National Junior College
2016 – 2017 H2 Mathematics

Revision: Exponential, Logarithmic and Modulus Functions and their Graphs

Key Questions to Answer:

- What is an exponential function?
- What is the range of values for which an exponential function is well-defined
- What laws do exponential functions follow?
- What does the graph of an exponential function look like?
- What are the key characteristics of the graph of an exponential function?
- What is a logarithmic function?
- What is the range of values for which a logarithmic function is well-defined?
- What laws do logarithmic functions follow?
- What does the graph of a logarithmic function look like?
- What are the key characteristics of the graph of a logarithmic function?
- What is the modulus function?
- When do we use the modulus function?
- How do I manipulate the modulus function?
- How can I draw a graph of a modulus function?

§1 Exponential Functions

Definition 1.0.1 (Exponential Function)

A function $y = a^x$, $a > 0$, $a \neq 1$, is known as an **exponential function**. It is a function used to model a relationship for which a constant increase in the independent variable (here denoted by *x*) gives the same proportional change in the dependent variable (here denoted by *y*).

The most common exponential function is $y = \mathrm{e}^x$.

UNDERSTAND
What is meant by 'increasing exponentially'?

1.1 Laws of Indices

If $a, b, m, n \in \mathbb{R}, a \neq 0, b \neq 0$

(i) $a^m \times a^n = a^{m+n}$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $\left(a^m\right)^n = a^{mn}$

(iv) $a^m \times b^m = \left(ab\right)^m$

(v) $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$

(vi) $\dfrac{1}{a^n} = a^{-n} \qquad , a \neq 0$

(vii) $a^0 = 1 \quad , \quad a \neq 0$

(viii) $\sqrt[n]{a} \ = \ a^{1/n}$

(ix) $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} \ = a^{m/n}$

(x) $a^x = a^n \Rightarrow x = n,$ where $a \neq -1, 0, 1$

WONDER
Why is that for (x), $a \neq -1, 0, 1$?

Exercise 1

Without the use of a calculator, simplify the following expressions:

(a) $9^{\frac{1}{3}} \times 9^{\frac{1}{6}}$

(b) $\dfrac{4^{2-n} \times 2^{n+1}}{\sqrt{2^n}}$

(c) $8^{\frac{1}{2}} \times 2^{0.5}$

(d) $12^3 \div 6^3$

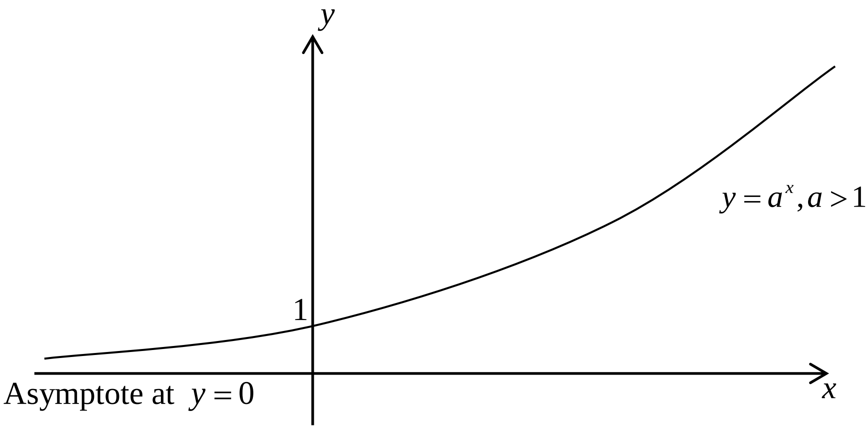
(a) $9^{\frac{1}{3}} \times 9^{\frac{1}{6}} = 9^{\frac{1}{2}} = 3$

(b)

(c) $8^{\frac{1}{2}} \times 2^{0.5} = 2^{\frac{3}{2}+\frac{1}{2}} = 2^2 = 4$

(d)

1.2 Graph of the Exponential Function



Key features of the graph of the exponential function:

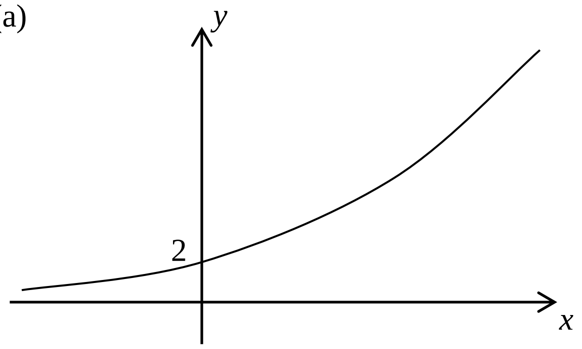
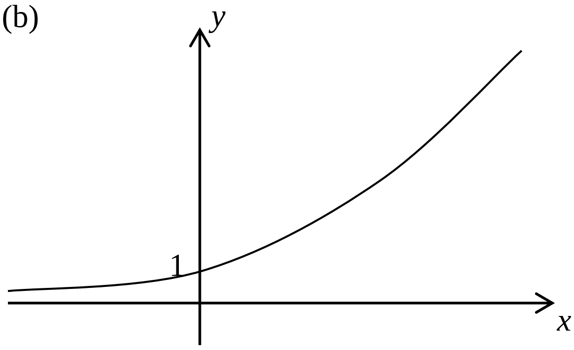
Axial Intercepts	$(0,1)$
Asymptotes	$y = 0$

Exercise 2

Identify the key features of the graphs of the following exponential functions and sketch them. (Hint: use your graphic calculator)

(a) $y = 2a^x,\ a > 1$ (b) $y = 3^{2x}$

(c) $y = e^{-x} - 1$ (d) $y = e^{2x+1} + 1$

(a) 	(b) 
(c)	(d)

WONDER

How should the graph of $y = a^x$ look like if $0 < a < 1$?

§2 Logarithmic Functions

Definition 2.0.1 (Logarithmic Function)

A function of the form $y = \log_a x$, where $a \in \mathbb{R}$, $a > 0$, $a \neq 1$ and $x > 0$.

$\log_a x$ is read as 'the logarithm of *x* to base *a*' or more simply, 'log, base *a*, *x*'.

Of special importance will be those with base e, i.e. those of the form $y = \log_e x = \ln x$.

WONDER
What are some practical uses for logarithmic functions?

2.1 Laws of Logarithms

If $a, b, c, x, y \in \mathbb{R}^+$ and $r \in \mathbb{R}$,

(i) $\log_a xy = \log_a x + \log_a y$

(ii) $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

(iii) $\log_a x^r = r \log_a x$

(iv) $\log_a b = \dfrac{\log_c b}{\log_c a}$

(v) $\log_a a = 1 \Leftrightarrow a^1 = a$

(vi) $\log_a 1 = 0 \Leftrightarrow a^0 = 1$

Take note that (i) is *not* equivalent to saying $\log_a (x+y) = \log_a x + \log_a y$,

In fact, $\log_a (x+y) \neq \log_a x + \log_a y$.

Exercise 3

Simplify and express the each of the following as a single logarithm.

(a) $2\log_x 5 - 3\log_x 2 + \log_x 4$

(b) $2\lg(x+2) + \lg(x+1) - \lg(x^2+3x+2)$

(c) $3 - 2\lg 5$

(d) $3\log_a 2 - 4 + \log_a a^3$

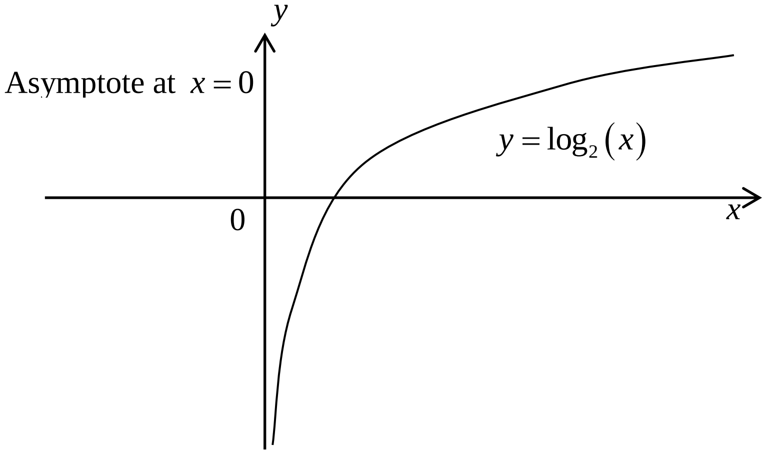
(a) $2\log_x 5 - 3\log_x 2 + \log_x 4 = \log_x\left(\frac{25}{8}\times 4\right) = \log_x\left(\frac{25}{2}\right)$

