RAFFLES INSTITUTION
H2 Mathematics (9758)
2024 Year 6

2024 Year 6 Timed Practice Revision Practice Paper 1 Solutions

Source: 2019 Year 6 Term 3 Common Test

1 The curve C_1 with equation $\frac{x^2}{a^2}+\frac{y^2}{4a^2}=1$ is transformed to curve C_2 by a translation of a units in the positive x-direction, followed by a stretch with scale factor 2 parallel to the x-axis, and followed by a reflection in the y-axis, where a is a positive constant.

(i) Find the equation of C_2. [3]

(ii) Describe the shape of C_2 geometrically. [2]

1(i) [3]	$\frac{x^2}{a^2}+\frac{y^2}{4a^2}=1$ $\downarrow$ a translation of a units in the positive x-direction i.e replace x with $x-a$ $\frac{(x-a)^2}{a^2}+\frac{y^2}{4a^2}=1$ $\downarrow$ a stretch with scale factor 2 parallel to the x-axis i.e replace x with $\frac{x}{2}$ $\left(\frac{x}{2}-a\right)^2\frac{1}{a^2}+\frac{y^2}{4a^2}=1 \quad \Rightarrow \quad \frac{(x-2a)^2}{4a^2}+\frac{y^2}{4a^2}=1$ $\downarrow$ a reflection in the y-axis i.e replace x with $-x$ $\frac{(-x-2a)^2}{4a^2}+\frac{y^2}{4a^2}=1 \quad \Rightarrow \quad (x+2a)^2+y^2=(2a)^2$	It is good practice to simplify as you go along. Otherwise, you may end up with a very complicated expression which makes simplification much harder.
(ii) [2]	C_2 is a circle with the centre $(-2a, 0)$, with a radius of $2a$ units.	

2 **Do not use a calculator in answering this question.**

The equation $z^2 - 2\sqrt{3}\,z + 4 = 0$ has two complex roots z_1 and z_2, where $0 < \arg(z_1) < \dfrac{\pi}{2}$.

(i) Find z_1 and z_2 in polar form. [3]

(ii) Show that $\dfrac{z_1^{\,4}}{z_2^{\,2}} = -4$. [1]

(iii) Find the set of possible values of n, $n \in \mathbb{Z}$, for which $\dfrac{z_1^{\,n}}{1+\sqrt{3}\,\mathrm{i}}$ is a real number. [3]

2(i) **[3]**	$z^2 - 2\sqrt{3}\,z + 4 = 0$ $\Rightarrow z = \dfrac{2\sqrt{3} \pm \sqrt{12-4(4)}}{2}$ $= \dfrac{2\sqrt{3} \pm \sqrt{-4}}{2}$ $= \sqrt{3} \pm \mathrm{i}$ $\lvert z_1 \rvert = \lvert z_2 \rvert = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$ $\arg(z_1) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ $\therefore z_1 = \sqrt{3} + \mathrm{i} = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}$ and $z_2 = \sqrt{3} - \mathrm{i} = 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$	Avoid the tedious method of replacing z by $x + \mathrm{i}y$ in the equation and comparing real and imaginary parts.
(ii) **[1]**	$\dfrac{z_1^{\,4}}{z_2^{\,2}} = \dfrac{\left(2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}\right)^4}{\left(2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}\right)^2}$ $= \dfrac{2^4}{2^2}\mathrm{e}^{\mathrm{i}\left[\frac{4}{6}-\left(-\frac{2}{6}\right)\right]\pi} = 4\mathrm{e}^{\mathrm{i}(\pi)} = -4$ (Shown)	
(iii) **[3]**	$\arg\left(\dfrac{z_1^{\,n}}{1+\sqrt{3}\,\mathrm{i}}\right) = n\arg(z_1) - \arg(1+\sqrt{3}\,\mathrm{i}) = \dfrac{n\pi}{6} - \dfrac{\pi}{3}$ For $\dfrac{z_1^{\,n}}{1+\sqrt{3}\,\mathrm{i}}$ to be a real number, $\dfrac{n\pi}{6} - \dfrac{\pi}{3} = k\pi$, where $k \in \mathbb{Z}$ Setting $\dfrac{n\pi}{6} - \dfrac{\pi}{3} = k\pi \Rightarrow n = 6k + 2, \quad k \in \mathbb{Z}$ Therefore the set of possible values is $\{n \in \mathbb{Z} \mid n = 6k+2, \quad k \in \mathbb{Z}\}$.	

3 The function f is defined as follows.

$$\mathrm{f}: x \mapsto x^2 - 3x, \text{ for } |x| < a.$$

(i) State the largest value of a for which the function f^{-1} exists. Hence find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1} for this value of a. [3]

In the rest of the question, use $a = 1$.

The function g is defined on an interval A as follows.

$$\mathrm{g}: x \mapsto \ln(3x+2), \text{for } x \in A.$$

(ii) A student suggests 2 possible intervals for A as follows.

