Register No. | Class

Name: marking scheme

‘Perseverance Yields Success’

Ping Yi Secondary School
Preliminary Examination 2021

Sec 4 Express
Additional Mathematics (Paper 1)

4049 / 01

2 hours 15 minutes

FOR EXAMINER’S USE	
TOTAL	90

INSTRUCTIONS TO CANDIDATES

Do not open this booklet until you are told to do so.

Write your name, class and register number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** questions.
Write your answers and working in the spaces provided.
Calculators may be used in this paper.
All workings must be clearly shown. Omission of essential working may result in loss of marks.
For π, use the calculator value or 3.142, unless the question requires the answer in terms of π.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

The intended marks for the question are given in the brackets [ ] at the end of each question or part question.

Expected Grade	☐ A1	☐ A2	☐ B3	☐ B4	☐ C5	☐ C6
Teacher’s Comment						
Student’s Comment						
Parent’s Comment and Signature						

This document consists of 16 printed pages including the cover page

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

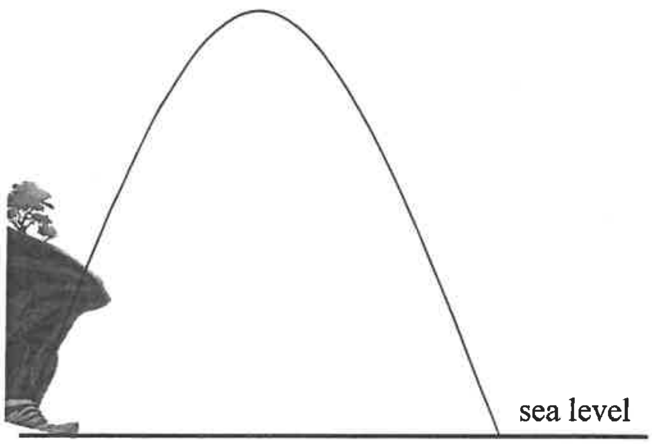
$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\Delta = \frac{1}{2}ab\sin C$$

Answer **all** questions.

1 A ball is thrown from a cliff overlooking the sea. The vertical height of the ball above sea level, h metres, is given by $h = -8t^2 + 36t + 20$, where t is the time in seconds after the ball is thrown.



(i) Find the height of the cliff. [1]

when $t=0$, $h = 20\text{ m}$. [A1]

(ii) By expressing h in its completed square form, determine whether the ball can reach a height of 65 metres above sea level. [3]

$h = -8t^2 + 36t + 20$

$= -8\left[t^2 - \frac{9}{2}t + \left(\frac{9}{4}\right)^2\right] + 20 + 8\left(\frac{9}{4}\right)^2$

$= -8\left(t - \frac{9}{4}\right)^2 + 60.5$ [M1]

since $\left(t-\frac{9}{4}\right)^2 \geq 0$ $\therefore$ $-8\left(t-\frac{9}{4}\right)^2 \leq 0$

$-8\left(t-\frac{9}{4}\right)^2 + 60.5 \leq 60.5$

$h \leq 60.5$ [M1]

The maximum height of the ball is 60.5 m.

$\therefore$ It cannot reach 65 m above sea level. [A1]

2 Without using a calculator, find the value of 10^x, given that $8^x \times 25^x = 2^{2x+4} \times 5^{x-2}$. [4]

$$8^x \times 25^x = 2^{2x+4} \times 5^{x-2}$$

$$2^{3x} \times 5^{2x} = 2^{2x} \cdot 2^4 \times \frac{5^x}{5^2} \quad [M1]$$

$$\frac{2^{3x} \times 5^{2x}}{2^{2x} \times 5^x} = \frac{2^4}{5^2} \quad [M1]$$

$$2^x \cdot 5^x = \frac{16}{25} \quad [M1]$$

$$\therefore 10^x = \frac{16}{25} \quad [A1]$$

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3 A is NO Given that $\sin A = -\frac{4}{5}$, where $180^\circ \leq A \leq 270^\circ$, and that $\cos B = \frac{5}{13}$, where B is acute. Without using a calculator, find the value of

(i) $\cos(A-B)$, [2]

