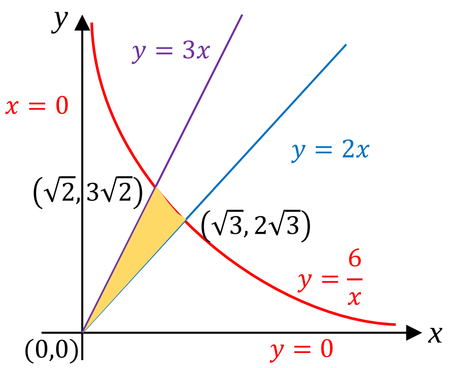
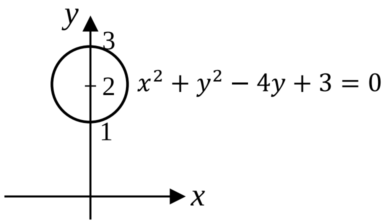
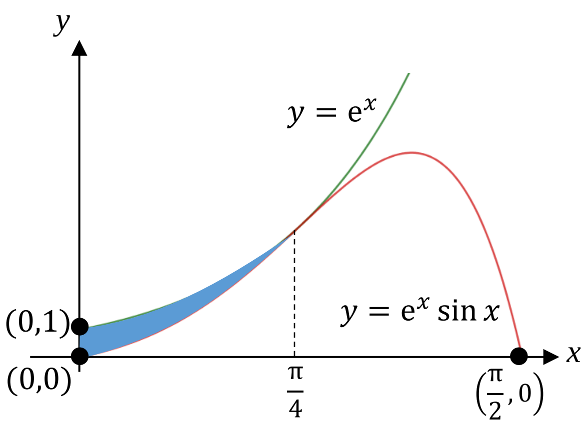
H2 Mathematics (9758)
Chapter 11 Definite Integrals
Extra Practice Solutions

Qn 1	**2009/VJC Prelim/1/10**
	$\int xe^{2x}\ \mathrm{d}x = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\ \mathrm{d}x$ $= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$ $= \frac{e^{2x}}{4}(2x-1) + c$ (shown)
(a)	Required area $= \int_0^4 \left(\frac{1}{2}e^4 x - e^x\sqrt{x}\right)\mathrm{d}x + \int_4^8 \left(e^x\sqrt{x} - \frac{1}{2}e^4 x\right)\mathrm{d}x$ $= 7239.2$ units^2 (1 dp)
(b)	Required Volume $= \frac{1}{3}\pi\left(2e^4\right)(4) - \int_0^4 \pi\left(e^{2x}x\right)\mathrm{d}x$ $= \frac{16\pi e^8}{3} - \pi\left[\frac{e^{2x}}{4}(2x-1)\right]_0^4$ $= \frac{16\pi e^8}{3} - \pi\left(\frac{e^8}{4}\right)(7) - \frac{1}{4}\pi$ $= \pi e^8\left(\frac{43}{12}\right) - \frac{\pi}{4}$ units^3 $A = \frac{43}{12}$ and $B = \frac{1}{4}$

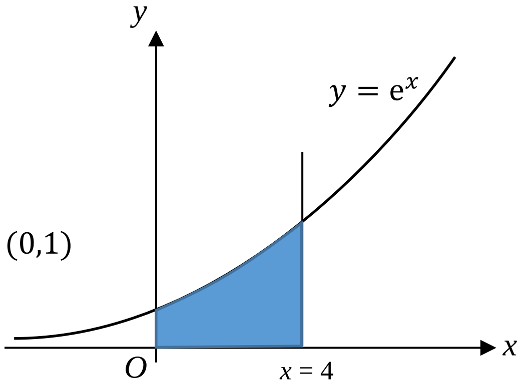
Qn 2	2011/TJC Prelim/1/10
	 Area $=\frac{1}{2}\left(\sqrt{2}\right)\left(3\sqrt{2}\right)+\int_{\sqrt{2}}^{\sqrt{3}}\frac{6}{x}\,\mathrm{d}x-\frac{1}{2}\left(\sqrt{3}\right)\left(2\sqrt{3}\right)$ $=\left[6\ln x\right]_{\sqrt{2}}^{\sqrt{3}}$ $=3\left(\ln 3-\ln 2\right)$ units^2
	 $x^2+y^2-4y+3=0 \Rightarrow x^2+\left(y-2\right)^2=1$ Volume $=2\int_0^1\left(\pi {y_1}^2-\pi {y_2}^2\right)\mathrm{d}x$ where $y_1=2+\sqrt{1-x^2}$ and $y_2=2-\sqrt{1-x^2}$ $=2\pi\int_0^1\left[\left(2+\sqrt{1-x^2}\right)^2-\left(2-\sqrt{1-x^2}\right)^2\right]\mathrm{d}x$ ≈ 39.5 units^3 (3 s.f.)

