Chapter 10

OSCILLATIONS

Content

- Simple harmonic motion
- Energy in simple harmonic motion
- Damped and forced oscillations: resonance

Learning Outcomes

Candidates should be able to:

Simple Harmonic Motion (SHM)

(a) describe simple examples of free oscillations.

(b) investigate the motion of an oscillator using experimental and graphical methods.

(c) show an understanding of and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency.

(d) recall and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion.

(e) recognise and use $x = x_0 \sin\omega t$ as a solution to the equation $a = -\omega^2 x$.

(f) recognise and use $v = v_0 \cos\omega t$ and $v = \pm\omega\sqrt{(x_0^2 - x^2)}$.

(g) describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion.

Energy in SHM

(h) describe the interchange between kinetic and potential energy during simple harmonic motion.

Damped and Forced Oscillations, Resonance

(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and the importance of critical damping in cases such as a car suspension system.

(j) describe practical examples of forced oscillations and resonance.

(k) describe graphically how the amplitude of a forced oscillation changes with driving frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and sharpness of the resonance.

(l) show an appreciation that there are some circumstances in which resonance is useful and other circumstances in which resonance should be avoided.

10.0 Oscillations and Waves

Links Between Sections and Topics

Periodic motion, where the pattern of movement repeats over time, is ubiquitous, and arises for example when objects are perturbed from a condition of stable equilibrium. While much of the motion we have considered is non-periodic, we have studied uniform circular motion, which is periodic and regular. Even in one spatial dimension, there can be complicated types of periodic motion. Nonetheless, we can gain a deep understanding of periodic motion by analysing the mathematically simplest case of free oscillations, known as simple harmonic motion (SHM). Such sinusoidally varying motion is essentially a projection of uniform circular motion, and provides a mathematical basis upon which to describe more complicated oscillations. Naturally, we revisit concepts in kinematics, dynamics, forces and energy in trying to understand SHM.

When we consider a system of connected particles, the idea of single particles undergoing oscillations is the starting point that leads on to the idea of waves within the system. While we have seen how powerful the particle picture is, it turns out that the wave picture, generalised beyond classical mechanics, is equally fundamental for describing and understanding the physical universe.

With waves, we move conceptually from physics of particles to the physics of continuous media. All waves are disturbances which result in oscillations. The oscillations then spread out as waves, which carry energy and can result in disturbances far away. Waves are a means of transmitting energy without the attendant transfer of matter. Remarkably, one of the many surprises of nature is that electromagnetic waves can travel through a vacuum, an example of field oscillations that do not require particles.

We can also discuss wave mechanics, as waves interact, though in a qualitatively different way from how particles interact. The principle of superposition allows accurate characterisation of interaction of waves. Interference and diffraction are important wave phenomena due to the superposition of waves. However, there is actually no clear distinction between interference and diffraction. The difference in the usage of the terms is mainly historical. Many of the ideas introduced during the study of waves in this section will later be important for appreciating the limitation of classical physics in explaining the behaviour of matter on the atomic scale and understanding quantum wave-particle duality.

Applications and relevance to daily life

Oscillations and waves play important roles in engineering and nature. In nature, molecules in a solid oscillate about their equilibrium position; electromagnetic waves consist of oscillating electric and magnetic fields, and waves are present everywhere, e.g. light travelling from the Sun to Earth, water waves and sound waves. The study and control of oscillation is needed to achieve important goals in engineering, e.g. to prevent the collapse of a building due to waves created by an earthquake. Furthermore, diffraction gratings allow us to determine the frequencies of light sources ranging from lamps to distant stars. Optical engineers also create optically variable graphics (OVG) on credit cards, which incorporate diffraction grating technology, as an anti-counterfeiting measure.

Links to Core Ideas

Systems and Interactions

- A wave is a source of disturbance that can transfer energy and momentum through time and space
- Interaction of electromagnetic wave with matter (e.g. reflection, refraction, diffraction, absorption, scattering)

Models and Representations

- Simple harmonic motion of a mass characterised by a restoring force that is proportional to its displacement
- Mechanical wave model
- Wave nature of electromagnetic radiation
- Superposition principle, which is used to explain wave phenomena (e.g. standing waves, two-source interference, diffraction)
- Common representations: e.g. wavefront diagrams, displacement-time graph (characteristic of every particle), displacement-position graph (snapshot of wave in time)
- Simplifying assumptions: e.g. ignore dissipative forces like friction and air resistance (negligible attenuation)

Conservation Laws

- Conservation of mechanical energy in an SHM system
- The relationship between intensity and distance for a point source
- The intensity distribution of a double-slit interference pattern obeys the conservation of energy

10.1 Simple Harmonic Motion (S.H.M.)

Introduction

A ***periodic*** motion is one in which an object continually retraces its path at *equal* time intervals. Many systems exhibit periodic motion. The molecules in a solid oscillate about their equilibrium positions; electromagnetic waves are characterised by oscillating electric and magnetic field vectors; and in alternating-current electrical circuits, voltage and current vary periodically with time.

An oscillation is a special periodic motion in which the oscillator moves to and fro about an equilibrium position. This is also called harmonic motion. Simple harmonic motion is a type of such a motion.

Definition

Simple harmonic motion is defined as (1) *the motion of a particle about a fixed point such that its acceleration is proportional to its displacement from the fixed point and* (2) *is always directed towards the point.*

Mathematically,

$$\boxed{a \propto -x}$$

i.e.

$$\boxed{a = -\omega^2 x}$$

x : displacement from equilibrium position
ω^2 : positive constant

where a is the acceleration, x is the displacement from the equilibrium position and ω^2 is a positive constant, where ω is the angular frequency of the oscillation.

Definition

Angular frequency is defined as the *rate of change of phase angle* of the oscillation, and is equal to the product of 2π and its frequency (i.e $\omega = 2\pi f$)

The unit of ω is radian per second (rad s^{-1}).

The negative sign in the equation indicates that the acceleration a acts in a direction opposite to that of the displacement x.

Since the acceleration of the object is <u>not</u> constant, it is <u>not</u> possible to apply the usual kinematics equations in solving SHM problems.

Note

scalar   vector

Do not confuse *angular frequency* with *angular velocity*. Even though the two have the same units of rad s^{-1} and are written with the same symbol ω, they are *not* the same. When dealing with simple harmonic motion, ω stands for *angular frequency*. When dealing with uniform circular motion, ω stands for angular velocity, which is the rate of change of angular displacement.

Characteristics of S.H.M.

Fig. 10.1.1 illustrates the motion of a particle N in simple harmonic motion. Particle N experiences a restoring force that is always directed towards the equilibrium position O and oscillates between points A and B. The displacement x from point O, the velocity v and acceleration a of particle N are also indicated in Fig. 10.1.1. The direction towards the right is taken to be positive.

Body vibrating in a straight line

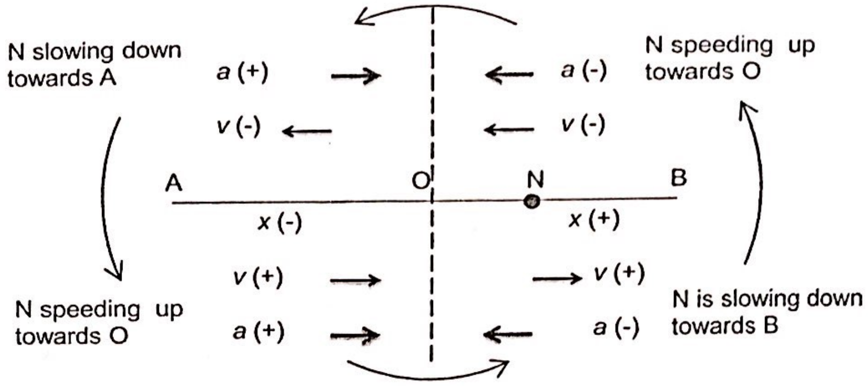


Fig. 10.1.1

A O B
x: max -ve, 0, max +ve
v: 0, max +ve or -ve, 0
a $= -\omega^2 x$: max +ve, 0, max -ve
in antiphase

The motion of particle N has the following characteristics:

- Its path is a straight line.
- It is symmetrical about the equilibrium position O. only 1 value
- $|OA| = |OB| = x_0$, where x_0 is the **amplitude** of the motion. amplitude ≠ displacement

Definition

- Amplitude is the *magnitude of the maximum displacement of the particle from its equilibrium position.*

Definition

- **Period *T*** is the *time taken to complete one oscillation.* (E.g. BOAOB)

$$\boxed{T = \frac{2\pi}{\omega}} \Rightarrow \boxed{\omega = \frac{2\pi}{T}}$$

Formula

Definition

- **Frequency *f*** is the *number of oscillations per unit time.*

  Thus T and f are related by the equation: $\boxed{T = \frac{1}{f}} \Rightarrow \boxed{\omega = 2\pi f}$

  If T is measured in seconds s, then f will be in hertz, Hz. (S.I. unit), where 1 Hz = 1 s^{-1}.

Note

- The speed at O is a maximum and zero at A and B.
- Acceleration is always directed towards O.
- The magnitude of the acceleration is zero at O and maximum at A and B.

Displacement-Time Graph

The variation of displacement from the equilibrium position x with time t of particle N in Fig. 10.1.1, can be represented by a displacement-time graph.

