1

$$\frac{3}{1-x} \le 5-4x$$

$$\frac{3}{1-x} - (5-4x) \le 0$$

$$\frac{3-5+9x-4x^2}{1-x} \le 0$$

$$\frac{4x^2-9x+2}{x-1} \le 0$$

$$\frac{(4x-1)(x-2)}{x-1} \le 0$$

$$\therefore \quad x \le \frac{1}{4} \quad \text{or} \quad 1 < x \le 2$$

Let $x = \sin y$,

$$\therefore \quad \sin y \le \frac{1}{4} \quad \text{or} \quad 1 < \sin y \le 2 \text{ (rejected)}$$

$$\therefore \quad -\pi \le y \le 0.253 \quad \text{or} \quad 2.89 \le y \le \pi$$

2

(i) $\int 2e^x \sin 2x \, dx = 2e^x \sin 2x - 4\int e^x \cos 2x \, dx$

$$= 2e^x \sin 2x - 4\left[e^x \cos 2x + \int 2e^x \sin 2x \, dx\right]$$

$$= 2e^x \sin 2x - 4e^x \cos 2x - 4\int 2e^x \sin 2x \, dx$$

$$\therefore 5\int 2e^x \sin 2x \, dx = 2e^x \sin 2x - 4e^x \cos 2x + C$$

$$\therefore \int 2e^x \sin 2x \, dx = \frac{1}{5}\left(2e^x \sin 2x - 4e^x \cos 2x + C\right)$$

(ii) $\int \frac{2x+1}{x^2+2x+5} \, dx = \int \frac{2x+2}{x^2+2x+5} \, dx - \int \frac{1}{x^2+2x+5} \, dx$

$$= \ln\left|x^2+2x+5\right| - \int \frac{1}{(x+1)^2+2^2} \, dx$$

$$= \ln\left|x^2+2x+5\right| - \frac{1}{2}\tan^{-1}\frac{x+1}{2} + C$$

3

(a)

$$z^5 + 32(1+i) = 0$$

$$z^5 = 2^{\frac{11}{2}} e^{i\left(-\frac{3\pi}{4}+2k\pi\right)}$$

$$z = 2^{\frac{11}{10}} e^{i\left(-\frac{3\pi}{20}+\frac{2k\pi}{5}\right)}, \text{ where } k = -2, -1, 0, 1, 2$$

(b)

$$z = \sqrt{3} - i$$

$$= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$z^n = \left[2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right]^n$$

$$= 2^n\left(\cos\frac{\pi n}{6} - i\sin\frac{\pi n}{6}\right)$$

Since z^n is purely imaginary,

$$\cos\frac{\pi n}{6} = 0$$

$$\frac{\pi n}{6} = \frac{(2k+1)\pi}{2}, \text{ where } k \in \mathbb{Z}$$

$$\therefore \{n = 6k+3,\ k \in \mathbb{Z}\}$$

4

(i)

$$\frac{1}{\sqrt{4-x}} = (4-x)^{-\frac{1}{2}}$$

$$= 4^{-\frac{1}{2}}\left(1 - \frac{1}{4}x\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1 + \frac{1}{8}x + \frac{3}{128}x^2 + \cdots\right)$$

$$= \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \cdots$$

Range of x: $-4 < x < 4$

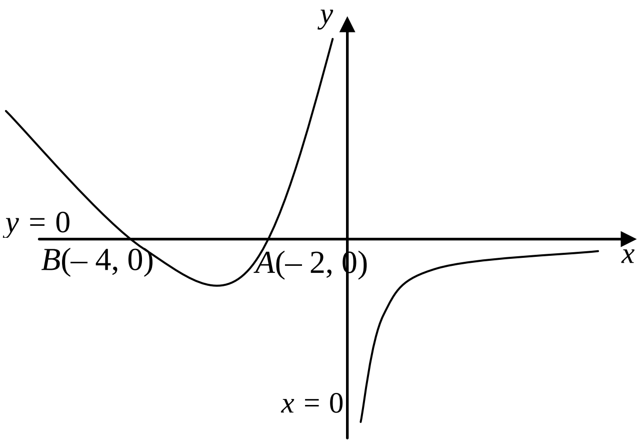
(ii)

$$\frac{1}{\sqrt{4-\frac{4}{5}}} = \frac{1}{2} + \frac{1}{16}\left(\frac{4}{5}\right) + \frac{3}{256}\left(\frac{4}{5}\right)^2 + \cdots$$