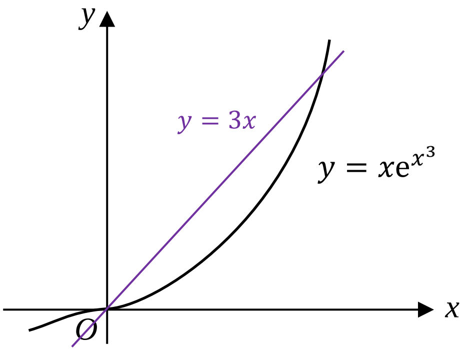
Qn3	2013/VJC Prelim/1/10
(i)	$\int_0^{\frac{1}{\sqrt{2}}} \cos^{-1} x \, dx = \left[x\cos^{-1} x \right]_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} \left(-\frac{x}{\sqrt{1-x^2}} \right) dx$ $= \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) - \left[\sqrt{1-x^2}\right]_0^{\frac{1}{\sqrt{2}}}$ $= \frac{\pi}{4\sqrt{2}} - \left(\sqrt{\frac{1}{2}} - 1\right)$ $= \frac{\pi}{4\sqrt{2}} + 1 - \frac{1}{\sqrt{2}}$
(ii)	$\text{Volume} = \pi\int_0^{\frac{1}{\sqrt{2}}} y^2 \, dx - \pi\left(\frac{\sqrt{\pi}}{2}\right)^2\left(\frac{1}{\sqrt{2}}\right)$ $= \pi\int_0^{\frac{1}{\sqrt{2}}} \cos^{-1} x \, dx - \frac{\pi^2}{4\sqrt{2}}$ $= \frac{\pi^2}{4\sqrt{2}} + \pi - \frac{\pi}{\sqrt{2}} - \frac{\pi^2}{4\sqrt{2}}$ $= \pi - \frac{\pi}{\sqrt{2}} \text{ unit}^3$
(iii)	Required Equation: $y = \sqrt{\cos^{-1} x} - \frac{\sqrt{\pi}}{2}$. $\text{Volume} = \pi\int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{\cos^{-1} x} - \frac{\sqrt{\pi}}{2}\right)^2 dx$ ≈ 0.116 (3 s.f.)

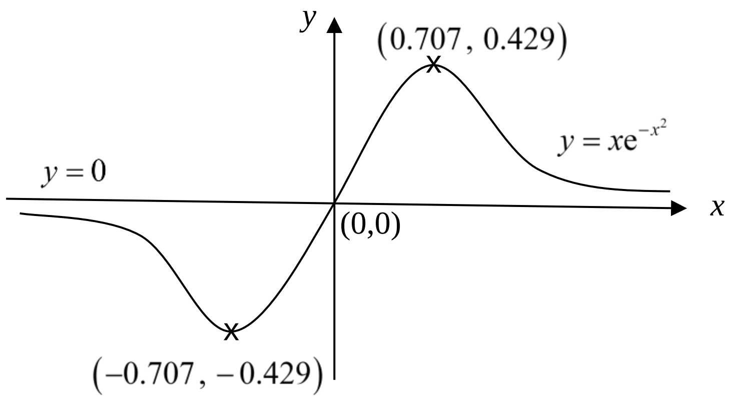
Qn 4	2013/TJC Prelim/1/9
(i)	$\int e^{2x}\cos 4x \, dx = \frac{1}{2}e^{2x}\cos 4x - \frac{1}{2}\int e^{2x}(-4\sin 4x) \, dx$ $= \frac{1}{2}e^{2x}\cos 4x + 2\int e^{2x}\sin 4x \, dx$ $= \frac{1}{2}e^{2x}\cos 4x + 2\left(\frac{1}{2}e^{2x}\sin 4x - \frac{1}{2}\int e^{2x}(4\cos 4x) \, dx\right)$ $= \frac{1}{2}e^{2x}\cos 4x + e^{2x}\sin 4x - 4\int e^{2x}\cos 4x \, dx$ $5\int e^{2x}\cos 4x \, dx = \frac{1}{2}e^{2x}(\cos 4x + 2\sin 4x) + C$ $\int e^{2x}\cos 4x \, dx = \frac{1}{10}e^{2x}(\cos 4x + 2\sin 4x) + C$