(b)

(c) $3 - 2\lg 5 = 3\lg 10 - \lg 25 = \lg\frac{10^3}{25} = \lg\frac{1000}{25} = \lg 40$

(d)

2.2 Graph of the Logarithmic Function



Key features of the graph of the logarithmic function:

Axial Intercepts $(1,0)$

Asymptotes $x=0$

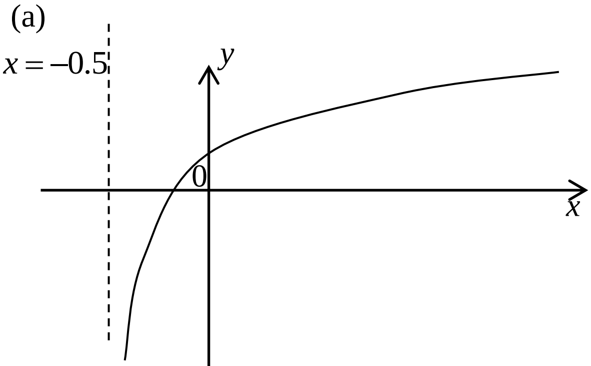
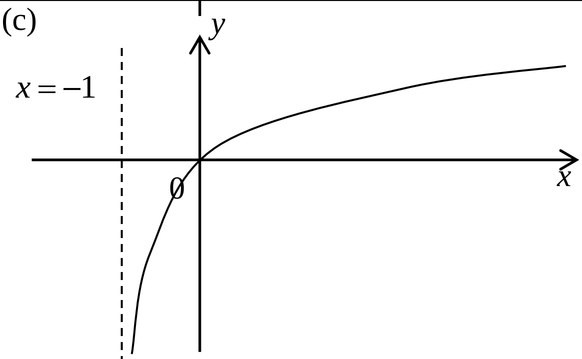
For $y=\log_a(bx+c)$, graph does not exist for $bx+c \le 0$

Exercise 4

Identify the key features of the graphs of the following logarithmic functions and sketch them:

(a) $y=\ln(2x+1)$ (b) $y=2\ln(-x+1)$

(c) $y=\log_2(x+1)$ (d) $y=-\ln(2x+3)+1$

(a) 	(b)
(c) 	(d)

WONDER

What do you observe about the graphs in Exercise 4 as compared to that in the graph at the top of the page?

§3 Modulus Functions

Definition 3.0.1 (Modulus Function)

The **absolute value** or **modulus** of a real number x is denoted by $|x|$.
Formally,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

UNDERSTAND
What are some alternative interpretations of the modulus function?

WONDER
The modulus function is an example of a piece-wise function. Can you think of any other piece-wise functions?

3.1 Properties of the Modulus Function

For all $x, y \in \mathbb{R}$,

(i) $|x| \geq 0$,

(ii) $|xy| = |x||y|$. Hence,

(a) $|-x| = |-1||x| = |x|$

(b) $|x^n| = |\underbrace{x \cdot x \cdots x}_{n \text{ times}}| = \underbrace{|x| \cdot |x| \cdots |x|}_{n \text{ times}} = |x|^n$ for any positive integer n,

(iii) $|x| = |y| \Leftrightarrow x = y$ or $x = -y$

(iv) For a general function f(x), $|\mathrm{f}(x)| = \begin{cases} \mathrm{f}(x) & \text{if} \quad \mathrm{f}(x) \geq 0 \\ -\mathrm{f}(x) & \text{if} \quad \mathrm{f}(x) < 0 \end{cases}$

CHECK

Is the following correct? $|\mathrm{f}(x)| = \begin{cases} \mathrm{f}(x) & \text{if} \quad x \geq 0 \\ -\mathrm{f}(x) & \text{if} \quad x < 0 \end{cases}$

In general, $|x| = k \Rightarrow x = k,$ or $x = -k,$ where $k \geq 0$, and $|a| = |b| \Rightarrow a = b$ or $a = -b$.