If at $t = 0$, $x = +x_0$ (i.e. N is at B), the motion of N is given by $x = x_o \cos \omega t$.

displacement amplitude angular frequency

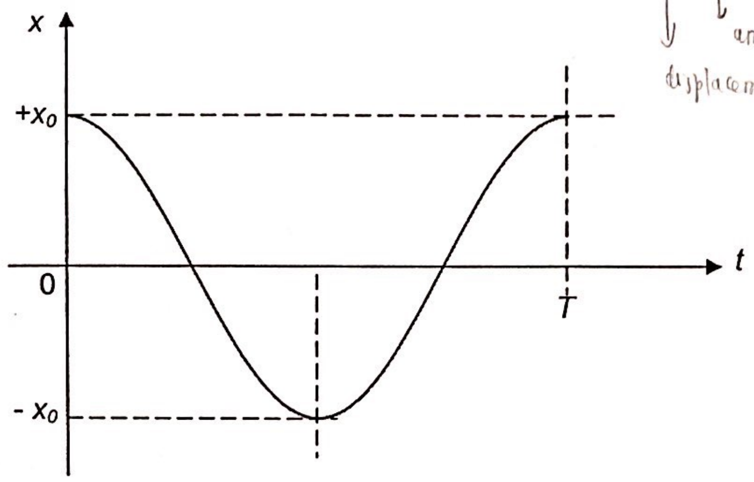


Fig. 10.1.2

If at $t = 0$, $x = 0$ (i.e. N is at O), the motion of N is given by $x = x_o \sin \omega t$.

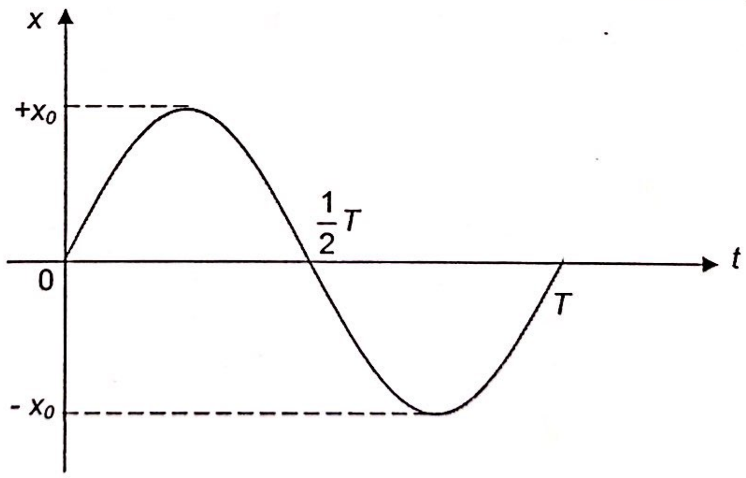


Fig. 10.1.3

Changes in x, v and a during S.H.M.

Consider the case $x = x_0$ at $t = 0$, ie at point B, displacement is given by

x* varies with *t

$$\boxed{x = x_o \cos \omega t}$$ (Refer to Fig. 10.1.4)

$x = x_o \sin \omega t$
$v = \omega x_o \cos \omega t$
$a = -\omega^2 x_o \sin \omega t$

By calculus, we have

$$v = \frac{dx}{dt}$$
$$= \frac{d}{dt}(x_o \cos \omega t)$$

v* varies with *t

$$\boxed{v = -x_o \omega \sin \omega t}$$ (Refer to Fig. 10.1.5)

and

$$a = \frac{dv}{dt}$$
$$= \frac{d}{dt}(-x_o \omega \sin \omega t)$$

a* varies with *t

for $a = -\omega^2 x$, must be a sinusoid (aka cos, sin)

$$\boxed{a = -x_o \omega^2 \cos \omega t}$$ (Refer to Fig. 10.1.6)

Also,

$$v = -x_o \omega \sin \omega t$$
$$= -\omega x_o \sin \omega t$$ ← not unique, time dependent
$$= -\omega x_o \left(\pm\sqrt{1-\cos^2 \omega t}\right)$$
$$= \pm\omega\sqrt{x_o^2\left(1-\cos^2 \omega t\right)}$$
$$= \pm\omega\sqrt{\left(x_o^2 - x_o^2\cos^2 \omega t\right)}$$

Formula

familiarise ← $$\boxed{v = \pm\omega\sqrt{\left(x_o^2 - x^2\right)}}$$ free v from t · in terms of displacement & amplitude ↓ different graph

(Refer to Fig. 10.1.7)

Formulae Sheet

and

$$a = -x_o \omega^2 \cos \omega t$$
$$= -\omega^2 (x_o \cos \omega t)$$

Formula

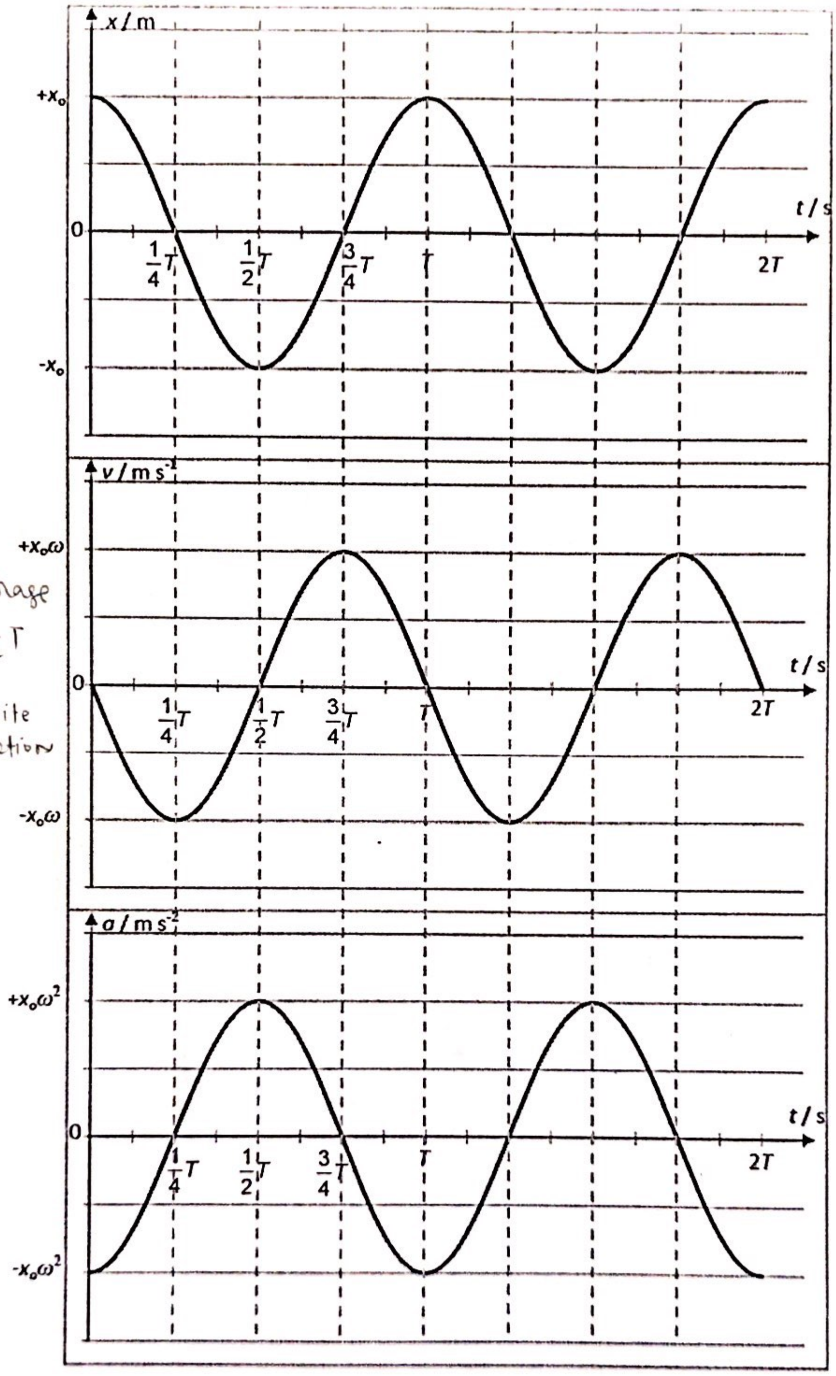
$$\boxed{a = -\omega^2 x}$$ (Refer to Fig. 10.1.8)

Graphical Illustrations

The graphs on the next page show how the displacement, velocity and acceleration of N in Fig. 10.1.1 vary with time during two complete cycles.

max occurs when $\sin\theta$ or $\cos\theta = 1$

not unique



$x = x_o \cos \omega t$ — Fig. 10.1.4

antiphase at $\frac{1}{2}T$

opposite direction

$(90-\theta)$

$\cos\theta = \sin$ ~~θ~~

$v = -x_o\omega \sin \omega t$ — Fig. 10.1.5

$a = -x_o\omega^2 \cos \omega t$ — Fig. 10.1.6

1st quarter of cycle	When N is speeding up from B towards O, both *v* and *a* are directed towards O (both negative).
2nd quarter of cycle	When N is slowing down from O to A, *v* and *a* are in opposite directions: *v* is negative but *a* is positive.
3rd quarter of cycle	When N is speeding up from A towards O, both *v* and *a* are directed towards O (both positive).
4th quarter of cycle	When N is slowing down from O to B, *v* and *a* are in opposite directions: *v* is positive but *a* is negative.
	The cycle then repeats.

The following graphs show how the velocity and acceleration of N in Fig. 10.1.1 vary with displacement.