(ii)	At the point of intersection, $e^x \sin 2x = e^x \Rightarrow e^x(\sin 2x - 1) = 0$ either $e^x = 0$ or $\sin 2x = 1$ (reject, as $e^x > 0$) or $2x = \frac{\pi}{2}$ $x = \frac{\pi}{4}$
(iii)	 Volume of solid generated $= \pi\int_0^{\frac{\pi}{4}} \left(e^x\right)^2 \, dx - \pi\int_0^{\frac{\pi}{4}} \left(e^x \sin 2x\right)^2 \, dx$ $= \pi\int_0^{\frac{\pi}{4}} e^{2x} \, dx - \pi\int_0^{\frac{\pi}{4}} e^{2x} \sin^2 2x \, dx$ $= \pi\int_0^{\frac{\pi}{4}} e^{2x} \, dx - \pi\int_0^{\frac{\pi}{4}} e^{2x} \left(\frac{1-\cos 4x}{2}\right) dx$ $= \frac{1}{2}\pi\int_0^{\frac{\pi}{4}} \left(e^{2x} + e^{2x}\cos 4x\right) dx$ $= \frac{1}{2}\pi\left(\left[\frac{1}{2}e^{2x}\right]_0^{\frac{\pi}{4}} + \left[\frac{1}{10}e^{2x}\left(\cos 4x + 2\sin 4x\right)\right]_0^{\frac{\pi}{4}}\right)$ $= \frac{1}{2}\pi\left(\frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{1}{10}\left(-e^{\frac{\pi}{2}} - e^0\right)\right)$ using **(i)** answer $= \frac{1}{10}\pi\left(2e^{\frac{\pi}{2}} - 3\right)$ units^3

Qn 5	**2017/MJC Promo/4a**
	$\int_0^3 \left\| x^3-5x^2+4x \right\| \text{ d}x = \int_0^1 x^3-5x^2+4x \text{ d}x - \int_1^3 x^3-5x^2+4x \text{ d}x$ $= \left[\frac{x^4}{4}-\frac{5x^3}{3}+\frac{4x^2}{2}\right]_0^1 - \left[\frac{x^4}{4}-\frac{5x^3}{3}+\frac{4x^2}{2}\right]_1^3$ $= \frac{7}{12} - \left[-\frac{27}{4}-\frac{7}{12}\right]$ $= \frac{95}{12}$