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) \quad [M1]$$

$$= -\frac{63}{65} \quad [A1]$$

(ii) $\operatorname{cosec} 2A$. [2]

$$\operatorname{cosec} 2A = \frac{1}{\sin 2A}$$

$$= \frac{1}{2\sin A \cos A}$$

$$= \frac{1}{2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right)} \quad [M1]$$

$$= \frac{25}{24} \quad [A1]$$

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4 A curve is such that $\frac{d^2y}{dx^2} = 12e^{2x} - 7e^x$. The curve passes through the point $P(0, -4)$ and the gradient of the curve at P is 2. Find the equation of the curve. [6]

$$\frac{d^2y}{dx^2} = 12e^{2x} - 7e^x$$

$$\frac{dy}{dx} = \frac{12e^{2x}}{2} - 7e^x + c$$

$$= 6e^{2x} - 7e^x + c \qquad \text{[m1]}$$

Given that $\frac{dy}{dx} = 2$ when $x = 0$:

$$2 = 6e^0 - 7e^0 + c \qquad \text{[m1]}$$

$$2 = 6 - 7 + c$$

$$\therefore c = 3$$

$$\frac{dy}{dx} = 6e^{2x} - 7e^x + 3 \qquad \text{[m1]}$$

$$y = \int 6e^{2x} - 7e^x + 3 \, dx$$

$$= \frac{6e^{2x}}{2} - 7e^x + 3x + d \qquad \text{[m1]}$$

At $(0, -4)$: $-4 = 3e^0 - 7e^0 + 0 + d$ [m1]

$$-4 = 3 - 7 + d$$

$$\therefore d = 0$$

Equation of the curve: $y = 3e^{2x} - 7e^x + 3x$ [A1]

5 **(a)** The first 3 terms in the expansion, in ascending powers of x, of $(1-3x)^n$, is $1-21x+ax^2$, where a is a constant and n is a positive integer greater than 2. Find the value of n and of a. [4]

$(1-3x)^n = 1 + \binom{n}{1}(-3x) + \binom{n}{2}(-3x)^2 + \ldots$ [M1]

$= 1 + n(-3x) + \frac{n(n-1)}{2}(9x^2) + \ldots$ [M1]

$= 1 - 3nx + 9\binom{n}{2}x^2 + \ldots$

since $1-21x+ax^2 = 1 - 3nx + \frac{9}{2}n(n-1)x^2$

Comparing x: $-21 = -3n$

$n = 7$ [A1]

$a = \frac{9}{2}(7)(7-1)$ or $9\binom{7}{2}$

$= 189$ [A1]

(b) Using your values of n and a, find the coefficient of x^2 in the expansion ~~of x^2 in the expansion of~~ $(2+x)^2(1-3x)^n$. [2]

$(2+x)^2(1-3x)^n = (4+4x+x^2)(1-21x+189x^2+\ldots)$

$\therefore$ Coefficient of $x^2 = 4(189) + 4(-21) + 1$ [M1]

$= 673$ [A1]

6 The variables x and y are related by the equation $y = \ln\left(\frac{x+3}{x-3}\right)$, where $x > 3$.

(a) Express $\frac{dy}{dx}$ in the form $\frac{k}{x^2-9}$ where k is a constant. [3]

$y = \ln(x+3) - \ln(x-3)$

$\frac{dy}{dx} = \frac{1}{x+3} - \frac{1}{x-3}$ [M1]

$= \frac{x-3-(x+3)}{(x+3)(x-3)}$ [M1]

$= \frac{-6}{x^2-9}$ [A1]

(b) Explain whether y is an increasing or decreasing function. [2]

Given that $x > 3$, $\therefore x^2 - 9 > 0$ and $-6 < 0$

$\therefore \frac{-6}{x^2-9} < 0$ [M1]

$\frac{dy}{dx} < 0$

$\therefore$ y is a decreasing function. [A1]

(c) Given that y is increasing at the rate of 8 units/s, find the rate of change of x when $x = 3.5$. [2]

$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$

$= \frac{x^2-9}{-6} \times 8$

when $x = 3.5$, $\frac{dx}{dt} = \frac{(3.5)^2-9}{-6} \times 8$ [M1]

$= -4\frac{1}{3}$ units/s [A1]

7 **(a)** Using $\sin(A+B)=\sin A\cos B+\cos A\sin B$, show that $\sin 75° = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$. [2]

$$\sin 75° = \sin(30°+45°)$$
$$= \sin 30° \cos 45° + \cos 30° \sin 45°$$
$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} \quad [M1]$$
$$= \frac{\sqrt{2}(1+\sqrt{3})}{4}$$
$$= \frac{\sqrt{2}(1+\sqrt{3})}{4}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{1+\sqrt{3}}{2\sqrt{2}} \quad [A1]$$