(a) $\left(-\frac{2}{3}, -\frac{1}{3}\right)$

(b) $\left(-\frac{1}{2}, 0\right)$

Determine which of the above intervals will result in the existence of the composite function fg, justifying your answer. In the case(s) that fg exists, find the range of fg. [4]

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<td>3(i)
[3]</td>
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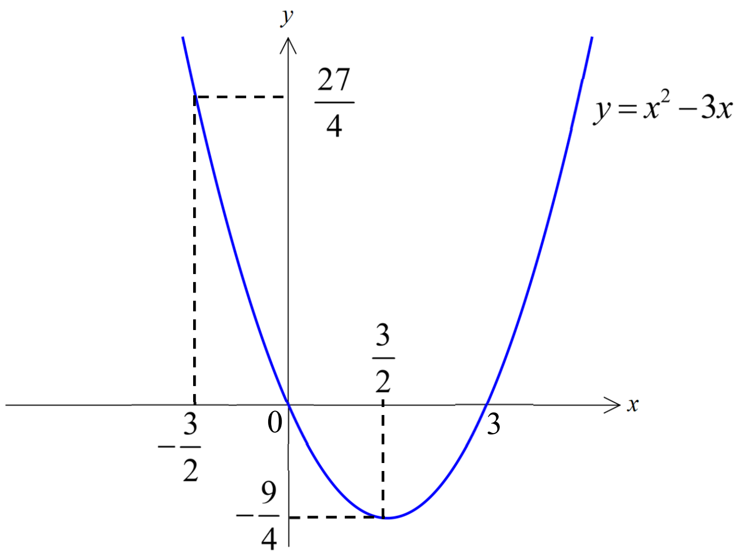

From the graph, largest value of $a = \frac{3}{2}$.

Let $y = \mathrm{f}(x)$

$= x^2 - 3x$

$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$

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	$x=\frac{3}{2}\pm\sqrt{y+\frac{9}{4}}$ $=\frac{3}{2}-\sqrt{y+\frac{9}{4}}$ $\left(\text{since } -\frac{3}{2}<x<\frac{3}{2}\right)$. Hence $f^{-1}(x)=\frac{3}{2}-\sqrt{x+\frac{9}{4}}$ and domain of $f^{-1}=\left(-\frac{9}{4},\ \frac{27}{4}\right)$.	

(ii) **[4]**	For fg to exist, we need $R_g \subseteq D_f=(-1,1)$.
	For (a) where $D_g=\left(-\frac{2}{3},\ -\frac{1}{3}\right)$, $R_g=(-\infty,0)\not\subset D_f$.
	Hence fg does not exists for (a).
	
	For (b) where $D_g=\left(-\frac{1}{2},\ 0\right)$,
	$R_g=(-\ln 2,\ \ln 2)=(-0.693,0.693)\subseteq D_f$.
	
	Hence fg exists for (b).
	
	$D_g=\left(-\frac{1}{2},\ 0\right)$
	$\downarrow$ g
	$R_g\left(-\ln 2,\ \ln 2\right)$
	$\downarrow$ f
	$R_{fg}=\left((\ln 2)^2-3\ln 2,\ (-\ln 2)^2+3\ln 2\right)$
	$=(-1.60,\ 2.56)$ (to3s.f.)
		Take note that f is a decreasing function in this domain.

4 **(a)** A geometric series has first term $2\sqrt{2}\sin\theta$ and second term $\sin 2\theta$.

(i) Show that the series is convergent for $\frac{\pi}{2} < \theta < \pi$. [2]

(ii) It is given that $\theta = \frac{3\pi}{4}$ and S_n denotes the sum of the first n terms of the series. Find S_n and hence determine the exact value of S_∞. [3]

(b) **(i)** Show that $\frac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1}-\sqrt{n}$, where $n \in \mathbb{Z}^+$. [1]

(ii) Hence find the least possible value of N such that $\sum_{n=4}^{N} \frac{1}{\sqrt{n+1}+\sqrt{n}} > 100$. [2]

4(a) (i) [2]	Common ratio, $r = \frac{\sin 2\theta}{2\sqrt{2}\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sqrt{2}\sin\theta} = \frac{\cos\theta}{\sqrt{2}}$. Since $\lvert r\rvert = \left\lvert\frac{\cos\theta}{\sqrt{2}}\right\rvert = \frac{1}{\sqrt{2}}\lvert\cos\theta\rvert < \frac{1}{\sqrt{2}} < 1 \left(\text{since } \lvert\cos\theta\rvert < 1 \text{ for } \frac{\pi}{2} < \theta < \pi\right)$, the series is convergent (shown).
(ii) [3]	Now, first term, $a = 2\sqrt{2}\sin\frac{3\pi}{4} = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$ and $r = \frac{1}{\sqrt{2}}\cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$. Hence $S_n = \frac{2\left[1-\left(-\frac{1}{2}\right)^n\right]}{1-\left(-\frac{1}{2}\right)} = \frac{4}{3}\left[1-\left(-\frac{1}{2}\right)^n\right]$. As $n \to \infty$, $\left(-\frac{1}{2}\right)^n \to 0$, thus $S_n \to S_\infty = \frac{4}{3}$.
(b) (i) [1]	$\frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$ $= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$ $= \sqrt{n+1}-\sqrt{n}$ (Shown).