Qn 6	**N2018/RVHS Promo/9**
(i)	$\int (\ln x)^2 \text{ d}x$ Let $u=(\ln x)^2 \Rightarrow \frac{\text{d}u}{\text{d}x}=\frac{2(\ln x)}{x}$ $\qquad v=1 \Rightarrow \int v \text{ d}x = x$ $\int (\ln x)^2 \text{ d}x = x(\ln x)^2 - \int \frac{2\ln x}{x} x \text{ d}x$ $= x(\ln x)^2 - 2\int \ln x \text{ d}x$ Let $u=\ln x \Rightarrow \frac{\text{d}u}{\text{d}x}=\frac{1}{x}$ $\qquad v=1 \Rightarrow \int v \text{ d}x = x$ $\int (\ln x)^2 \text{ d}x = x(\ln x)^2 - 2\int \ln x \text{ d}x$ $= x(\ln x)^2 - 2\left[x\ln x - \int \frac{1}{x} x \text{ d}x\right]$ $= x(\ln x)^2 - 2x\ln x + 2x + C$
(ii)	Volume $= \pi(4)^2\left(\text{e}^4\right) - \pi\int_1^{\text{e}^4} x^2 \text{ d}y$ $= 16\pi\text{e}^4 - \pi\int_1^{\text{e}^4} (\ln y)^2 \text{ d}y$ $= 16\pi\text{e}^4 - \pi\left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^{\text{e}^4}$ $= 16\pi\text{e}^4 - \pi\left[\left(\text{e}^4\cdot 4^2 - 8\text{e}^4 + 2\text{e}^4\right) - 2\right]$ $= 16\pi\text{e}^4 - \pi\left[10\text{e}^4 - 2\right]$ $= 6\pi\text{e}^4 + 2\pi$ units^3 

Qn7	**2018/YJC Promo/11**
	 $3x = xe^{x^3}$ From GC, $x = 1.03185$, $y = 3.09554$ Volume $= \frac{1}{3}\pi(3.09554)^2(1.03186) - \pi\int_0^{1.03185}\left(xe^{x^3}\right)^2 \mathrm{d}x$ $= 6.17$

Qn 8	**N2009/P1/11**
(i)	
(ii)	$\mathrm{f}(x) = x\mathrm{e}^{-x^2}$ $\mathrm{f}'(x) = \mathrm{e}^{-x^2} + x\mathrm{e}^{-x^2}(-2x) = \mathrm{e}^{-x^2} - 2x^2\mathrm{e}^{-x^2}$ $\mathrm{e}^{-x^2} - 2x^2\mathrm{e}^{-x^2} = 0$ $\mathrm{e}^{-x^2}\left(1-2x^2\right) = 0$ $\mathrm{e}^{-x^2} = 0$ or $x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ (N.A $\because \mathrm{e}^{-x^2} > 0$) When $x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}\mathrm{e}^{-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2\mathrm{e}}}$ When $x = -\frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}\mathrm{e}^{-\left(-\frac{1}{\sqrt{2}}\right)^2} = -\frac{1}{\sqrt{2\mathrm{e}}}$ Coordinates of the turning points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2\mathrm{e}}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2\mathrm{e}}}\right)$
(iii)	$\int_0^n \mathrm{f}(x)\,\mathrm{d}x = \int_0^n x\mathrm{e}^{-x^2}\,\mathrm{d}x$ $= \int_0^{n^2} \sqrt{u}\mathrm{e}^{-u}\frac{1}{2\sqrt{u}}\,\mathrm{d}u$ $= \frac{1}{2}\int_0^{n^2} \mathrm{e}^{-u}\,\mathrm{d}u$ $= \frac{1}{2}\left[-\mathrm{e}^{-u}\right]_0^{n^2}$ $= \frac{1}{2}\left(-\mathrm{e}^{-n^2} + \mathrm{e}^0\right)$ $= \frac{1}{2}(1-\mathrm{e}^{-n^2})$ $u = x^2 \Rightarrow x = \sqrt{u} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2\sqrt{u}}$ When $x = 0 \Rightarrow u = 0$ When $x = n \Rightarrow u = n^2$ Area of region between C and the positive x-axis $= \lim_{n\to\infty}\int_0^n \mathrm{f}(x)\,\mathrm{d}x = \frac{1}{2}(1-0) = 0.5 \text{ units}^2$

(iv)	$\int_{-2}^{2}\lvert f(x)\rvert\,dx = 2\int_{0}^{2} f(x)\,dx$ (by symmetry) $= 2\times\frac{1}{2}(1-e^{-2^2})$ $= 1-e^{-4}$
(v)	Vol of revolution $= \pi\int_{0}^{1}\left(f(x)\right)^2\,dx$ $= \pi\int_{0}^{1}\left(xe^{-x^2}\right)^2\,dx$ $= 0.11570\pi$ $= 0.36349$ ≈ 0.363 units^3