$v = \pm\omega\sqrt{(x_o^2 - x^2)}$

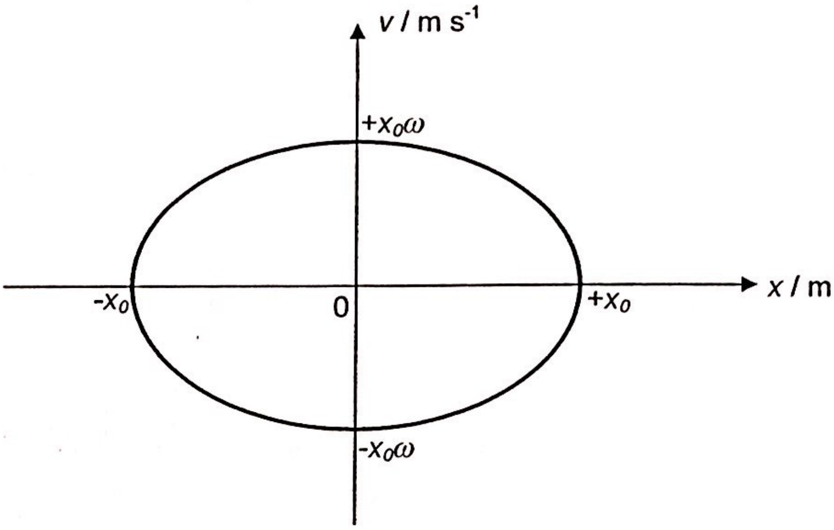


Fig. 10.1.7

$a = -\omega^2 x$

gradient = $-\omega^2$

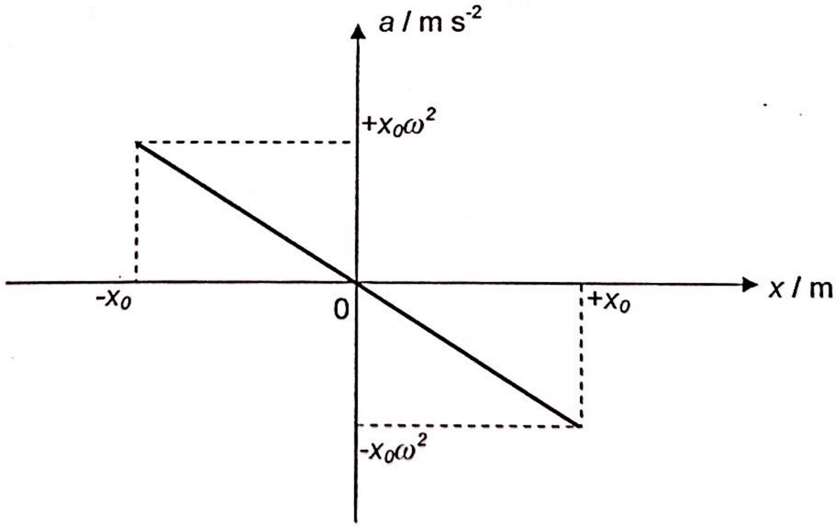
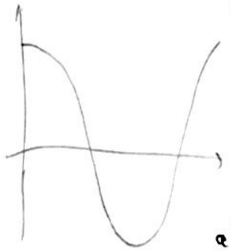


Fig. 10.1.8

Example 1

t=0 equilibrium
t=0 amplitude



A particle describes S.H.M. in which the displacement is given by

$$x = 0.05\cos(4\pi t)$$

where x is in metres and t in seconds.

(a) What is the amplitude of the motion?
(b) What is the period of the motion?
(c) What is the maximum velocity of the particle?
(d) What is the maximum acceleration of the particle?

Solution

a) amplitude = 0.05 m

b) $\omega = 4\pi$

$T = \frac{2\pi}{\omega} = 0.50$ seconds

c) max. velocity $= \omega\sqrt{x_0^2}$

$= \omega x_0$

$= (0.05)(4\pi)$

$= 0.2\pi$ ms^{-1} //

$|v| = \omega\sqrt{x_0^2 - x^2}$

maximum velocity $= \omega x_0$

$= 0.628 ms^{-1}$

d) max. acceleration $= (4\pi)^2(0.05)$

$= 0.8\pi^2$ ms^{-2},

$= 7.90 ms^{-1}$

mod

$|a| = \omega^2 x$

max acceleration $= \omega^2 x_0$

Example 2
(J82/II/10)

Figures (a) and (b) below show how the displacement x and the acceleration a of a body vary with time when it is oscillating with simple harmonic motion.

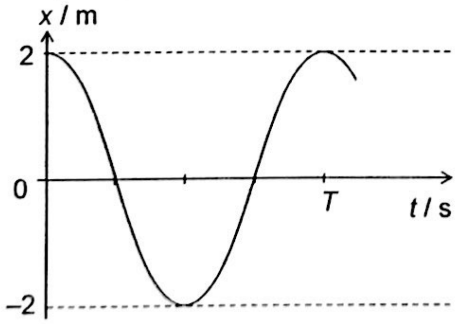


Figure (a)

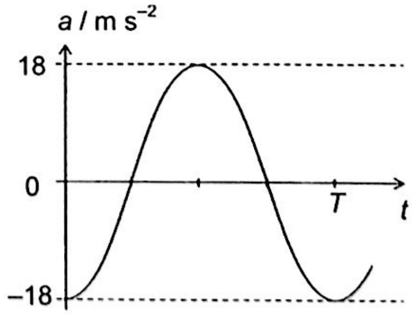


Figure (b)

What is the value of T?

Solution

From Figure (a), at $t = 0$, $x_0 = 2$ m and from Figure (b), $a_{max} = -18$ m s^{-2}

$-\omega^2 x = -18$

$-\omega^2 = \frac{-18}{2}$

$\omega = 3$

$T = \frac{2\pi}{\omega}$

$T = 2.09 s$ //

$a = -\omega^2 x \Rightarrow a_{max} = -\omega^2 x_0$

$-18 = -\left(\frac{2\pi}{T}\right)^2 (2)$

$T = 2.09 s$

Free Oscillations

If an object is displaced from its equilibrium position and then released, it oscillates at its *natural frequency* about the equilibrium position.

Free oscillation occurs when an object oscillates with no resistive and driving force acting on it. Its total energy and amplitude remain constant with time.

Examples: a swinging simple pendulum and a loaded spiral spring bobbing up and down

Models for S.H.M.

Two common mechanical systems are used to illustrate simple harmonic motion: spring-mass system and simple pendulum.

Spring-Mass System

Horizontal

Consider a block of mass m attached to the end of a spring of negligible mass and force constant k, with the block free to move on a horizontal frictionless surface. When the spring is neither stretched nor compressed it is at its equilibrium position as shown in Fig. 10.1.9.

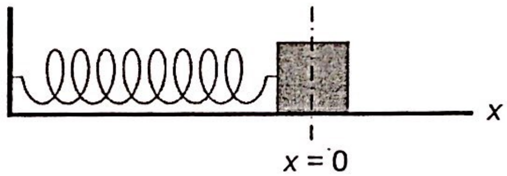


Fig. 10.1.9

The block is displaced a distance x to the right in Fig. 10.1.10.

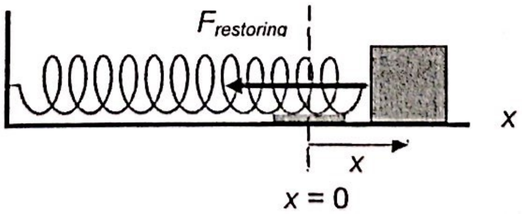


Fig. 10.1.10

The restoring force exerted towards the left by the spring on the block is

$$F_{restoring} = -kx$$

It is the resultant force acting on the block, hence by Newton's 2nd law of motion:

$$F_{restoring} = ma$$

~~changing length~~

$$-kx = ma$$

$$\boxed{a = -\left(\frac{k}{m}\right)x}$$

Formula

Comparing with $a = -\omega^2 x$, $\boxed{\omega = \sqrt{\frac{k}{m}}}$ and $\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$

proportional to $\sqrt{m}$ and inv. prop to $\sqrt{k}$

<u>Vertical</u>

A vertically suspended spring of negligible mass and force constant k is stretched by an amount e when a block of mass m is hung on it and remains at rest at the equilibrium position. The block is then given an additional downward displacement y (positive direction downward) and released as shown in Fig. 10.1.11.

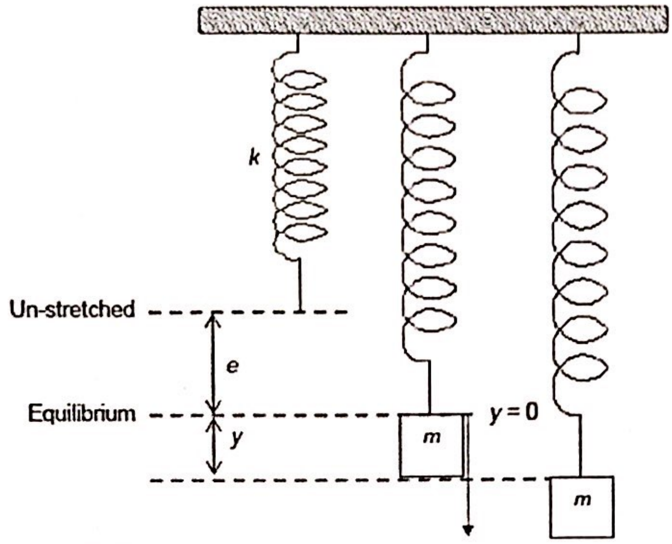
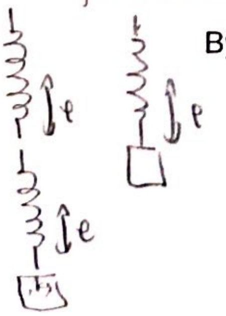


Fig. 10.1.11

The initial static equilibrium is characterised by a balance between the elastic force and the weight of the block:

$$* \quad mg = ke \quad (kx)$$

changing length affects T even though not in formulae ie in series. will Δk.



Once the block is pulled down and released, the restoring force is

$$F_{restoring} = -k(e+y) + mg$$

By Newton's 2nd law:

$$F_{restoring} = ma$$

$$-k(e+y) + mg = ma$$

$$-ky = ma$$

$$\boxed{a = -\left(\frac{k}{m}\right)y}$$

same form

regardless of vertical/horizontal

Formula

Comparing with $a = -\omega^2 y$, $\boxed{\omega = \sqrt{\frac{k}{m}}}$ and $\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$ $= 2\pi\sqrt{\frac{e}{g}}$

By comparing the horizontal and vertical spring-mass systems, it can be concluded that <u>only the mass and spring constant determine their motions</u>.