(ii) **[2]**	$\sum_{n=4}^{N} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ $= \sum_{n=4}^{N} \left(\sqrt{n+1}-\sqrt{n}\right)$ $= \begin{Bmatrix} \sqrt{5}-\sqrt{4} \\ +\sqrt{6}-\sqrt{5} \\ +\sqrt{7}-\sqrt{6} \\ +\dots \quad \dots \\ +\sqrt{N}-\sqrt{N-1} \\ +\sqrt{N+1}-\sqrt{N} \end{Bmatrix}$ $= \sqrt{N+1}-2$ $\sqrt{N+1}-2 > 100 \Rightarrow N > 10403$ Therefore the least possible value of N is 10404.

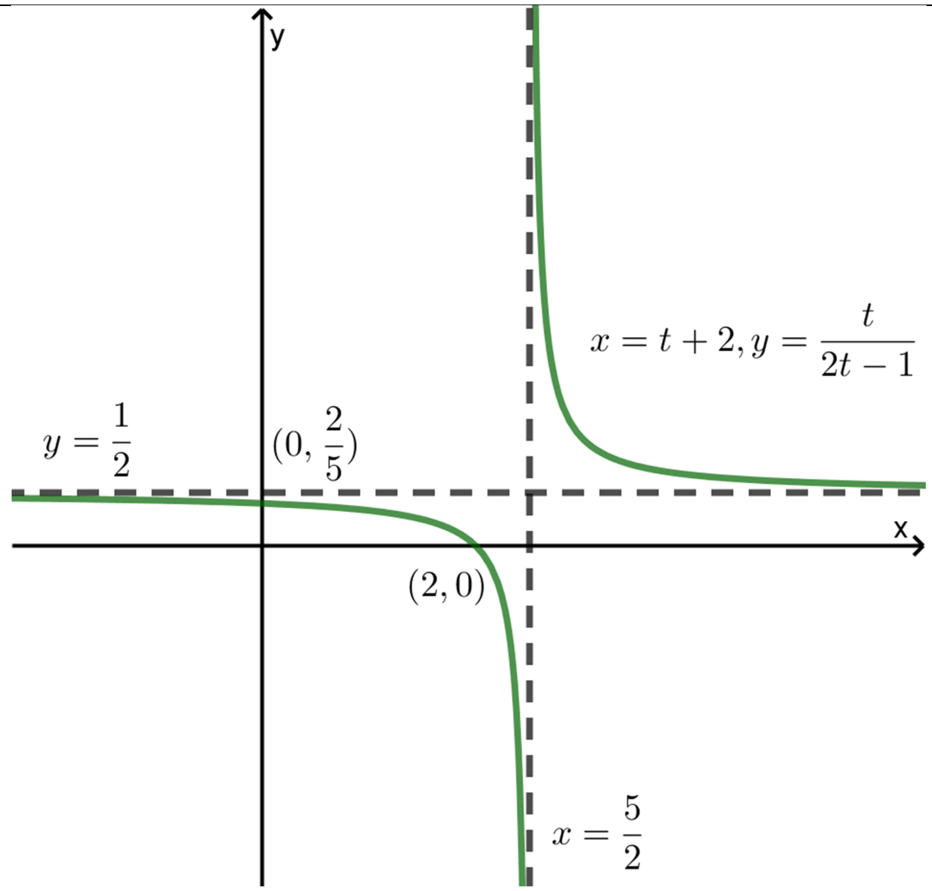
5 The parametric equations of a curve are

$$x = t + 2, \quad y = \frac{t}{2t-1},$$

where $t \in \mathbb{R}$, $t \neq \frac{1}{2}$.

(i) Sketch the curve, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the axes. [3]

(ii) The tangent to the curve at the point $\left(p+2, \dfrac{p}{2p-1}\right)$ intersects the x-axis at point A and the y-axis at point B. Find, in terms of p, an expression for the area of the triangle OAB. [5]

5(i) **[3]**		Note that you should find the Cartesian equation from the set of parametric equations to derive the asymptotes. You should also indicate the intercepts and equations of the asymptotes clearly on your sketch.
(ii) **[5]**	$\frac{dx}{dt} = 1$ $\frac{dy}{dt} = \frac{(2t-1) - 2t}{(2t-1)^2} = -\frac{1}{(2t-1)^2}$ $\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{1}{(2t-1)^2}$ When $t = p$, $\frac{dy}{dx} = -\frac{1}{(2p-1)^2}$	

	$y-\frac{p}{2p-1}=-\frac{1}{(2p-1)^2}\left(x-\left(p+2\right)\right)$ $\Rightarrow y=-\frac{1}{(2p-1)^2}x+\frac{2(p^2+1)}{(2p-1)^2}$ When $y=0,\ x=2(p^2+1)$, therefore $A\left(2(p^2+1),0\right)$. When $x=0,\ y=\frac{2(p^2+1)}{(2p-1)^2}$, therefore $B\left(0,\frac{2(p^2+1)}{(2p-1)^2}\right)$. It follows that the area of the triangle *OAB* is $\frac{2(p^2+1)^2}{(2p-1)^2}$.	

