Sketch the path of P, for $0 \le \theta \le 2\pi$, indicating clearly the coordinates of the x-intercepts. [3]

(c) Without the use of a calculator, find the length of the path of P in terms of a. [6]

(d) A student commented that a point P' with the initial position $(3a, 0)$ (as shown in Figure 1) is further away from O than point P and therefore travels a longer path as the circular gear makes a full revolution around the fixed circular axle. Do you agree with the comment, justifying your answer? [2]

10. To promote Earth Day, the Earth Day Network organises an event that will be held for n consecutive days. Cash prizes will be awarded to participants throughout the event, with one lucky participant chosen each day. Let the budget for the cash prizes be $\$m$.

The amount of cash prize given out on the first day is a sum of $\$10$ and $\frac{1}{7}$ of the remaining budget, i.e. $\$\left(10+\frac{1}{7}(m-10)\right)$. The amount of cash prize given out on the second day is a sum of $\$20$ and $\frac{1}{7}$ of the remaining budget. In general, the amount of cash prize given out on the k^{th} day is a sum of $\$10k$ and $\frac{1}{7}$ of the remaining budget. Let u_k denotes the amount of cash prize given out on the k^{th} day, with $1 \le k \le n$, $k \in \mathbb{Z}^+$.

(a) Write down the expression for u_k, in terms of $u_1, u_2, ..., u_{k-1}$. [1]

(b) By considering $u_{k+1} - u_k$, show that $u_{k+1} = \frac{6}{7}u_k + \frac{60}{7}$. [2]

(c) Find u_k in the form $u_k = p\left(\frac{6}{7}\right)^k (m-360)+q$, where p and q are constants to be determined. [3]

It is given that $m = 4000$.

(d) Find an expression of the total amount cash prizes given out in n days. Hence or otherwise, explain if it is possible for the Earth Day Network to host this event for 2 weeks. [4]

(e) Find the set of values of k for which the daily cash prize will be less than 5% of the initial budget. [2]

(f) Determine the minimum budget that the Earth Day Network needs to have so that they can host this event for 2 weeks. [3]