(b) Hence, express $\text{cosec}^2\, 75°$ in the form $a+b\sqrt{3}$, where a and b are integers. [4]

$$\text{cosec}^2\, 75° = \frac{1}{\sin^2 75°}$$
$$= \left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^2 \quad [M1]$$
$$= \frac{4(2)}{1+2\sqrt{3}+3} \quad [M1]$$
$$= \frac{8}{4+2\sqrt{3}}\cdot\frac{4-2\sqrt{3}}{4-2\sqrt{3}} \quad [M1]$$
$$= \frac{32-16\sqrt{3}}{16-4(3)}$$
$$= 8-4\sqrt{3} \quad [A1]$$

8 **(a)** Given that $2x^3-5x^2+13x-1 \equiv A(x-2)(x^2+3)+B(x^2+3)+(Cx+D)(x-2)$ for all values of x, find the values of A, B, C and D. [4]

Comparing x^3 : $2 = A$ [A1]

Let $x = 2$: $2(2)^3 - 5(2)^2 + 13(2) - 1 = B(2^2+3)$

$7B = 21$

$B = 3$ [A1]

Comparing constant : $-1 = A(-2)(3) + 3B + D(-2)$

$-1 = -6(2) + 3(3) - 2D$

$D = -1$ [A1]

Let $x = 1$: $2 - 5 + 13 - 1 = 2(1-2)(1+3) + 3(1+3) + (C-1)(1-2)$

$9 = -8 + 12 - (C-1)$

$C - 1 = -5$

$C = -4$ [A1]

(b) When a polynomial $\mathrm{f}(x)$ is divided by $(x-1)$, the remainder is 4.

When the same polynomial is divided by $(x+3)$, the remainder is -24.

Find the remainder when $\mathrm{f}(x)$ is divided by (x^2+2x-3), leaving your answer in the form $Ax+B$, where A and B are constants. ~~[4]~~ [5]

$f(x) = (x^2+2x-3)Q(x) + Ax+B$

$= (x+3)(x-1)Q(x) + Ax+B$ [m1]

$f(1) = A(1)+B = 4$

$A+B = 4$ .... (1) [m1]

$f(-3) = A(-3)+B = -24$

$-3A + B = -24$ .... (2) [m1]

(1)−(2): $4A = 28$

$\left.\begin{array}{l} A = 7 \\ B = -3 \end{array}\right\}$ [m1]

∴ The remainder is $7x-3$. [A1]

9 An object is heated until it reaches a temperature of T_0 °C. It is then allowed to cool. Its temperature T °C, when it has been cooled for t minutes, is given by the equation $T = 36 + 17e^{-0.75t}$.

(a) Find the value of T_0. [1]

when $t = 0$, $T_0 = 36 + 17e^0$

$= 53°C$ [A1]

(b) Find the value of t when $T = 47$ °C. [2]

$47 = 36 + 17e^{-0.75t}$

$e^{-0.75t} = \frac{11}{17}$

$-0.75t = \ln\left(\frac{11}{17}\right)$ [M1]

$t = 0.580$ mins [A1]

(c) Calculate the minimum number of complete minutes required for the temperature to reach 40 °C. [4]

$36 + 17e^{-0.75t} \leq 40$

$e^{-0.75t} \leq \frac{4}{17}$ [M1]

$-0.75t \ln e \leq \ln\left(\frac{4}{17}\right)$

$t \geqslant \dfrac{\ln\left(\frac{4}{17}\right)}{-0.75}$ [M1]

$t \geqslant 1.93$ [M1]

∴ The minimum number of minutes for the temperature to reach 40°C is 2 mins. [A1]

(d) Explain whether the temperature of the object will reach 36 °C. [1]

As t gets large, $17e^{-0.75t}$ approaches 0. [B1]

∴ $36 + 17e^{-0.75t}$ approaches 36°C.

∴ The temperature of the object will not reach 36°C.

10 **(a)** Show that $\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x} = \sec x$. [4]

LHS : $\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x}$

$= \frac{2\sin x \cos x}{\sin x} - \frac{2\cos^2 x - 1}{\cos x}$ [M1] , [M1]

$= 2\cos x - 2\cos x + \frac{1}{\cos x}$ [M1]

$= \sec x$ [A1]

$=$ RHS (shown)

(b) Hence find, for $0 \le x \le 6$, the values of x for which $\frac{3\cos 2x}{\cos x} - \frac{3\sin 2x}{\sin x} = 5$ [4]