Simple Pendulum

A simple pendulum consists of a bob suspended by a light string. The forces acting on the bob are the tension T in the string and the weight mg of the bob. When the bob is displaced by a small angle θ (<10°), it is displaced by a distance $s = L\theta$.

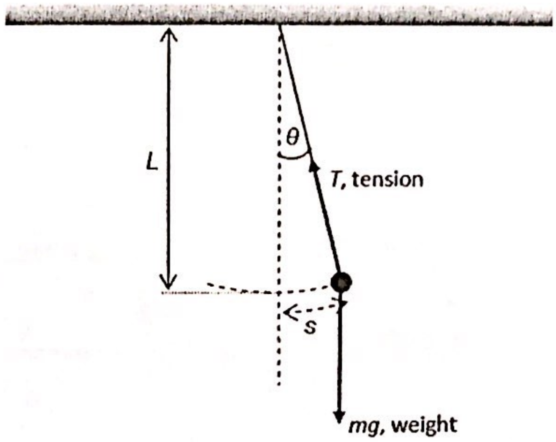
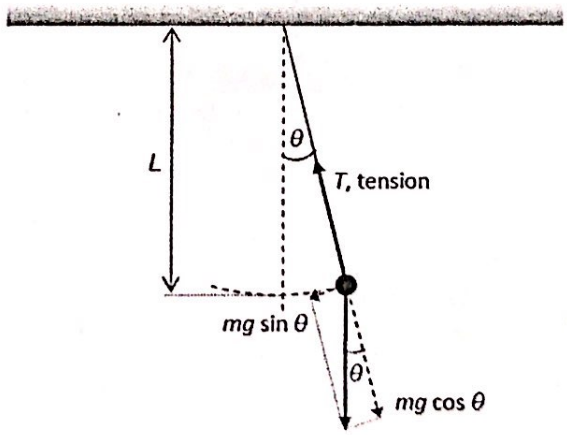


Fig. 10.1.12

The restoring force is the tangential component of mg:

$$F_{restoring} = -mg\sin\theta$$



changing mass of pendulum doesn't affect T

Fig. 10.1.13

Since θ is very small, by small angle approximation,

$\sin\theta \approx \theta$ and $F_{restoring} \approx -mg\theta = -mg\frac{s}{L}$ radians

By Newton's 2nd law:

θ is small

$$F_{restoring} = ma$$

$$-mg\frac{s}{L} = ma$$

$$\boxed{a = -\left(\frac{g}{L}\right)s}$$

$T \propto \frac{L}{g}$

Formula

Comparing with $a = -\omega^2 s$, $\boxed{\omega = \sqrt{\frac{g}{L}}}$ and $\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$

Example 3

An object of mass 0.20 kg is hung from the lower end of a spring. When the object is pulled down 5.0 cm below its equilibrium position O and released, it vibrates with S.H.M. with a period of 2.0 s.

(a) What is the extension of the spring when the mass is hung at rest from the lower end of the spring?
(b) What is the speed of the mass as it passes through O?
(c) What is the magnitude of its acceleration when it is 2.5 cm above O?
(d) Through what distance will the object move in the first 0.75 s?

Solution

$m = 0.20\,kg$ $\quad x_0 = 5.0\,cm = 0.05\,m$ $\quad T = 2.0\,s$

a) $mg = ke$

$T = 2\pi\sqrt{\frac{m}{k}}$

$2.0 = 2\pi\sqrt{\frac{0.20}{k}}$

$\therefore k = 1.97\,N\,m^{-1}$ (3s.f.)

$e = \frac{mg}{k}$

$= \frac{9.81 \cdot 0.2}{1.974}$

$= 0.996\,m$ (3s.f.) $\quad 0.99\,m$ (2s.f.) ✓

b) When mass passes through O, $v = \pm\omega x_0$ (3s.f.)

$\omega = \frac{2\pi}{2.0}$

$x_0 = 0.05\,m$

$v = \left(\frac{2\pi}{2.0}\right)(0.05)$

$= 0.16\,ms^{-1}$

c) $|a| = |-\omega^2 x|$

$= \left(\frac{2\pi}{2.0}\right)^2 (2.5\times10^{-2})$

$= 0.25\,ms^{-2}$

d) taking downwards and displacement below eqm posi[tion] as negative,

$x = -5.0\cos\left(\frac{2\pi}{2.0}t\right)$

$= -5.0\cos(\pi t)$

at $t = 0.75$, $x = -5.0\cos(\pi(0.75))$

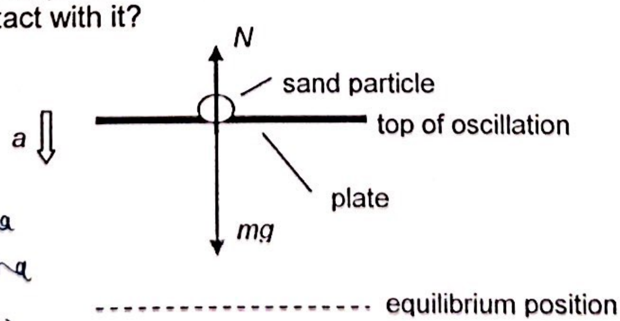
$= 3.5\,cm$

i.e. the object is 3.5 cm above eqm position. Hence it moves by $(5.0 + 3.5) = 8.5\,cm$ in the first 0.75 s.

Example 4

A horizontal plate is vibrating vertically in S.H.M. at a frequency of 20 Hz. What is the maximum amplitude of vibration so that fine sand on the plate always remains in contact with it?

Solution



$T = \frac{1}{f} = \frac{1}{20}$

$= 0.05\,s$

$mg - N = ma$

$N = mg - ma$

$= m(g-a)$

$N > 0 \Rightarrow$ For sand to remain in contact with plate, normal contact force N should be more than 0.

$\therefore g - a > 0$

$a < g$

$\omega^2 x < g$

$x < \frac{9.81}{(2\pi(20))^2}$

$< 0.00062\,m$

10.2 Energy in Simple Harmonic Motion

Interchange between Kinetic and Potential Energy

For a simple harmonic oscillator, the system does no work against dissipative forces, and so its total energy remains constant with time.

The kinetic energy of an oscillating body clearly varies during the cycle. There is a continual change of energy from kinetic energy to potential energy and *vice-versa*. At any instant during the motion, the total energy of the system is equal to the sum of its kinetic energy and potential energy.

gravitational or elastic or both

Variation of Energy in S.H.M.

Suppose that a body of mass *m*, which is attached to a light spring, oscillates on a horizontal frictionless surface about the equilibrium position O:

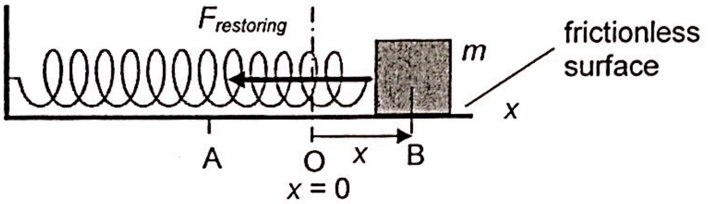


Fig. 10.2.1

Let us consider an ideal oscillator, which would continue to oscillate forever once it is started. We are assuming that there is no energy loss as a result of friction or other resistance to motion.

Let us now see how the kinetic and potential energies of the oscillator vary with distance from the equilibrium position.

E_k varies with x

The kinetic energy E_k at a distance x from its equilibrium position is given by

$$E_k = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left(\pm\omega\sqrt{\left(x_o^2 - x^2\right)}\right)^2$$

Formula

$$\boxed{E_k = \frac{1}{2}m\omega^2\left(x_o^2 - x^2\right)}$$

where x_o is the amplitude of the motion.

E_p varies with x

The potential energy E_p at a displacement x from its equilibrium position is given by

$$E_p = \frac{1}{2}kx^2$$

Formula

$$\boxed{E_p = \frac{1}{2}m\omega^2x^2}$$

$E_{k\,max}$ / $E_{p\,max}$ = total energy

The total energy E at displacement x is given by

E is constant

Formula

$$E = E_k + E_p$$

$$= \frac{1}{2}m\omega^2\left(x_o^2 - x^2\right) + \frac{1}{2}m\omega^2 x^2$$

$$\boxed{E = \frac{1}{2}m\omega^2 x_o^2}$$

The following graph in Fig. 10.2.2 shows the variation of E_k, E_p and E with displacement x:

Variation of E_k, E_p and E with displacement x

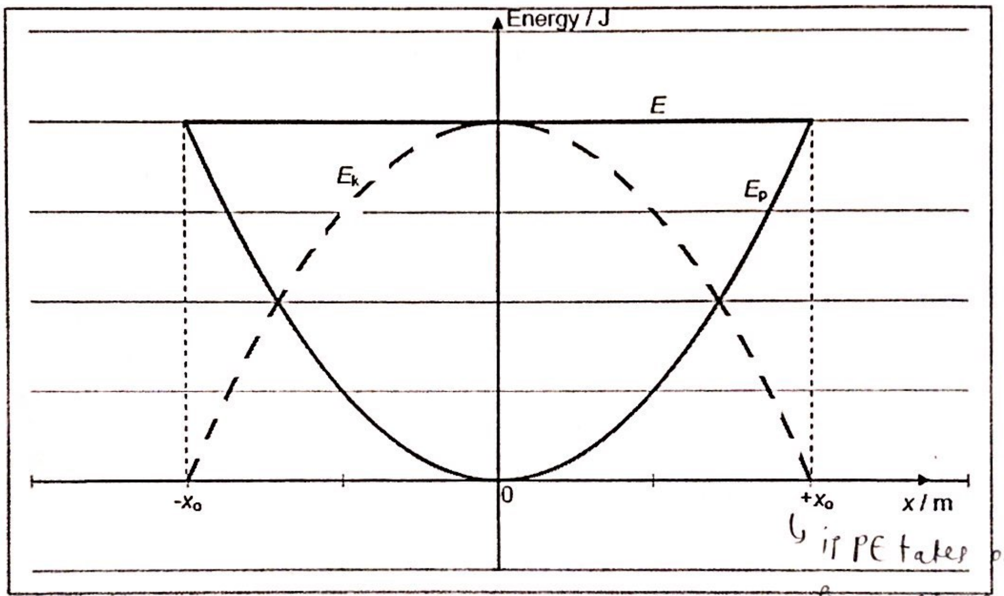


Fig. 10.2.2

↳ if PE takes on only 1 form: simple pendulum/ horizontal spring mass system

Let us now see how the kinetic and potential energies of the oscillator vary with time.