6 **(a)** Given **a** and **b** are two non-zero and non-parallel vectors and $\mathbf{c} = |\mathbf{b}|\mathbf{a} + |\mathbf{a}|\mathbf{b}$, show that the length of projection of **c** onto **a** and the length of projection of **c** onto **b** have the same magnitude. [3]

(b) The equations of line l_1, planes π_1 and π_2 are

$$l_1 : \frac{x-3}{2} = \frac{1-y}{2} = -z,$$
$$\pi_1 : 2x + 3y - z = 5,$$
$$\pi_2 : -ax - 3y + 2z = b,$$

respectively.

(i) If l_1 lies on π_2, find the values of a and b. [2]

For the rest of the question, l_1 does not lie on π_2 and l_1 intersects π_1 at point F.

(ii) Find the coordinates of F. [3]

(iii) Find, in terms of a, a direction vector of the line of intersection l_2 between π_1 and π_2. [2]

(iv) Find the relationship between a and b if F also lies on π_2. State, in terms of a, a vector equation of l_2. [2]

6(a) **[3]**	Length of projection of **c** onto **a**: $|\mathbf{c}\cdot\hat{\mathbf{a}}| = \left|\left(|\mathbf{b}|\mathbf{a} + |\mathbf{a}|\mathbf{b}\right)\cdot\frac{\mathbf{a}}{|\mathbf{a}|}\right| = \left|\frac{|\mathbf{b}|\mathbf{a}\cdot\mathbf{a}}{|\mathbf{a}|} + \mathbf{b}\cdot\mathbf{a}\right| = \left|\frac{|\mathbf{b}||\mathbf{a}|^2}{|\mathbf{a}|} + \mathbf{b}\cdot\mathbf{a}\right| = \left||\mathbf{b}||\mathbf{a}| + \mathbf{b}\cdot\mathbf{a}\right|$ Length of projection of **c** onto **b**: $|\mathbf{c}\cdot\hat{\mathbf{b}}| = \left|\left(|\mathbf{b}|\mathbf{a} + |\mathbf{a}|\mathbf{b}\right)\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right| = \left|\mathbf{a}\cdot\mathbf{b} + \frac{|\mathbf{a}|}{|\mathbf{b}|}\mathbf{b}\cdot\mathbf{b}\right| = \left|\mathbf{a}\cdot\mathbf{b} + \frac{|\mathbf{a}||\mathbf{b}|^2}{|\mathbf{b}|}\right| = \left|\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}||\mathbf{b}|\right|$ Since $|\mathbf{b}||\mathbf{a}| = |\mathbf{a}||\mathbf{b}|$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$, then $|\mathbf{c}\cdot\hat{\mathbf{a}}| = |\mathbf{c}\cdot\hat{\mathbf{b}}|$ i.e. the length of projection of **c** onto **a** and of **c** onto **b** have the same magnitude.	

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$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$

For line l_1 to lie on plane π_2, its direction vector must be parallel to π_2 (i.e. perpendicular to the normal of π_2) and the point $(3,1,0)$ on l_1 must lie on π_2.

Direction vector perpendicular to normal:

$\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -a \\ -3 \\ 2 \end{pmatrix} = 0$

$\Rightarrow a = 2$

Point $(3,1,0)$ lies on the plane π_2:

$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} = b$

$\Rightarrow b = -9$

<u>Alternative Method:</u>

$\begin{pmatrix} 3+2\lambda \\ 1-2\lambda \\ -\lambda \end{pmatrix} \bullet \begin{pmatrix} -a \\ -3 \\ 2 \end{pmatrix} = b \quad \forall \lambda \in \mathbb{R}$

$\Rightarrow -3a - 2a\lambda - 3 + 6\lambda - 2\lambda = b$

$\Rightarrow \begin{cases} -3a - 3 = b \\ -2a + 6 - 2 = 0 \end{cases}$

$\Rightarrow \begin{cases} a = 2 \\ b = -9 \end{cases}$

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<td>Exercise extra care in converting the equation from cartesian to vector form. Making mistakes here is very costly.</td>
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<td>(b)(ii)
[3]</td>
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$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$

then $\overrightarrow{OF} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$, for some $\lambda \in \mathbb{R}$

$\begin{pmatrix} 3+2\lambda \\ 1-2\lambda \\ -\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 5 \Rightarrow \lambda = 4$

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	$\overrightarrow{OF}=\begin{pmatrix}3+2(4)\\1-2(4)\\-4\end{pmatrix}=\begin{pmatrix}11\\-7\\-4\end{pmatrix}$ The coordinates of F is $(11,-7,-4)$.	
(b) (iii) **[2]**	Direction vector of l_2 is $\begin{pmatrix}2\\3\\-1\end{pmatrix}\times\begin{pmatrix}-a\\-3\\2\end{pmatrix}=\begin{pmatrix}3\\a-4\\-6+3a\end{pmatrix}$	
(b)(iv) **[2]**	F lies on $\pi_2 \Rightarrow \begin{pmatrix}11\\-7\\-4\end{pmatrix}\bullet\begin{pmatrix}-a\\-3\\2\end{pmatrix}=b$ $\Rightarrow b=-11a+13$ $l_2:\mathbf{r}=\begin{pmatrix}11\\-7\\-4\end{pmatrix}+\mu\begin{pmatrix}3\\a-4\\3a-6\end{pmatrix}, \mu\in\mathbb{R}$	Note that F lies on π_2 and π_1 and therefore should lie on the line of their intersection l_2.