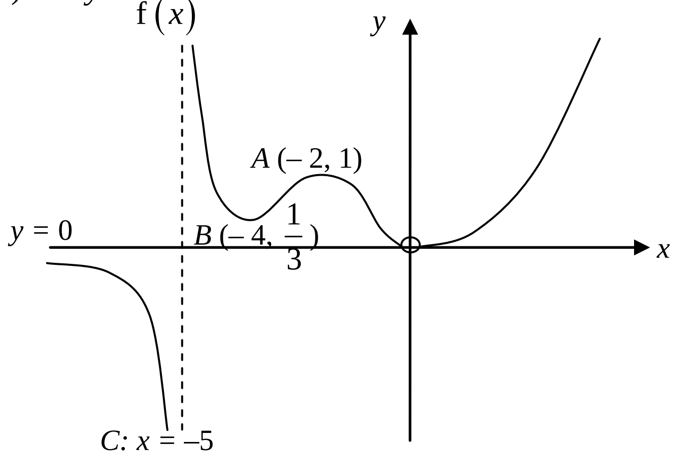
$$\frac{1}{\sqrt{\frac{16}{5}}} = \frac{1}{2} + \frac{1}{20} + \frac{3}{400} + \cdots$$

$$\frac{\sqrt{5}}{4} = \frac{223}{400} + \cdots$$

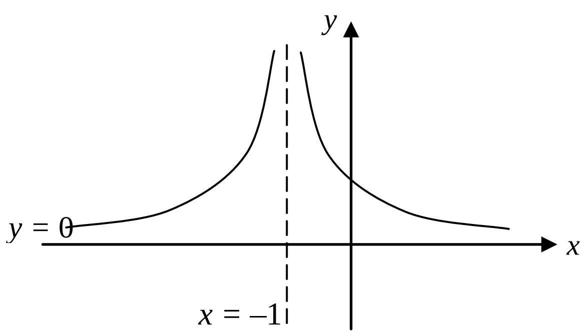
$$\sqrt{5} \approx \frac{223}{100}$$

5	(i)	$40 = 2y + 3x$ $y = 20 - \frac{3}{2}x$ Area, $A = xy + \frac{1}{2}x^2 \sin(60^\circ)$ $= xy + \frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ $= x\left(20 - \frac{3}{2}x\right) + \frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ $= 20x - \frac{3}{2}x^2 + \frac{\sqrt{3}}{4}x^2$ $= 20x + \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)x^2$ (Shown)
	(ii)	$\frac{\mathrm{dA}}{\mathrm{d}x} = 20 + 2\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)x$ $20 + 2\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)x = 0$ $x \approx 9.37218$ $\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = 2\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right) < 0$ $\therefore$ A has a maximum value when $x \approx 9.37218$. Max $A = 20(9.37218) + \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)(9.37218)^2$ ≈ 93.7218 ≈ 93.72 (to 2 d.p)
6	(i)	$y = \mathrm{f}'(x)$ 

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$



(iii) $y = \mathrm{f}\left(|x+1|\right)$



7 (i) $x = t + \ln t, \quad y = t + 1, \quad t > 0$

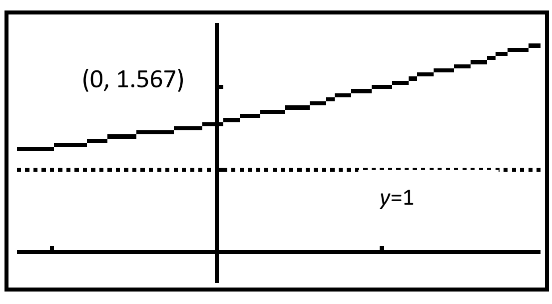
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \frac{1}{t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 1,$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+\frac{1}{t}} = \frac{t}{t+1}$$