Suppose $x = x_o \cos\omega t$ and $v = -x_o\omega \sin\omega t$, the kinetic energy E_k and potential energy E_p at time t are given by

$$E_k = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left(-x_o\omega \sin\omega t\right)^2$$

E_k varies with t

$$\boxed{E_k = \frac{1}{2}m\omega^2 x_o^2 \sin^2\omega t}$$

→ might not be unique

and

$$E_p = \frac{1}{2}kx^2$$

$$= \frac{1}{2}k\left(x_o \cos\omega t\right)^2$$

$$= \frac{1}{2}kx_o^2 \cos^2\omega t$$

E_p varies with t

$$\boxed{E_p = \frac{1}{2}m\omega^2 x_o^2 \cos^2\omega t}$$

E is constant with t

Thus at any instant the total energy is given by $E = \frac{1}{2}m\omega^2 x_o^2$, as before.

The graph in Fig. 10.2.4 shows the variation of E_k, E_p and E with time t:

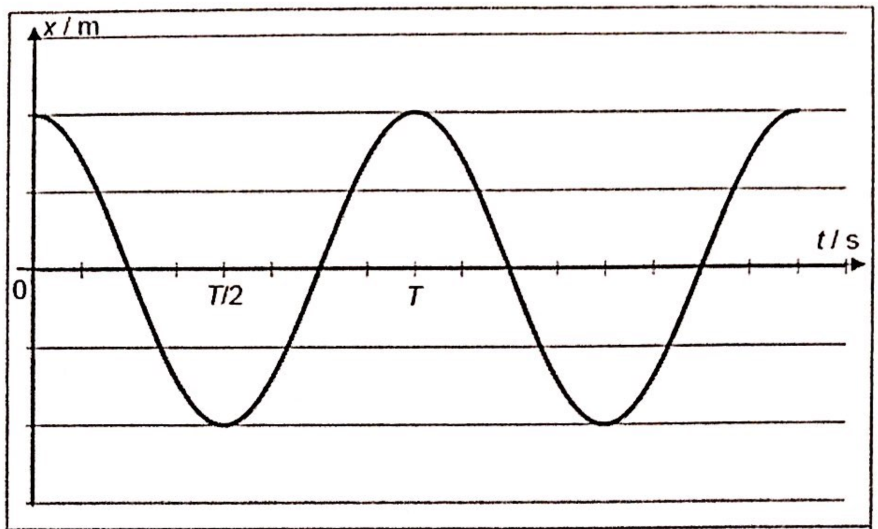


Fig. 10.2.3

Variation of E_k, E_p and E with time t

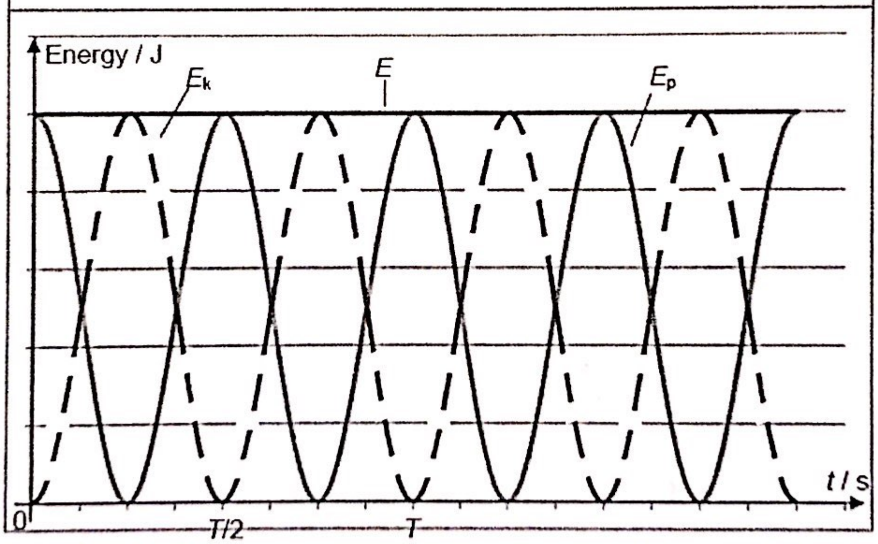


Fig. 10.2.4

The frequency of the energy variation is *twice* that of the motion.

Example 5

Refer to Fig. 10.1.11

A mass of 0.50 kg is attached to a light spring which has a force constant of 20 N m^{-1}. The mass is displaced a distance 5.0 cm below the equilibrium position and then released. Calculate

(a) the maximum value of the potential energy of the oscillating system, assuming it is zero at the equilibrium position.

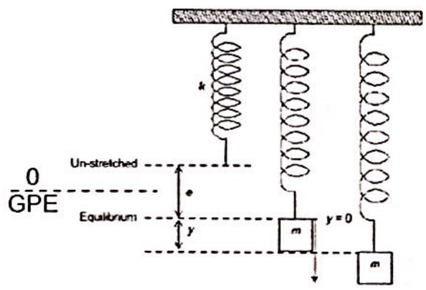
(b) the maximum velocity of the mass,

(c) the distance from the equilibrium position when the kinetic energy is one quarter of its maximum value.

(Take the zero level for gravitational potential energy to be midway between the un-stretched and equilibrium position.)

Solution

(a) Assuming potential energy of the oscillating system at equilibrium is zero.



At eqm: $ke = mg \Rightarrow e = \dfrac{mg}{k}$

Max. potential energy when mass is at its max. displacement below eqm position = work done by external force to bring mass to that position

$= \overset{\Delta EPE}{\text{increase}}$ in E.P.E. – decrease in G.P.E.

$$= \left[\frac{1}{2}k(e+0.050)^2 - \frac{1}{2}ke^2\right] - mg(0.050)$$

$$= \left[\frac{1}{2}(20)\left(\frac{0.50\times 9.81}{20}+0.050\right)^2 - \frac{1}{2}(20)\left(\frac{0.50\times 9.81}{20}\right)^2\right] - (0.50\times 9.81)(0.050)$$

$= 0.025$ J

b) max K.E. = max P.E.

$= 2.5\times 10^{-2}$ J

$\frac{1}{2}(0.50)v_{max}^2 = 2.5\times 10^{-2}$

$v_{max} = \pm\sqrt{0.10} = \pm 0.32$ m s^{-1}

c) K.E. $= 2.5\times 10^{-2} \times \frac{1}{4}$

$E_k = \frac{1}{4}(E_k)_{max}$

$\frac{1}{2}m\omega^2(y_0^2 - y^2) = \frac{1}{4}\left(\frac{1}{2}m\omega^2 y_0^2\right)$

$y_0^2 - y^2 = \frac{1}{4}y_0^2$

$y = \frac{\sqrt{3}}{2}y_0$

$= \frac{\sqrt{3}}{2}(0.050)$

$= 0.043$ m

Note: See self-check Q9 for the energy-displacement graphs

10.3 Damped and Forced Oscillations: Resonance

Damped Oscillations

The oscillatory motions that have been discussed so far have been for *ideal* systems.

A real oscillating system is opposed by dissipative forces, such as friction and viscous forces, which cause the amplitude of the motion to decrease with time. The system then does positive work: the energy to do this work is taken from the energy of the oscillation, and usually appears as internal energy of the surroundings and the system.

Definition of damping

Damping is the process whereby *energy is removed from an oscillating system.*

For a damping force which is proportional to the velocity of the mass, the decay in amplitude is *exponential.* This means that the amplitude decreases by the same fraction during each vibration. $\frac{x_1}{x_2} = \frac{x_2}{x_3} = \ldots = \text{constant}$

A full mathematical analysis of damped harmonic motion shows that the frequency of the damped motion is *less than* the undamped frequency.

Degrees of Damping

The degree of damping depends on the magnitude of the retarding force (or the amount of resistance to the oscillation). In practice, the motion of an oscillator will depend on the magnitude of the damping. In certain cases the damping may prevent the system from oscillating and it will just return to its equilibrium position.