7 The S-I model is used to study the spread of infectious diseases across different scenarios. In this question, it is assumed that we are studying a homogeneous population in a closed community of constant size N throughout the period of consideration. The population is divided into two groups – one group of infected individuals who have the diseases and another group of healthy and susceptible individuals who becomes infected when they come into some form of contact with the other group of infected individuals. Using x and y to represent the number of infected individuals and number of healthy and susceptible individuals at time t respectively, we can model the situation using the following differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kxy, \qquad \textbf{(I)}$$

where k is a positive constant.

(i) By expressing y in terms of N and x, show that equation (**I**) can be rewritten in the form of a first order differential equation in terms of x and t. Given that $x = \alpha$ when $t = 0$, solve this differential equation and show that the solution can be expressed in the form

$$x = \frac{A}{1+(B-1)\mathrm{e}^{-Nkt}},$$

where A and B are constants expressed in terms of N and α. [8]

(ii) This model was tested using some data from the 2003 SARS epidemic in Singapore where $\alpha = 1$, $N = 206$, k was estimated to be 8.1835×10^{-4} and time was measured in days.

Write down the equation of the corresponding solution curve and sketch the part of the curve which is relevant to this context. (Your sketch should be suitably labelled on the axes.) State what happens to x for large values of t. [4]

7(i) **[8]**	Since $x + y = N$, $y = N - x$. Then we have $$\frac{\mathrm{d}x}{\mathrm{d}t} = kxy = kx(N-x)$$ $$\frac{1}{x(N-x)}\frac{\mathrm{d}x}{\mathrm{d}t} = k$$ $$\frac{1}{N}\left(\frac{1}{x}+\frac{1}{N-x}\right)\frac{\mathrm{d}x}{\mathrm{d}t} = k$$ Integrating w.r.t. t, we have $\int\left(\frac{1}{x}+\frac{1}{N-x}\right)\mathrm{d}x = \int Nk\,\mathrm{d}t$ $\ln x - \ln(N-x) = Nkt + c$ ($\because x > 0$ and $N - x > 0 \Rightarrow \lvert x\rvert = x$ and $\lvert N-x\rvert = N-x$) $\ln\left(\frac{x}{N-x}\right) = Nkt + c$ $\frac{x}{N-x} = D\mathrm{e}^{Nkt}$ where $D = \mathrm{e}^{c}$	

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$$x = De^{Nkt}(N-x)$$
$$x\left(1+De^{Nkt}\right) = DNe^{Nkt}$$
$$x = \frac{DNe^{Nkt}}{\left(1+De^{Nkt}\right)}$$
When $t = 0$, $x = \alpha$. $D = \frac{\alpha}{N-\alpha}$.
$$x = \frac{\frac{\alpha}{N-\alpha}Ne^{Nkt}}{\left(1+\frac{\alpha}{N-\alpha}e^{Nkt}\right)} = \frac{N}{1+\left(\frac{N-\alpha}{\alpha}\right)e^{-Nkt}} = \frac{N}{1+\left(\frac{N}{\alpha}-1\right)e^{-Nkt}}$$
where $A = N$, $B = \frac{N}{\alpha}$.
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<u>Alternatively:</u>
$$\frac{dx}{dt} = kxy = kx(N-x)$$
$$\frac{1}{x(N-x)}\frac{dx}{dt} = k$$
$$\int\left(\frac{1}{x(N-x)}\right)dx = \int k\ dt$$
$$\int\left(\frac{1}{x^2\left(\frac{N}{x}-1\right)}\right)dx = \int k\ dt$$
$$-\frac{1}{N}\ln\left|\frac{N}{x}-1\right| = kt+c$$
$$\ln\left(\frac{N}{x}-1\right) = -Nkt-Nc\ (\because 0 < x < N \Rightarrow \frac{N}{x} > 1)$$
$$\frac{N}{x}-1 = e^{-Nkt-Nc} = De^{-Nkt} \quad \text{where } D = e^{-Nc}$$
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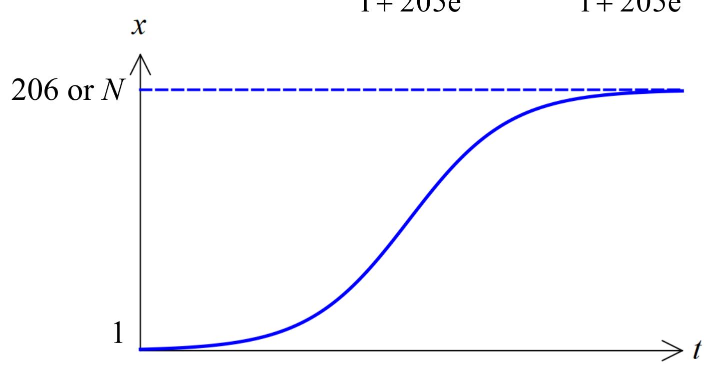
<tr><td>(ii)
[4]</td><td>
Equation of solution curve is $x \approx \frac{206}{1+205e^{-0.16858t}} \approx \frac{206}{1+205e^{-0.169t}}$