Qn 9	**2016/RVHS/Promo/11**
(a)	$\int \frac{x^2}{(x^2+9)^2}\,dx,$ $x = 3\tan\theta \Rightarrow \frac{dx}{d\theta} = 3\sec^2\theta,$ $= \int \frac{9\tan^2\theta}{(9\tan^2\theta+9)^2} 3\sec^2\theta\ d\theta$ $= \int \frac{9\tan^2\theta}{81\sec^4\theta} 3\sec^2\theta\ d\theta$ $= \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\ d\theta$ $= \frac{1}{3}\int \sin^2\theta\ d\theta$ $= \frac{1}{6}\int (1-\cos 2\theta)\ d\theta$ $= \frac{1}{6}\left[\theta - \frac{\sin 2\theta}{2}\right] + c$ $= \frac{1}{6}\theta - \frac{\sin 2\theta}{12} + c$ $= \frac{1}{6}\theta - \frac{2\sin\theta\cos\theta}{12} + c$ $= \frac{1}{6}\tan^{-1}\frac{x}{3} - \frac{1}{12}(2)\frac{x}{\sqrt{x^2+9}}\frac{3}{\sqrt{x^2+9}} + c$ $= \frac{1}{6}\tan^{-1}\frac{x}{3} - \frac{x}{2(x^2+9)} + c$ $x = 3\tan\theta$ $\Rightarrow \tan\theta = \frac{x}{3}$ $\Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+9}}$ $\Rightarrow \cos\theta = \frac{3}{\sqrt{x^2+9}}$

(b)(i)	When $a = 3$, Volume generated $= 2\times\pi\int_0^3 \left(\frac{x}{x^2+9}\right)^2 \mathrm{d}x$ $= 2\pi\left[\frac{1}{6}\tan^{-1}\frac{x}{3}-\frac{x}{2(x^2+9)}\right]_0^3$ $= 2\pi\left[\frac{1}{6}\cdot\frac{\pi}{4}-\frac{3}{36}\right]$ $= \frac{1}{12}\pi^2-\frac{1}{6}\pi \quad \text{units}^3$
(ii)	In the actual hourglass, the neck connecting the two glass bulbs constitutes to a volume which is not accounted for in the theoretical working.
(iii)	$2\times\pi\int_0^a \left(\frac{x}{x^2+9}\right)^2 \mathrm{d}x$ $= 2\times(\frac{1}{12}\pi^2-\frac{1}{6}\pi)$ $\int_0^a \left(\frac{x}{x^2+9}\right)^2 \mathrm{d}x = \frac{1}{12}\pi-\frac{1}{6}$ $\left[\frac{1}{6}\tan^{-1}\frac{x}{3}-\frac{x}{2(x^2+9)}\right]_o^a = \frac{1}{12}\pi-\frac{1}{6}$ $\frac{1}{6}\tan^{-1}\frac{a}{3}-\frac{a}{2(a^2+9)}-\frac{1}{12}\pi+\frac{1}{6}=0$ Using GC, $a = 4.8600148 \approx 4.86$

Qn 10	
(a)	Using GC, intersection between $(y-2)^2 = x+1$ and $y+2x=6$ occur when $x=3$ or $x=\frac{5}{4}$. Also, $(y-2)^2 = x+1 \Rightarrow y = 2 \pm \sqrt{x+1}$ Volume generated $= \pi\int_{-1}^{\frac{5}{4}}\left(2+\sqrt{x+1}\right)^2 \,dx + \pi\int_{\frac{5}{4}}^{3}(6-2x)^2 \,dx - \pi\int_{-1}^{3}\left(2-\sqrt{x+1}\right)^2 \,dx$ $= 78.5725 = 78.6 \quad (3 \text{ s.f.})$
(b)	Points of intersection of curves are (−5, 9) and (0, 4). Volume $= \pi\int_0^9\left(-2-\sqrt{y}\right)^2 \,dy - \pi\int_0^4\left(-2+\sqrt{y}\right)^2 \,dy - \pi\int_4^9\left(\frac{16-y^2}{13}\right)^2 \,dy$ $= 466.52 - 8.3775 - 107.66$ $= 350.48$ $= 350$ (3 s.f)