The following graphs show how different degrees of damping affect the displacement of an oscillating body.

system↑ temperature

amplitude remains constant

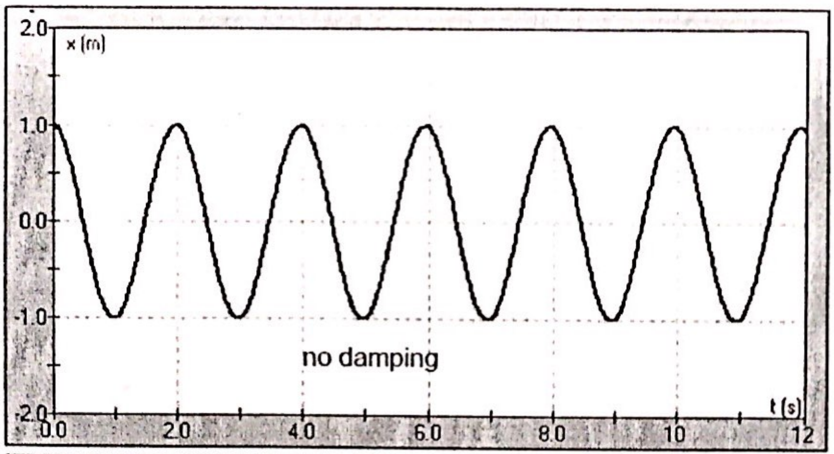


Fig. 10.3.1

amplitude decays exponentially

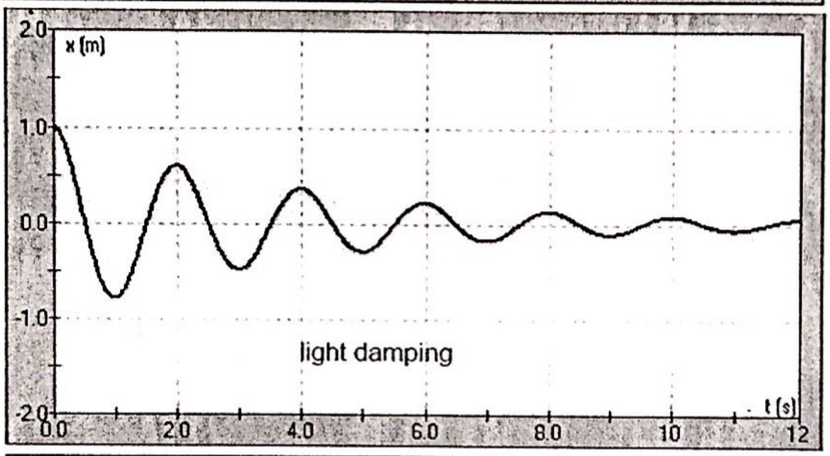


Fig. 10.3.2

no oscillation, returns to eqm in the shortest time

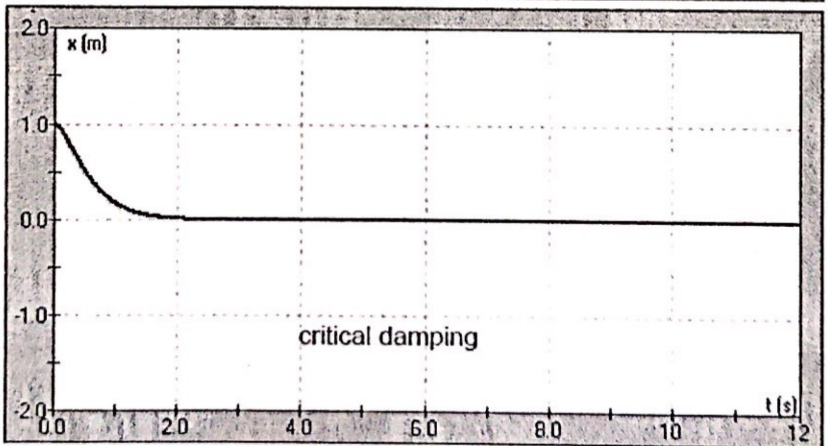


Fig. 10.3.3

no oscillation, returns to eqm after a long time

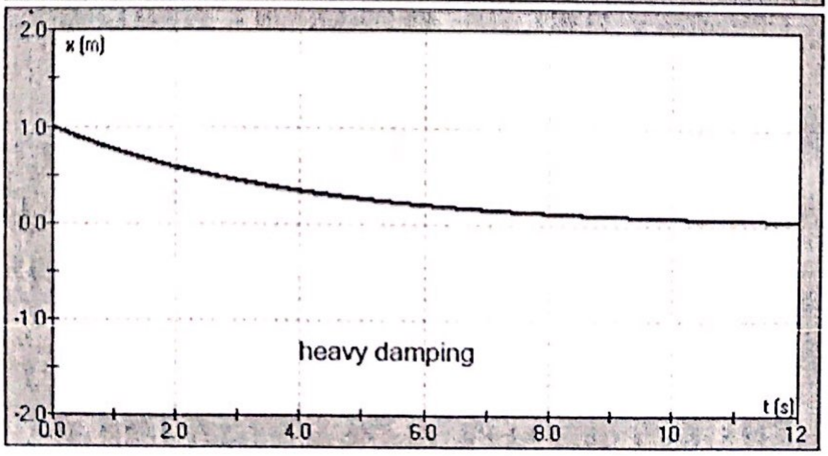


Fig. 10.3.4

Three Categories of Damping

Light Damping results in oscillations whereby the amplitude decays exponentially with time. The frequency of oscillations is slightly smaller than the undamped frequency. (Fig. 10.3.2)

Critical Damping results in no oscillation and the system returns to the equilibrium position in the shortest time. (Fig. 10.3.3)

Heavy Damping results in no oscillation and the system takes a long time to return to its equilibrium position. (Fig. 10.3.4)

Energy Variation of Damped Oscillations

Light Damping

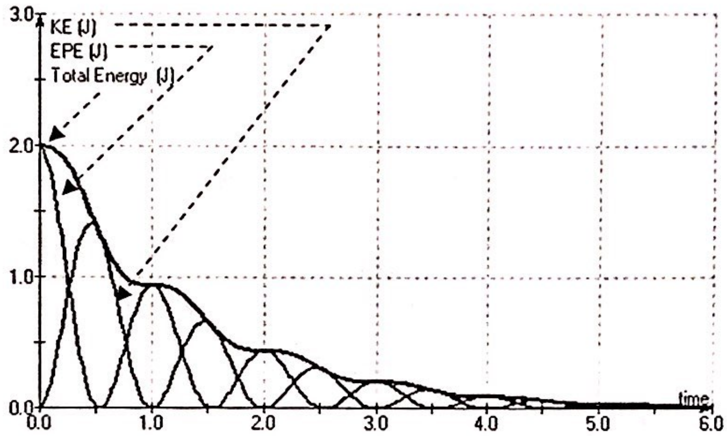


Fig. 10.3.5

Critical Damping

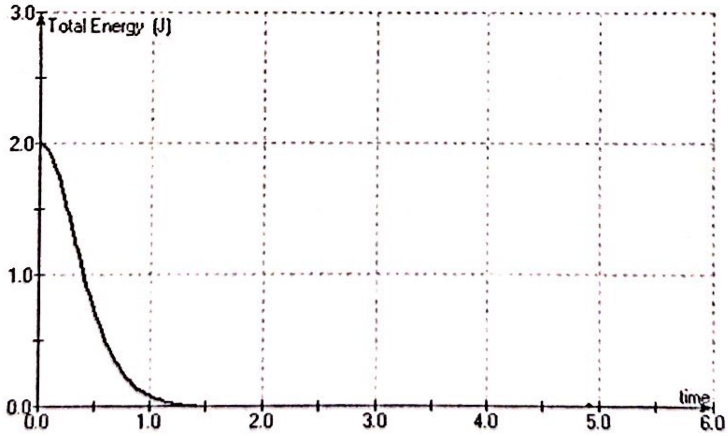


Fig. 10.3.6

Heavy Damping

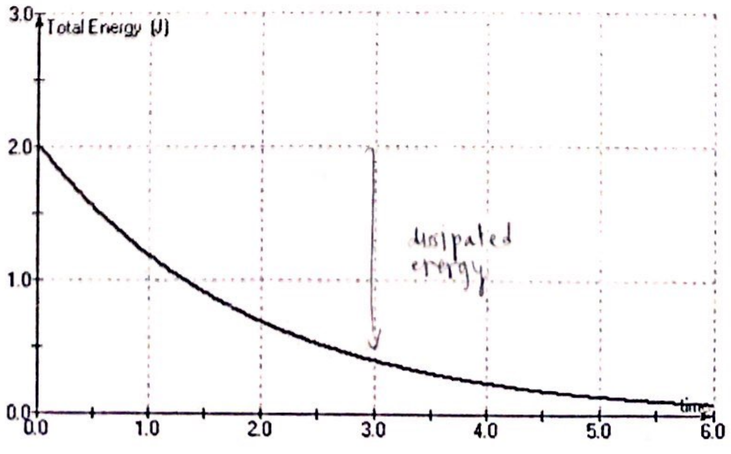


Fig. 10.3.7

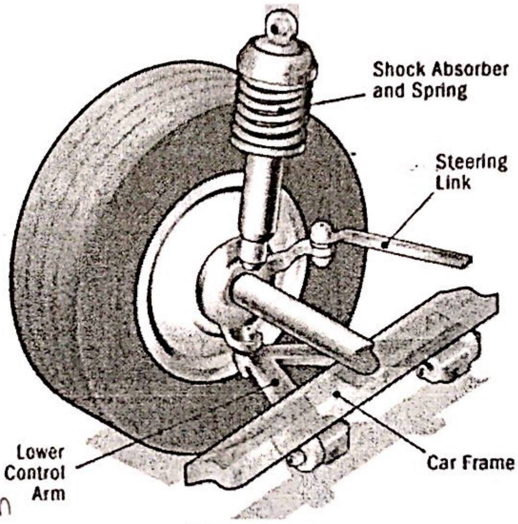
Importance of Critical Damping: Car Suspension System

The degree of damping of a mechanical system is important. Too little damping results in a large number of oscillations: too much damping leads to there being too long a time when the system cannot respond to further disturbances. This is illustrated well by the trouble which car manufacturers take with the suspension of cars. The suspension is the link between the wheels and axles of a car and the body and the passengers, and consists of a spring which is damped by a shock absorber.

Without the suspension system, the wheels' vertical motion, due to road imperfections (e.g. a bump), is transferred to the car frame, which moves upwards, and the tires can lose contact with the road completely. Then, under the downward force of gravity, the tires can slam back onto the road surface. The suspension system will absorb the energy of the vertically accelerated wheel, allowing the frame and body to ride nearly undisturbed while the wheels and tires follow the bump in the road.