Since $t > 0$, $t + 1 > 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{t}{t+1} > 0$ for all $t > 0$

Hence C does not have a stationary point

(ii)

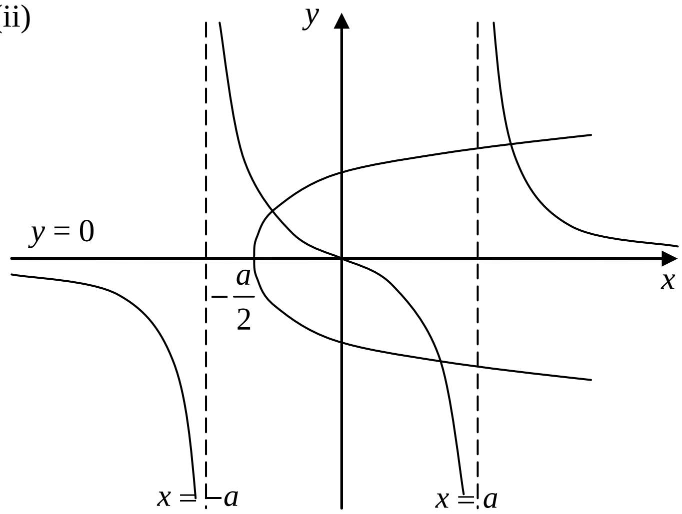


When $x = 0$, $t + \ln t = 0 \Rightarrow t = 0.5671432904$ (by g.c.)

	$y=1+0.5671432904=1.5671432904$ When $t\to 0,\ \ x\to -\infty,\ \mathrm{y}\to 0+1=1$ (iii) When t = 1, $x=1+\ln 1=1$ $y=1+1=2,\quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}$ Equation of normal : $y-2=-2(x-1)$ $\Leftrightarrow y=-2x+4$ (iv) Volume generated $=\pi\int_{0.5671432904}^{1}(t+1)^2\left(1+\frac{1}{t}\right)\mathrm{d}t+\frac{1}{3}\pi\left(2^2\right)(1)$ = 14.10 (2 decimal places)
8	$y=\ln\left(1+\tan^{-1}2x\right)$ $e^y=1+\tan^{-1}2x$ Diff w.r.t x, $e^y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+4x^2}(2)$ $\left(1+4x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2e^{-y}$ (shown) Diff w.r.t x, $\left(1+4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+8x\frac{\mathrm{d}y}{\mathrm{d}x}=-2e^{-y}\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(1+4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+8x\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(1+4x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ $\left(1+4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+8x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1+4x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$ $\left(1+4x^2\right)\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]+8x\frac{\mathrm{d}y}{\mathrm{d}x}=0$ (Shown) When x = 0, $y=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=2$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-4$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0$ $\therefore y=2x-2x^2+\cdots$ $\ln\left(\frac{1+\tan^{-1}2x}{1-x}\right)=\ln\left(1+\tan^{-1}2x\right)-\ln\left(1-x\right)$

