Section A: Pure Mathematics (40 marks)

1 A sequence $u_1, u_2, u_3, \ldots$ is such that $u_n = \frac{1}{n!}$ for $n \in \mathbb{Z}^+$ and $u_n = u_{n-1} + \frac{1-n}{n!}$ for all $n \geq 2$.

(i) Find $\sum_{n=2}^{N} \frac{1-n}{n!}$. [3]

(ii) Give a reason why the series in **(i)** is convergent and state the sum to infinity. [2]

(iii) Hence find $\sum_{n=8}^{N+5} \frac{2-n}{(n-1)!}$ in terms of N. [3]

2 The equations of three planes Π_1, Π_2 and Π_3 are

$$x - py + z = 9, \quad 3x - y - 2z = 10 \quad \text{and} \quad x - ay - z = 5$$

respectively, where a and p are constants.

The line l_1 has equation $\mathbf{r} = 2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$, where $\mu \in \mathbb{R}$.

(i) Given that l_1 does not intersect with Π_1, show that $p = -4$ and find the shortest distance between the line l_1 and Π_1. [3]

(ii) The line l_2 is the reflection of the line l_1 in Π_1. Hence, or otherwise, find the Cartesian equation of the plane that contains the line l_2 and is parallel to Π_1. [2]

(iii) Given that the line l_3 lies on both Π_2 and Π_3, find a vector equation of l_3, leaving your answer in terms of a. [3]

(iv) Let θ be the acute angle between l_3 and Π_1. Find the value(s) of a if $\sin\theta = \frac{\sqrt{3}}{18}$. [3]

3 In this question you may use expansions from the List of Formulae (MF26).

(i) Find the Maclaurin expansion of $\ln(1+\cos 3x)$ in ascending powers of x, up to and including the term in x^4, for $0 \le x < \frac{\pi}{3}$. [4]

(ii) Use your expansion from part **(i)** and to find an approximate value for $\int_0^{0.5} x\ln(1+\cos 3x)\,\mathrm{d}x$, giving your answer to 5 decimal places. [2]

(iii) Use your calculator to find the value of $\int_0^{0.5} x\ln(1+\cos 3x)\,\mathrm{d}x$ up to 5 decimal places. [1]

(iv) With the aid of a suitable diagram, comparing your answers in **(ii)** and **(iii)**, comment on the accuracy of your approximations. [2]

4 A curve C has parametric equations

$$x = t^2 - 9 \ , \ y = t^3 - 12t \ ,$$

where $t \in [-3, 3]$.

(i) Sketch the graph of C, indicating the coordinates of the end points. [2]

The normal to the curve C at the point P is given by $9y + 2x = 83$.

(ii) By finding the gradient of the curve C at the point with parameter t, calculate the value of t at the point P. [3]

Given that the normal to curve C at P intersects the curve C at another point Q with parameter q.

(iii) Show that $9q^3 + 2q^2 - 108q - 101 = 0$ at Q. Hence, find the coordinates of Q. [4]

(iv) Find the area bounded by the curve C and the normal to the curve at point P, giving your answers correct to 3 significant figures. [3]

Section B: Probability and Statistics (60 marks)

5 A group of 10 people consists of 9 men and 1 woman. Find the number of ways which the group can be seated at a round table with 10 identical chairs if

(i) 2 particular men, Caleb and James are not seated beside the woman, but are seated next to each other. [3]

The 10 identical chairs at the table are replaced with 10 chairs of different colours.

(ii) Find the number of ways in which the group can be seated at the round table if Caleb, James and the woman are not all seated next to one another. [3]

6 On average, 30% of the students in Saints Senior Institute could solve the differentiation question in Paper One of the Preliminary Examinations. The Head of Department randomly selects a class to analyse the results. You may assume that the number of students who could solve the differentiation question follows a binomial distribution.

Given that there are 30 students in the class, find

(i) the probability that at least 6 students in that class could solve that question. [2]

(ii) the probability that only 2 students among the first 8 selected students in that class could solve the question given that at least 6 students could solve that question. [3]

Another class with n students is then randomly selected.

(iii) It is known that the probability of no more than 5 students could solve the question in a randomly selected class exceeds 0.9. Find the largest possible number of students in that class. [3]

7 A study is being carried out to investigate the relationship between M, the BMI(Body Mass Index), and P, Diastolic Blood Pressure(DBP), for male adults aged 45 to 54 years old. A random sample of 9 male adults is taken and their data are shown in the table below.

BMI (M kg/m^2)	19.2	20.6	23.0	25.9	28.8	30.0	31.6	33.3	33.6
DBP (P mm Hg)	89	89	90	k	96	97	99	101	102

Given that the product moment correlation coefficient between M and P is 0.986 and the equation of the regression line of P on M is $P = 0.92588M + 69.804$.