When $t \to \infty$, $x \to 206$.
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8 **(i)** A code consists of 10 digits which are either zeros or ones, for example, 1011011010. Calculate the number of such codes if there is no restriction. [1]

Given further that the 10 digits consists of 4 zeros and 6 ones, calculate the number of such codes if

(ii) there is no other restriction, [1]

(iii) all the zeros must be separated and the first and last digits must be different, [2]

(iv) no more than 4 ones are together. [2]

8(i) **[1]**	Each digit has 2 choices. Total number of codes with no restriction is $2^{10}=1024$	
(ii) **[1]**	Out of 10 digits choose 4 to be zeros: ${}^{10}C_4=210$ possible codes **<u>Or</u>** Out of 10 digits choose 6 to be ones: ${}^{10}C_6=210$ possible codes **<u>Or</u>** Arrange 10 digits with 6 identical ones and 4 identical zeros: $\dfrac{10!}{6!4!}=210$ possible codes	
(iii) **[2]**	Case 1: <u>1</u> 1 1 1 1 <u>1 0</u> Start with one and end with zero, left with 5 positions between the ones to insert 3 zeros: ${}^{5}C_3=10$ Case 2: <u>01</u> 1 1 1 1 <u>1</u> Start with zero and end with 1, left with 5 positions between the ones to insert 3 zeros: ${}^{5}C_3=10$ Total number of such codes: ${}^{5}C_3\times 2=20$	
(iv) **[2]**	Total number of such codes from **(ii)**: 210 Number of ways to have exactly 5 ones together: ${}^{5}P_2=20$ Number of ways to have all 6 ones together: $\dfrac{5!}{4!}=5$ Therefore $210-{}^{5}P_2-\dfrac{5!}{4!}=185$ possible codes	For the 5 ones together case, we would like to have the unit of 5 ones and the last "one" to be separated so that it doesn't include the case of 6 ones together, so there are 5 slots between the 4 zeros to insert this unit of 5 ones and the last "one".

9 In an online shop, the time taken, in hours, to sell a watch is a normally distributed continuous random variable X. The standard deviation of X is 0.68 hours and the expected value of X is 1.75 hours. After an aggressive advertising campaign, the total time taken to sell 8 watches is found to be 11 hours. Test, at 5% level of significance, whether there is evidence that the mean time taken to sell a watch has decreased. State an assumption that you have used in your calculation. [6]

9 **[6]**	Let μ denote the mean time taken to sell a watch in hours. Null hypothesis, $H_0: \mu = 1.75$ Alternative hypothesis, $H_1: \mu < 1.75$ Perform a 1-tail test at 5% significance level. Under H_0, $\overline{X} \sim N\left(\mu_0, \frac{\sigma^2}{8}\right)$ where $\mu_0 = 1.75$ and $\sigma = 0.68$. From the sample, $\overline{x} = \frac{11}{8}$ Using a z-test, p-value $= 0.0594 > 0.05$ (3s.f.). Since p-value $= 0.0594 > 0.05$, we do not reject H_0 and conclude that there is insufficient evidence, at 5% significance level, that the time taken to sell a watch has decreased. The time taken to sell one watch is assumed to be independent of the time taken to sell another watch.	Please take note of the presentation and the notations used.

10 In this question you should state the parameters of any distributions that you use.

Crispy Cream Donut Shop sells 2 types of donuts: Ring Donuts and Filled Donuts. The masses in grams of Ring Donuts and Filled Donuts have normal distributions $\mathrm{N}\left(54,\ 1.5^2\right)$ and $\mathrm{N}\left(86,\ 2^2\right)$ respectively.

(i) Find the probability that the total mass of 3 randomly chosen Ring Donuts is more than twice the mass of a randomly chosen Filled Donut. [3]

12 donuts are packed into a paper box. The mass in grams of an empty paper box has a normal distribution $\mathrm{N}\left(80,\ 5^2\right)$.