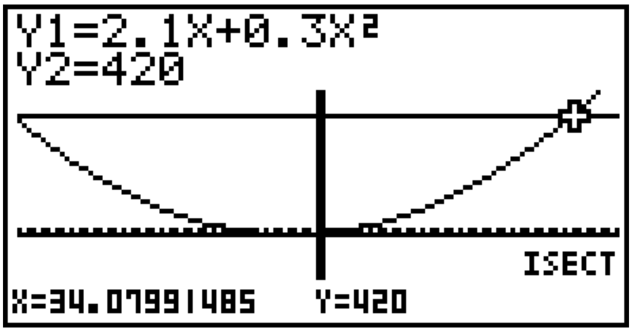
	$=\left(2x-2x^2+\cdots\right)-\left(-x-\frac{1}{2}x^2+\cdots\right)$ $=3x-\frac{3}{2}x^2+\cdots$
9	(i) Area $R=\int_1^2\sqrt{1-\frac{1}{x^2}}\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{3}}\sqrt{1-\cos^2\theta}\,\sec\theta\tan\theta\,\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{3}}\sqrt{\sin^2\theta}\,\sec\theta\tan\theta\,\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{3}}\sin\theta\,\frac{1}{\cos\theta}\tan\theta\,\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{3}}\tan^2\theta\,\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{3}}\left(\sec^2\theta-1\right)\mathrm{d}\theta$ $=\left[\tan\theta-\theta\right]_0^{\frac{\pi}{3}}$ $=\left(\tan\frac{\pi}{3}-\frac{\pi}{3}\right)-\left(\tan 0-0\right)$ $=\sqrt{3}-\frac{\pi}{3}$ units^2 (ii) Area $Q=\int_0^a\frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y$ $=\left[\sin^{-1}y\right]_0^a$ $=\sin^{-1}a$ $\left(\sin^{-1}a\right)+\left(\sqrt{3}-\frac{\pi}{3}\right)=\sqrt{3}$ $\sin^{-1}a=\frac{\pi}{3}$ $a=\frac{\sqrt{3}}{2}$

<table>
<tr><td>10</td><td>(i) $y = 0,\ x = a,\ x = -a$

(ii)


(iii) Add in the graph $y^2 = 2x + a$

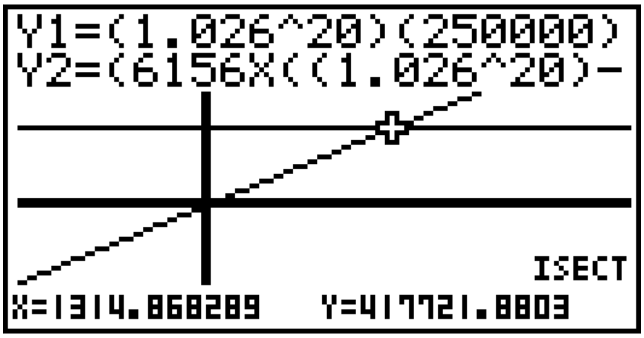
Number of real roots of the equation $2x + a = \left(\frac{x}{x^2 - a^2}\right)^2$ is 3.</td></tr>
<tr><td>11a</td><td>(i) Total distance $= \frac{n}{2}\left(2(2.4) + (n-1)(0.6)\right)$
$= 2.1n + 0.3n^2$

(ii) $2.1n + 0.3n^2 \geq 420$


$n \geq 34.0799$
Least $n = 35$</td></tr>
<tr><td>11b</td><td>(i) Amount of outstanding loan at the end of n months
$= 1.002^n(250000) - 1.002^{n-1}k - 1.002^{n-2}k - \cdots - k$
$= 1.002^n(250000) - \left[\frac{k(1.002^n - 1)}{1.002 - 1}\right]$
$= 1.002^n(250000) - 500k(1.002^n - 1)$</td></tr>
</table>

(ii) $n = 240$ months

$$1.002^{240}\left(250000\right) - 500k\left(1.002^{240} - 1\right) = 0$$



$\therefore k = 1312.61186 \approx \$1313$ (to nearest dollars)

12

(i) $\alpha \approx -2.414, \beta \approx 0.414, \gamma = 2$

(ii) If the sequence converges, then as $n \to \infty, x_n \to l$ and $x_{n+1} \to l$

$$l = \left(5l - 2\right)^{\frac{1}{3}}$$

$$0 = \sqrt[3]{5l - 2} - l$$

From part (i) , the roots of the equation $\sqrt[3]{5x-2} - x = 0$ are α, β, γ

$\therefore l = \alpha, l = \beta,$ or $l = \gamma$

$\therefore$ The sequence converges to either α , β or γ .