(i) Show that $k = 93$. [1]

(ii) Draw a scatter diagram to illustrate the data, labelling the axes clearly. [2]

(iii) The relationship between M and P can also be modelled by an equation of the form $\ln P = aM + b$, where a and b are constants. Using the scatter diagram in **(ii),** explain whether a is positive or negative. Find the product moment correlation coefficient between M and ln P. [2]

(iv) Using **(ii)** and **(iii)**, explain which of $P = 0.92588M + 69.804$ or $\ln P = aM + b$ is the better model. [2]

(v) John is 50 years old. His BMI is 35 kg/m^2. Comment on whether it is reliable to estimate his DBP using the better model in **(iv)**. [1]

8 In a game, two boxes have the following contents.

Box A contains 3 green balls and 4 white balls.

Box B contains 6 green balls and 5 white balls.

A fair die is thrown once. Two balls are then drawn in sequence in the following manner:

1. If the number that appears on the top face of the die is less than 3, the first ball is drawn from Box A and transferred to Box B. Otherwise, the first ball is drawn from Box B and transferred to Box A.
2. The second ball is then randomly picked from the box that received the transferred ball in step 1.

(i) Draw a probability tree diagram to represent the above information. [2]

(ii) If the second ball drawn is white, the player wins the game. Otherwise, the player loses the game. Find the probability that the first ball drawn is from Box A, given that the player wins the game. [5]

9 A random variable X has the probability distribution given as

$$\mathrm{P}(X = x) = \begin{cases} p, & x = 2, 5 \\ q, & x = 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

(i) Given that $\mathrm{E}\left(X^2\right) = 13.3$, find the values of p and q. Hence, without the use of the calculator, find Var(X). [5]

(ii) Thirty independent observations of X are taken. Using a suitable approximation, find the probability that the mean value of these observations exceeds 3.8. [2]

10 In this question you should state clearly the parameters of any distributions that you use.

A supermarket sells honeydews and watermelons. The masses, in kilograms, of the honeydews and the watermelons each follow a normal distribution. The means and standard deviations of these distributions are shown in the following table:

	Mean (kg)	**Standard deviation (kg)**
Honeydew	1.5	0.2
Watermelon	8.5	0.3

You may assume that the masses of the fruits (watermelon and honeydew) are independent of one another.

(i) Find the probability that for 3 randomly chosen honeydews, two of the honeydews each has mass less than 1.8 kg and one of the honeydews has mass more than 1.8 kg. [2]

(ii) Find the probability that the total mass of 5 randomly chosen honeydews is less than the mass of one randomly chosen watermelon. [3]

The supermarket wants to pack fruits into gift packs to be donated to needy families. Each gift pack consists of one randomly chosen honeydew and one randomly chosen watermelon.

(iii) 90% of the gift packs have masses differ from the mean mass of gift packs by less than m kg, find the value of m. You may assume that the packing material has negligible mass. [3]

The selling price of honeydews is \$3.50 per kilogram and the selling price of watermelons is \$0.70 per kilogram. Lam has a budget of \$10.

(iv) Lam intends to buy one honeydew and one watermelon. Find the probability that Lam is able to pay for his purchase. [3]

Let the probability that a honeydew and watermelon cost at most \$5 **each** be k.

(v) Explain, without any further calculation, why the probability in **(iv)** is at least k. [1]

11 The branch manager of a bank would like to find out about the satisfactory level of the customer services provided by the branch. He would like to survey 80 customers of the branch to find out about their opinion on the wait time.

(i) Describe how the branch manager could obtain a random sample of 80 customers to conduct his survey. [2]

The mean amount of time a customer needs to wait in the queue until they were served was known to be at least 15 minutes. After the survey was conducted, he realised that the waiting time for each customer before they were served at the counter was of a major concern. A change in processes at the branch was implemented. After a month, the branch manager of the bank decided to record the waiting time, *t* minutes, for a customer at the branch for 50 different customers to evaluate if the changes were effective. The results are summarised by

$$\sum(t-15)=-60\ ,\ \sum(t-15)^2=1168\,.$$

(ii) Find the unbiased estimates of the population mean and variance. [2]

(iii) Test, at the 5% level of significance, whether the change in processes have been effective. [4]

(iv) Explain what is meant by the *p*-value obtained in **(iii)** in the context of the question. [1]

(v) The quality service manager claimed that that the mean waiting time before there were any changes in processes was in fact k minutes. With the same data collected from the same 50 customers, find the range of values of k if there is insufficient evidence to conclude that there was a change in the mean waiting time at 2% level of significance. [3]