(ii) The probability that the total mass of a box containing 6 Ring Donuts and 6 Filled Donuts is more than m grams is 0.95. Find m. [4]

(iii) State an assumption that you have used in your calculations in parts **(i)** and **(ii)**. [1]

10 (i) [3]	Let R be the mass of a Ring Donut in grams. Then $R \sim \mathrm{N}\left(54,\ 1.5^2\right)$. Let F be the mass of a Filled Donut in grams. Then $F \sim \mathrm{N}\left(86,\ 2^2\right)$. Let $X = R_1 + R_2 + R_3 - 2F$. Then $\mathrm{E}(X) = 3(54) - 2(86)$ $= -10,$ and $\mathrm{Var}(X) = 3\left(1.5^2\right) + 2^2\left(2^2\right)$ $= 22.75.$ So $X \sim \mathrm{N}(-10,\ 22.75)$. Required probability $= \mathrm{P}\left(R_1 + R_2 + R_3 > 2F\right)$ $= \mathrm{P}\left(R_1 + R_2 + R_3 - 2F > 0\right)$ $= \mathrm{P}(X > 0)$ ≈ 0.0180157796 $= 0.0180$ (3 sf).	Random variables should be clearly defined, and their distributions clearly stated.
(ii) [4]	Let E be the mass of an empty box in grams. Then $E \sim \mathrm{N}\left(80,\ 5^2\right)$. Let $Y = \sum_{i=1}^{6} R_i + \sum_{i=1}^{6} F_i + E$. Then $\mathrm{E}(Y) = 6(54) + 6(86) + 80$ $= 920,$	

	and $\text{Var}(Y) = 6(1.5^2) + 6(2^2) + 5^2$ $= 62.5.$ So $Y \sim \text{N}(920, 62.5)$. We want to find m such that $\text{P}(Y > m) = 0.95$ Using GC, we have $m \approx 906.99629 = 907$ (3 s.f.)	
(iii) **[1]**	Assume that the masses of all donuts and the masses of all empty boxes are independent of one another.	Note that you need to state the "masses of …" It is incorrect to say "donuts are independent".

11 **(a)** Two digits X and Y are chosen independently at random from the set of 10 digits $\{0, 1, 2, \ldots, 9\}$. Events A and B are defined as follows:

A: $X = Y + 1$, .

B: X and Y are both less than 6.

Find

(i) P(A), [1]

(ii) P(B), [1]

(iii) $\mathrm{P}(A \cup B)$. [2]

(b) On a particular afternoon in June, 5 girls and 4 boys were in the Shaw Library and 6 girls and 9 boys were in the Hullett Library. A teacher selects at random 2 students from each library to distribute 4 free concert tickets.

(i) Calculate the probability that 2 girls and 2 boys received the tickets. [3]

(ii) Given that 2 girls and 2 boys received the tickets, calculate the probability that the 2 students selected from the Hullett Library are of the same gender. [2]

11(a) (i) [1]	Consider the ordered pairs (X, Y). Since each of X, Y has 10 possible values, there are $10^2 = 100$ possible pairs. Of these, 9 satisfy the condition $X = Y + 1$, i.e. (1, 0), (2, 1), . . . , (9, 8). $\therefore \mathrm{P}(A) = \dfrac{9}{100}$ or 0.09.
(ii) [1]	$\mathrm{P}(B) = \mathrm{P}(x < 6) \times \mathrm{P}(y < 6)$ $= \dfrac{6}{10} \times \dfrac{6}{10}$ $= \dfrac{9}{25}$
(iii) [2]	Since $A \cap B = \{ (1, 0), (2, 1), (3, 2), (4, 3), (5, 4) \}$ $\mathrm{P}(A \cap B) = \dfrac{5}{100} = \dfrac{1}{20}$ $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \dfrac{9}{100} + \dfrac{36}{100} - \dfrac{5}{100} = \dfrac{2}{5}$

(b) **(i)** **[3]**	P($2G2B$ received tickets) $= \text{P}(2B \text{ from } S, 2G \text{ from } H) + \text{P}(1G1B \text{ from } S, 1G1B \text{ from } H)$ $+ \text{P}(2G \text{ from } S, 2B \text{ from } H)$ $= \left(\frac{{}^4C_2}{{}^9C_2} \times \frac{{}^6C_2}{{}^{15}C_2}\right) + \left(\frac{{}^5C_1 \times {}^4C_1}{{}^9C_2} \times \frac{{}^9C_1 \times {}^6C_1}{{}^{15}C_2}\right) + \left(\frac{{}^5C_2}{{}^9C_2} \times \frac{{}^9C_2}{{}^{15}C_2}\right)$ $= \frac{1}{42} + \frac{2}{7} + \frac{10}{105}$ $= \frac{17}{42}$
	Alternative method: P($2G2B$ received tickets) $= \text{P}(2B \text{ from } S, 2G \text{ from } H) + \text{P}(1G1B \text{ from } S, 1G1B \text{ from } H)$ $+ \text{P}(2G \text{ from } S, 2B \text{ from } H)$ $= \left(\frac{4}{9} \times \frac{3}{8} \times \frac{6}{15} \times \frac{5}{14}\right) + \left[\left(2 \times \frac{5}{9} \times \frac{4}{8}\right) \times \left(2 \times \frac{9}{15} \times \frac{6}{14}\right)\right] + \left(\frac{5}{9} \times \frac{4}{8} \times \frac{9}{15} \times \frac{8}{14}\right)$ $= \frac{1}{42} + \frac{2}{7} + \frac{10}{105}$ $= \frac{17}{42}$
(ii) **[2]**	P(Both from H, same gender | $2G2B$ selected) $= \frac{\text{P}(2B \text{ from } H, 2G \text{ from } S) + \text{P}(2G \text{ from } H, 2B \text{ from } S)}{\text{P}(2G2B \text{ selected})}$ $= \frac{\frac{1}{42} + \frac{10}{105}}{\frac{17}{42}}$ $= \frac{5}{17}$

12 The National Aeronautics and Space Administration (NASA) compiles data on space shuttle launches and publishes them on its website. The following table displays the frequency distribution for the number of crew members on each of the 135 missions from April 1981 to July 2011.