(Diagram from HowStuffWorks website)

A shock absorber consists of a piston that moves in a cylinder containing a viscous fluid. Holes on the piston allow it to move up and down in a damped manner and the amount of damping is adjusted so that the suspension system is close to condition of critical damping.



car suspension system

Fig. 10.3.8

A good suspension system is one in which the damping is critical or slightly under critical as this results in a comfortable ride and also leaves the car ready to respond to further bumps in the road quickly.

Fig. 10.3.9 shows that by the time the car has reached P the shock absorbing system is ready for the drop in road surface. After Q, it is ready for another bump.

we want critical damp for shock absorber

The springing of a car suspension is critically damped when it goes over a bump; the passengers in the car quickly and smoothly regain equilibrium.

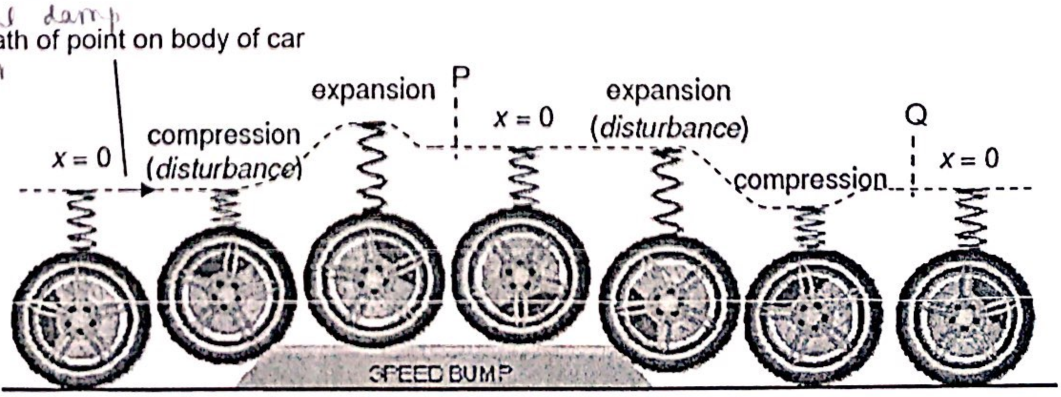


Fig. 10.3.9

other examples of critical damping: pointers in balances or electrical meters so that the pointer comes to the point[illegible]

recoil mechanisms (guns)

Without the shock absorber, a car spring, after a compression, will extend and release
the energy it absorbs from the rise of a bump at an uncontrolled rate. The spring will continue to bounce at its natural frequency until all of the energy originally put into it is used up. A suspension built on springs alone would make for an extremely bouncy and uncomfortable ride and, depending on the terrain, an uncontrollable car.

A heavily damped shock absorbing system would still have a compressed spring by the time P is reached and so would not be able to respond to the sudden drop in road surface. So long as there are bumps on a road then these must have an effect on a passenger in a car. The shock-absorbing system can only reduce the forces applied. It cannot eliminate them because, clearly, in the above diagram, the passenger must rise and drop eventually by the height of the bump.

Instruments such as analogue balances and electrical meters are also designed to be critically damped so that the pointer comes quickly to the correct position in the shortest possible time without oscillating.

Forced Oscillation and Resonance

Since all macroscopic mechanical oscillations are damped, energy is continually being lost from the system. If we wish to maintain the vibrations at constant amplitude, then energy must be supplied at the rate at which energy is being dissipated to the surroundings and within the system. A force must therefore be applied to oppose the damping forces.

Forced oscillations are produced when a body is subjected to a periodic external driving force. The force is also know as a driver.

Demonstration Using Barton's Pendulums

Fig. 10.3.10 shows a setup of Barton's pendulums. It consists of a number of very light pendulums (made from paper cones) of varying length (A, B, C, D and E) and one pendulum with a heavy bob (X). This massive pendulum is called the driver pendulum. All the pendulums are suspended from the same string.

The setup is used to demonstrate what happens when a system is made to vibrate at some frequency other than its own natural frequency of vibration.

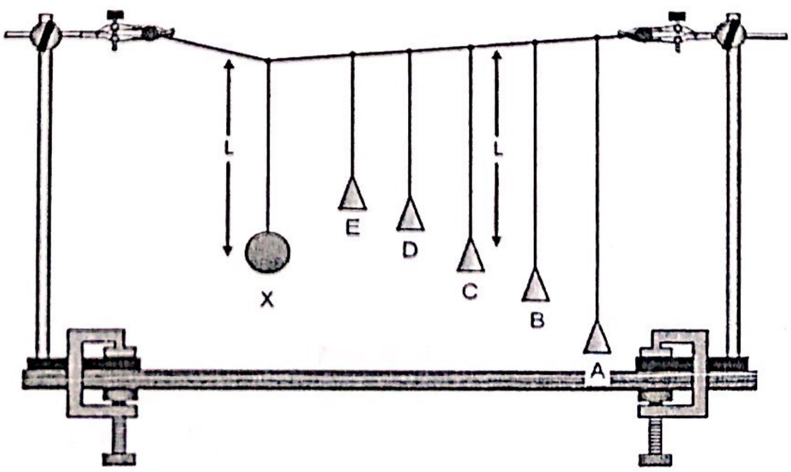


Fig. 10.3.10

Forced Oscillations

The motion of the Barton's pendulums can be divided into two distinct sections. Initially, it is very chaotic; the pendulums tend to oscillate at their own natural frequency (determined by their length) while the driving pendulum tries to make them all oscillate at its own frequency. Gradually, the driving pendulum wins and the pendulums are all forced to oscillate at a frequency which is not the same as their own natural frequency. Energy is being transferred from the driver pendulum to the driven pendulums. This is an example of *forced oscillations.*

Resonance

The Barton's pendulums experiment shows that the forced vibrations are at the maximum when the natural frequency of the driven system is equal to the frequency of the driving oscillator. Pendulum C, which has the same length and thus has the same natural frequency as pendulum X, is observed to oscillate with the largest amplitude. This is an example of *resonance*. At resonance, maximum energy is being transferred by way of the string from the driving system to the driven system.

Variation of Amplitude of a Forced Oscillation with Driving Frequency

Fig. 10.3.11 is a frequency response graph which shows how the amplitude x_o of a forced oscillation depends on the driving frequency f when the system is damped at different degrees.

① peak becomes lower, graph becomes flatter
② peak shifts left

(also represents energy)

$E_{max} = \frac{1}{2} m\omega A^2$

energy ∝ amplitude

think air vs water

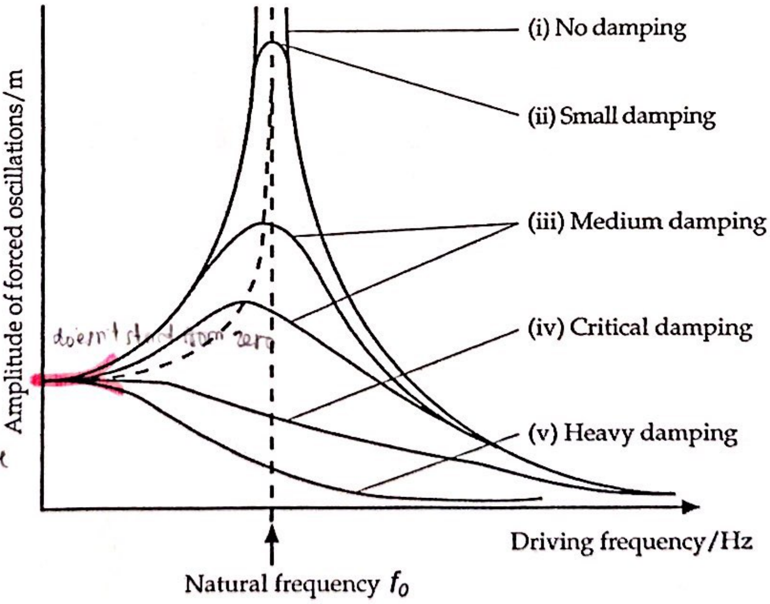


Fig. 10.3.11

For a forced oscillation, when conditions are *steady*, the following observations are made:

- The amplitude of a forced oscillation depends upon:
  1. the damping of the system,
  2. the relative values of the driving frequency f and the natural frequency f_o of the system (i.e. how far f is from f_o).
- The oscillations with largest amplitude (i.e. resonance) occur when f is approximately equal to f_o.

The sharpness of resonance is determined by the degree of damping:

1. When there is no damping, the amplitude of resonance becomes infinite. (Fig. 10.3.11(i))
2. When damping is light, the amplitude is large but falls off rapidly when the driving frequency of the body differs slightly from the natural frequency of the body. The resonance is sharp. (Fig. 10.3.11(ii))
3. When the degree of damping increases, the amplitude at resonance decreases. The curve falls off gradually and maximum amplitude occurs at a frequency that is lower than the natural frequency of the body. (Fig. 10.3.11(iii))
4. When damping is critical or heavy, the resonance is flat. (Fig. 10.3.11(iv),(v))

Definition

Resonance occurs when a system *responds at maximum amplitude to an external driving force. This occurs when the frequency of the driving force is equal to the natural frequency of the driven system.*

Circumstances in which resonance is useful

just quote no need to understand

electrical

Tuning a radio receiver (Electrical resonance)

The electrons in a radio receiving aerial are forced to vibrate by the radio wave passing the aerial. When we tune the receiver, we are making the natural frequency of the electrical circuit equal to the frequency of the signal. Hence the tuning circuitry uses resonance to isolate and amplify the signal of the required frequency.

Increasing the intensity of a note produced by a string in a musical instrument (Acoustic resonance)

This is done by coupling the vibrating string to a resonator. The air inside a cavity (e.g. a guitar body) and the material of the instrument (the thin wooden body of the guitar) all vibrate producing much greater vibrations in the surrounding air than would be produced by the string alone.