EXPLORE
How can we apply the definition of the modulus function to solve inequalities involving the modulus function?

Exercise 5

Solve the following equations.

(a) $|x - 6| = 7$ (b) $|x^2 - 5x - 1| = 5$ (c) $|x^2 - 5x + 1| = -5$

Solution:

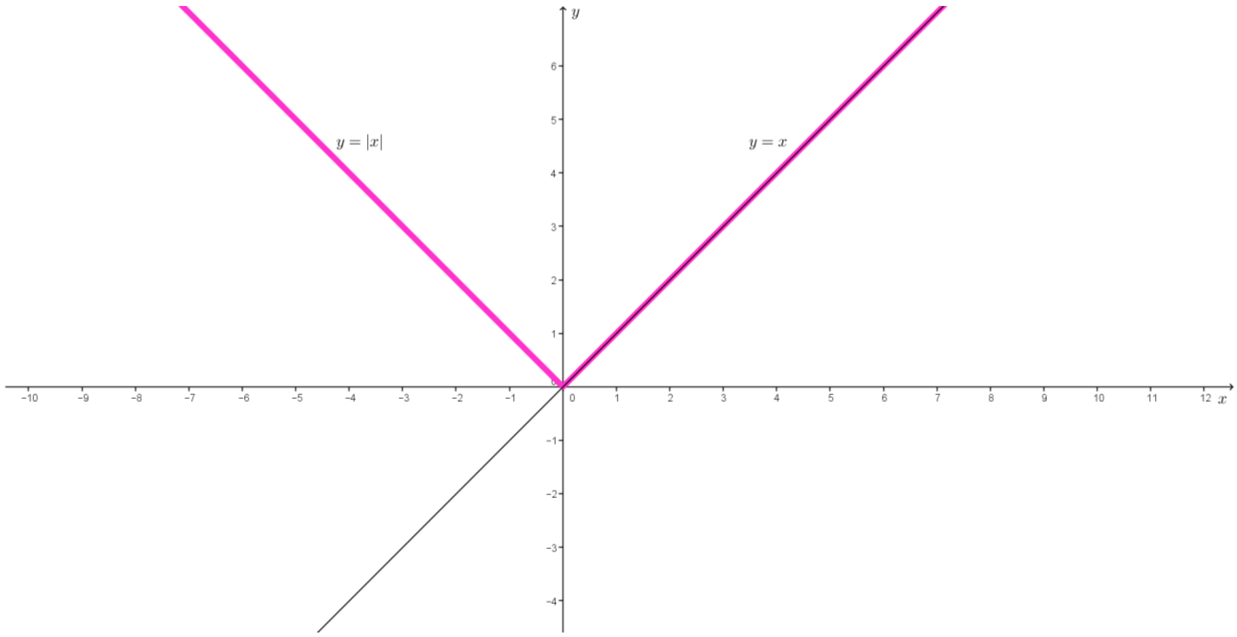
(a) $|x - 6| = 7$ $\Rightarrow$ $x - 6 = 7$ or $x - 6 = -7$

$\therefore$ $x = 13$ or $x = -1$

(b)

(c)

3.2 Graph of the Modulus Function ($y = |f(x)|$)



Notice that when $x \geq 0$, the graph of $y = |x|$ is the same as that of $y = x$, and when $x < 0$, the graph of $y = |x|$ is the same as that of $y = -x$, which agrees with the definition of $|x|$.

In general, for any curve $y = \text{f}(x)$,

(i) the curve $y = -\text{f}(x)$ is a reflection of $y = \text{f}(x)$ about the x-axis.
(ii) the curve $y = |\text{f}(x)|$ is obtained by keeping the part of the graph of $y = \text{f}(x)$ that is above the x-axis, and reflecting the part of the graph of $y = \text{f}(x)$ below the x–axis about the x–axis.

Exercise 6

Sketch the graph of $y = |\text{f}(x)|$ for the following graph of $y = \text{f}(x)$.