Crew Size	2	3	4	5	6	7	8	9	10
Frequency	4	0	3	36	24	48	14	0	6

(i) Let the random variable C denote the crew size of a randomly selected mission between April 1981 to July 2011. Obtain the probability distribution of C and find $\mathrm{E}(C)$. [3]

A group of students are doing research on the data collected by the crew members of these missions. Each student is randomly assigned one mission, and the student is required to write one report for each crew member in the mission.

(ii) Find the probability that the total number of reports written by 2 randomly chosen students is 8. [2]

(iii) Show that the probability of the total number of reports written by 2 randomly chosen students exceeding 8 is 0.971. [2]

There are n mentors and each mentor is randomly assigned 2 students to grade their reports. Let T denote the number of mentors who need to grade more than 8 reports in total.

(iv) Find the value of n if $\mathrm{E}(T) = 20.391$. [1]

(v) Use the value of n found in part **(iv)** to calculate the probability that more than 19 mentors need to grade more than 8 reports each. [2]

(vi) Find the value of n if the most probable value of T is 25. [2]

12(i) [3]

The probability distribution of C is as follows:

c	2	3	4	5	6	7	8	9	10
$\mathrm{P}(C=c)$	$\frac{4}{135}$	0	$\frac{3}{135}$	$\frac{36}{135}$	$\frac{24}{135}$	$\frac{48}{135}$	$\frac{14}{135}$	0	$\frac{6}{135}$

$\mathrm{E}(C) = \sum c\,\mathrm{P}(C=c) = \dfrac{284}{45}$ or 6.31 (3 s.f.)

(ii) [2]

Let S denote the total number of reports written by 2 randomly chosen students.

$$\mathrm{P}(S=8) = 2\times \mathrm{P}(C=2)\times \mathrm{P}(C=6) + \mathrm{P}(C=4)\times \mathrm{P}(C=4)$$

$$= 2\times \frac{4}{135}\times\frac{24}{135}+\frac{3}{135}\times\frac{3}{135}$$

$$= \frac{67}{6075} \text{ or } 0.0110 \text{ (3 s.f.)}$$

(iii) [2]

$$\begin{aligned}P(S>8)&=1-P(S\le 8)\\&=1-P(S=4,6,7,8)\\&=1-\left(\frac{4}{135}\times\frac{4}{135}+2\times\frac{4}{135}\times\frac{3}{135}+2\times\frac{4}{135}\times\frac{36}{135}+\frac{201}{135^2}\right)\\&=1-\frac{529}{135^2}\\&=0.97097\text{ (5 s.f.)}\\&=0.971\text{ (3 s.f.)}\ \text{(Shown)}\end{aligned}$$

(iv) [1]

$T \sim B\ (n, 0.971)$

$E(T) = 0.971 \times n = 20.391$

$\Rightarrow n = 21$ (to the nearest whole number)

(v) [2]

$T \sim B\ (21, 0.971)$

$P\ (T > 19) = 1 - P\ (T \le 19) = 0.877$ (to 3 s.f.)

Alternatively

$P\ (T > 19) = P\ (T = 20) + P\ (T = 21) = 0.877$ (to 3 s.f.)

(vi) [2]

$T \sim B(n, 0.971)$

From GC:

n	$P(T=24)$	$P(T=25)$	$P(T=26)$
25	0.3578	0.4792	0
26	0.1349	0.3613	0.4653

Therefore, $n = 25$.

Alternative method – if question requires forming of inequalities.
Given that the most probable value of T is 25.

Therefore $n \ge 25$ and

$$\ldots < P(X=24) < P(X=25) > P(X=26) > \ldots$$

We consider

$$P(X=24) < P(X=25)$$

$$\frac{n!}{24!(n-24)!}(0.971)^{24}(0.029)^{n-24} < \frac{n!}{25!(n-25)!}(0.971)^{25}(0.029)^{n-25}$$

$$25(0.029) < (n-24)(0.971)$$

$$n > 24.75$$

For $n \ge 26$, we also consider

$$P(X=25) > P(X=26)$$

$$\frac{n!}{25!(n-25)!}(0.971)^{25}(0.029)^{n-25} > \frac{n!}{26!(n-26)!}(0.971)^{26}(0.029)^{n-26}$$

	$$26(0.029) > (n-25)(0.971)$$ $$n < 25.78$$ Which means that it is impossible for 25 to be the mode for $n \geq 26$. Hence, the value of n required is 25.	