Magnetic resonance

Energy from strong oscillating magnetic fields is used to cause the nuclei of atoms to oscillate and emit radio frequency signals. In any given molecule there will be many resonant frequencies, and whenever resonance occurs energy is absorbed. The pattern of energy absorption can be used to detect the presence of particular molecules within any specimen and biochemists are using the technique to study complex molecules and the part they play in biological processes.

Magnetic resonance is also being used instead of X-rays as an imaging system (MRI) in the medical field. The radio frequency signals emitted are made to encode position information by varying the magnetic field. The contrast between different tissues is determined by the rate at which excited atoms return to the equilibrium state. One major advantage of magnetic resonance used in this way is that no ionising radiation is involved.

Circumstances in which resonance should be avoided

However, resonance is not always useful.

All mechanical structures have one or more natural frequencies, and if a structure is subjected to a strong external driving force that matches one of these frequencies, resonance is said to occur and the resulting oscillations of the structure may rupture it.

At Angers, France in 1850, a French infantry battalion was marching over a suspension bridge when it collapsed, resulting in the deaths of 220 men. Since that time, it has been common practice to order soldiers to break step when crossing a bridge. The soldiers' marching caused sufficient vibration and twisting to break the bridge.

A more modern bridge disaster occurred in 1940 when wind-induced oscillations caused the collapse of the Tacoma Narrows Bridge in the U.S. state of Washington. The bridge's natural mode of vibration coupled with the wind forces, produced unstable oscillations with increased amplitude that were beyond the strength of the suspender cables.

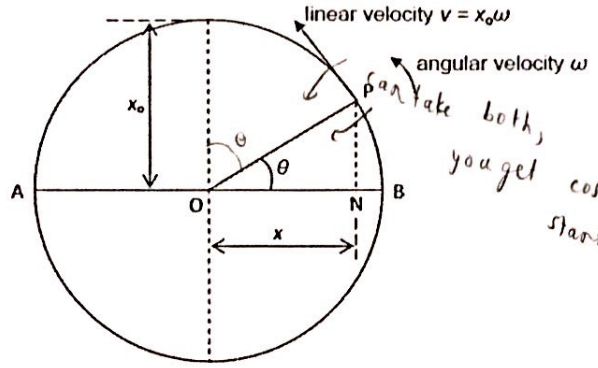
Resonance was also the cause for the collapse of some buildings during a major earthquake in Mexico in 1985. Many intermediate-height buildings collapsed because their natural frequency matched that of the seismic waves, whereas taller or shorter buildings were unaffected.

A more mundane example of resonance is the way in which the bodywork of a bus can vibrate violently at a particular engine speed.

As such, engineers have to carry out elaborate vibration tests on model structures of, for example, bridges, buildings and aeroplanes before they are satisfied that the design features will prevent extremely large amplitudes from building up in the system.

APPENDIX

Relationship between Uniform Circular Motion and S.H.M.



can take both, you get cos(θ) and sin(90° − θ)
starting position matters

Point P moves in a circle of radius x_o at a steady angular velocity ω. N is the projection of P to the diameter AOB of the circle. As P moves steadily round the circle, N moves to and fro along AOB.

The centripetal acceleration of P is $x_o\omega^2$, directed towards O.

Assume that $t = 0$ when $\theta = 0$ (i.e. $t = 0$ when $x = x_o$ or when the point N is at B). After a time t,

$$\theta = \omega t$$

$$x = x_o \cos\theta = x_o \cos\omega t$$

The acceleration of N is the component of the acceleration of P parallel to AB:

$$a = -x_o\omega^2 \cos\theta$$

The negative sign indicates that the acceleration is directed towards O.
We can write

$$\begin{aligned} a &= -x_o\omega^2 \cos\theta \\ &= -x_o\omega^2 \cos\omega t \\ &= -\omega^2 (x_o \cos\omega t) \\ &= -\omega^2 x \end{aligned}$$

Thus N is in S.H.M.

The period of N (time taken for N to go from A to B and back again) is given by

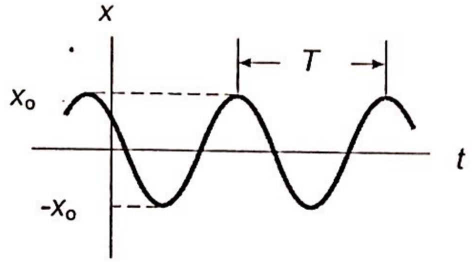
$$T = \frac{2\pi}{\omega}$$

The model shows that when a point moves in a uniform circular motion, the projection of that point to the diameter of the circle moves in S.H.M.

circular motion can be mapped to a SHM when projected perpendicularly

Phase Constant

In general, $x = x_o \cos(\omega t + \phi)$



ϕ is the *phase constant* or the initial phase angle.

The phase of the motion is the quantity $(\omega t + \phi)$.

x_0 and ϕ are determined uniquely by the position and velocity of the particle at $t = 0$. E.g. if the particle is at $x = x_0$ at $t = 0$, then $\phi = 0$.

Phase Difference

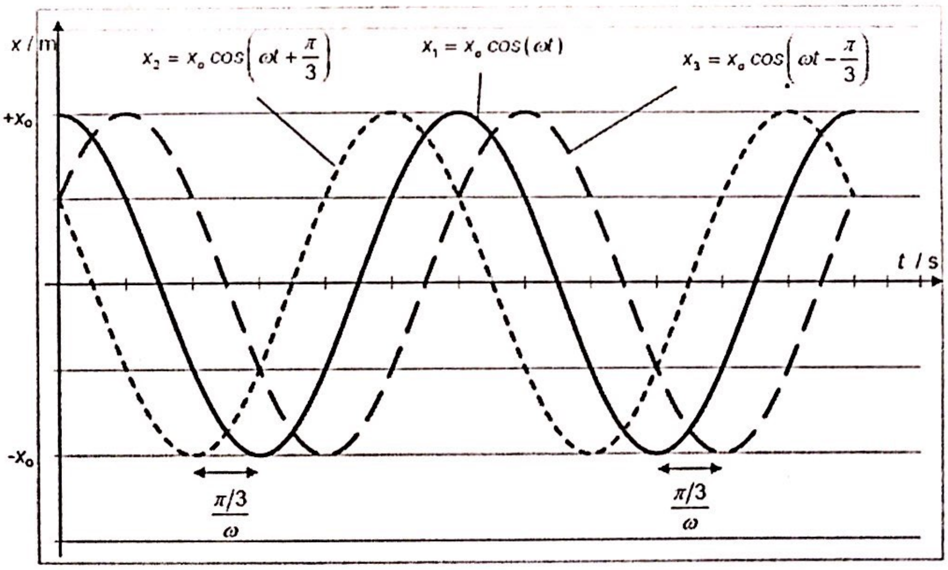
A graph of $x = x_o \cos(\omega t + \phi)$ is the graph of $x_o \cos \omega t$ displaced *to the left* by a time interval $\frac{\phi}{\omega}$.

The motion described by $x = x_o \cos(\omega t + \phi)$ is *not in phase* with that described by $x_o \cos \omega t$. It is out of phase by angle ϕ (radian) or time $\frac{\phi}{\omega}$. The plus sign indicates that this motion *leads* by time $\frac{\phi}{\omega}$ and so the graph is displaced to the left.

If the motion was described by $x = x_o \cos(\omega t - \phi)$, the graph would be displaced to the right. This motion would be said to *lag* by time $\frac{\phi}{\omega}$.

Phase difference between two oscillators is the fraction of a complete oscillation by which one is ahead of the other. It can be expressed as a fraction of an oscillation, or, more usually, as an angle, measured in radians.

Example

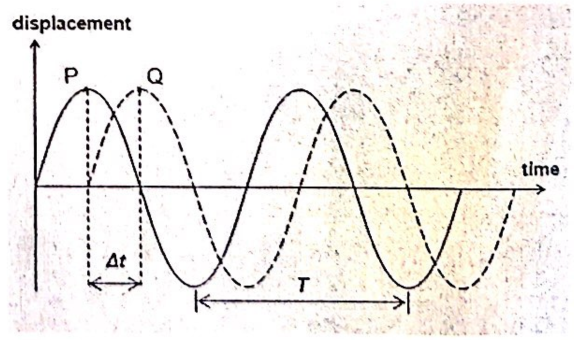


The motion described by $x_2 = x_0 \cos\left(\omega t + \frac{\pi}{3}\right)$ is not in phase with that described by $x_1 = x_o \cos \omega t$. It is out of phase by $\frac{\pi}{3}$ radian or time $\frac{\pi/3}{\omega}$. The plus sign indicates that x_2 *leads* x_1 by time $\frac{\pi/3}{\omega}$ = T/6 and so the graph is displaced to the *left*. The motion described by $x_3 = x_o \cos\left(\omega t - \frac{\pi}{3}\right)$ is the graph of x_1 displaced to the *right*. x_3 is said to *lag* x_1 by time $\frac{\pi/3}{\omega}$ = T/6.

The phase difference ϕ between two waveforms P and Q having the same period can be calculated using

$$\frac{\Delta t}{T} = \frac{\phi}{2\pi}$$

$$\Rightarrow \boxed{\phi = \frac{2\pi}{T}\Delta t}$$



Example

Refer to the graphs in Fig. 10.1.4, Fig. 10.1.5 and Fig. 10.1.6. What are the *time difference* and the *phase difference* between (i) v and x, (ii) v and a, and (iii) a and x?

Solution

(i) The time difference is $T/4$, the phase difference is $\pi/2$ rad, and v leads x.

(ii) The time difference is $T/4$, the phase difference is $\pi/2$ rad, and a leads v.

(iii) The time difference is $T/2$, the phase difference is π rad, and x and a are in 'antiphase'.