(iii) $x_{n+1} - x_n = \left(5x_n - 2\right)^{\frac{1}{3}} - x_n = \sqrt[3]{5x_n - 2} - x_n$

From the given graph,

if $\beta < x_n < \gamma$, $\sqrt[3]{5x_n - 2} - x_n > 0$

$RHS > 0 \Rightarrow LHS = x_{n+1} - x_n > 0$

$\therefore \; x_{n+1} > x_n$

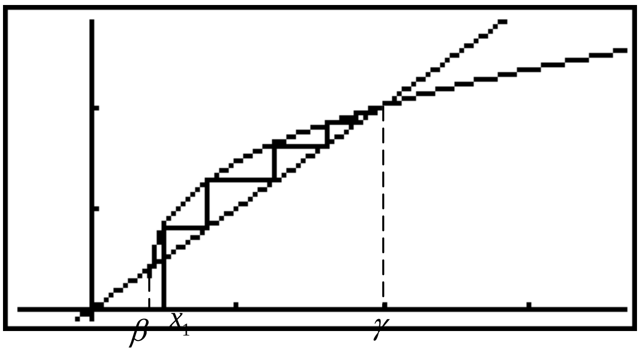
From the given graph,

if $x_n < \beta$ or $x_n > \gamma$, $\sqrt[3]{5x_n - 2} - x_n < 0$

$RHS < 0 \Rightarrow LHS = x_{n+1} - x_n < 0$

$\therefore \; x_{n+1} < x_n$

(iv)



13

(i) Sub $x=-3, y=5, z=2$ into l_2,

$b-(-3)=2-1$

$b=-2$

(ii) $1-s=-2-\lambda$ -------- (1)

$4+s=5 \Rightarrow s=1$

$-3+2s=1+\lambda$ -------- (2)

From (1) and (2),

$\lambda=-2$

$$\therefore \overrightarrow{OB}=\begin{pmatrix}1\\4\\-3\end{pmatrix}+\begin{pmatrix}-1\\1\\2\end{pmatrix}=\begin{pmatrix}0\\5\\-1\end{pmatrix}$$

Coordinates of B = (0, 5, –1)

(iii) $$\therefore \overrightarrow{OC}=\begin{pmatrix}1-s\\4+s\\-3+2s\end{pmatrix}$$

$$\overrightarrow{CA}=\begin{pmatrix}-3\\5\\2\end{pmatrix}-\begin{pmatrix}1-s\\4+s\\-3+2s\end{pmatrix}=\begin{pmatrix}-4+s\\1-s\\5-2s\end{pmatrix}$$

$$\overrightarrow{CA}\bullet\begin{pmatrix}-1\\0\\1\end{pmatrix}=0$$

$$\begin{pmatrix}-4+s\\1-s\\5-2s\end{pmatrix}\bullet\begin{pmatrix}-1\\0\\1\end{pmatrix}=0$$

$4-s+5-2s=0$

$s=3$

$$\therefore \overrightarrow{OC}=\begin{pmatrix}1-3\\4+3\\-3+2(3)\end{pmatrix}=\begin{pmatrix}-2\\7\\3\end{pmatrix}$$

(iv) Let C' be the point on l_3 where C is reflected about l_2

$\therefore \overrightarrow{OC'}=2\overrightarrow{OA}-\overrightarrow{OC}$

$$=2\begin{pmatrix}-3\\5\\2\end{pmatrix}-\begin{pmatrix}-2\\7\\3\end{pmatrix}$$

$$=\begin{pmatrix}-4\\3\\1\end{pmatrix}$$

	$\overrightarrow{BC'} = \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix}, \quad \overrightarrow{BD} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$ Normal vector of $\pi = \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix}$ Cartesian equation of π : $\mathbf{r} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $y + z = 4$