$\frac{3\cos 2x}{\cos x} - \frac{3\sin 2x}{\sin x} = 5$

$-3\left(\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x}\right) = 5$

$-3\sec x = 5$

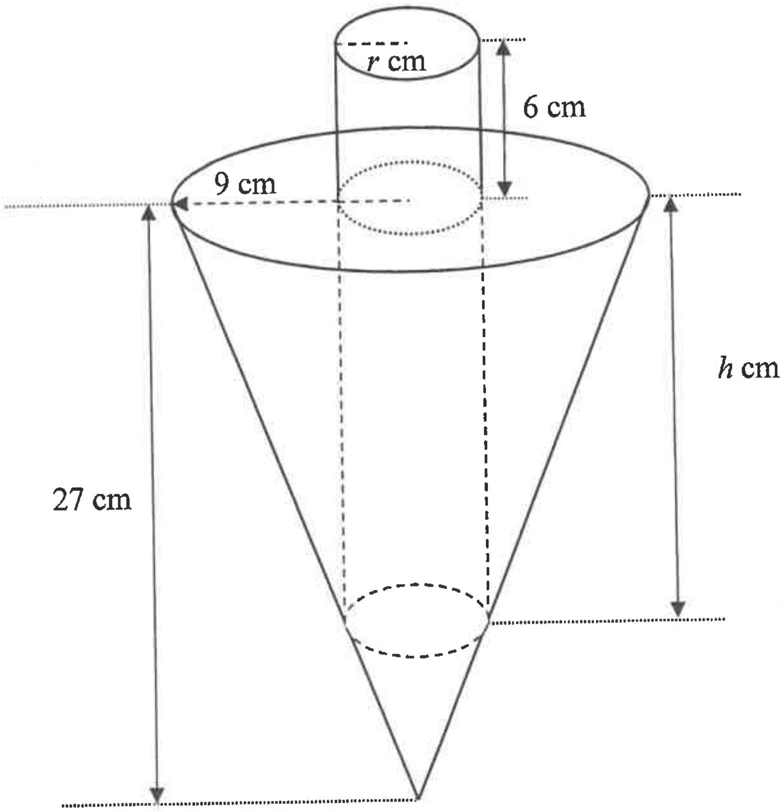
$\cos x = -\frac{3}{5}$ [M1]

Basic angle $= 0.927295$ radians

$x = \pi - 0.927295 , \pi + 0.927295$ [M1] , [M1]

$= 2.21 , 4.07$ radians [A1]

11



A cylinder of radius r cm is placed upright inside a cone so that the top of the cylinder is 6 cm above the top of the cone as shown in the figure above. The cone has a radius of 9 cm and a height of 27 cm. The part of the cylinder inside the cone is h cm deep.

(i) Show that $h = 27 - 3r$. [2]

Using similar triangles,

$$\frac{9}{27} = \frac{r}{27-h} \quad [M1]$$

$$243 - 9h = 27r$$

$$\therefore \; h = 27 - 3r \quad \text{(shown)} \quad [A1]$$

Question 11 continues on the next page.

(ii) Find an expression in terms of r for the volume of the cylinder. [2]

Volume of cylinder, $V = \pi r^2 (h+6)$ [M1]

$= \pi r^2 (27-3r+6)$

$= \pi r^2 (33-3r)$ [A1]

$= 33\pi r^2 - 3\pi r^3$

(iii) Given that h and r can vary, find the value of h for which the volume of the cylinder is a maximum. [4]

For V to be a maximum, $\frac{dV}{dr} = 0$

$66\pi r - 9\pi r^2 = 0$ [M1]

$\pi r(66-9r) = 0$

$r = 0$ or $r = 7\frac{1}{3}$ cm [M1]

$\frac{d^2V}{dr^2} = 66\pi - 18\pi r$

when $r = 7\frac{1}{3}$, $\frac{d^2V}{dr^2} < 0$ [M1]

$\therefore$ Volume is a maximum

$\therefore h = 27 - 3(7\frac{1}{3})$

$= 5$ cm [A1]

12 A particle ~~moves in~~ moving a straight line passes a fixed point O with a velocity of 9 m/s. Its acceleration, a m/s^2, is given by $a = 6t - 12$, where t is the time in seconds after passing O.

(i) Find the minimum velocity of the particle. [3]

$$v = \int 6t - 12 \; dt$$
$$= \frac{6t^2}{2} - 12t + c$$
$$= 3t^2 - 12t + c$$

when $t = 0, v = 9$: $c = 9$

$v = 3t^2 - 12t + 9$ [M1]

For minimum velocity, $\frac{dv}{dt} = 0$

$\therefore 6t - 12 = 0$

$t = 2$ [M1]

when $t = 2$, $v = 3(2)^2 - 12(2) + 9$

$= -3$ m/s [A1]

(ii) Obtain an expression, in terms of t, for the displacement of the particle after passing O. Hence, find its displacement when it is first instantaneously at rest. [4]

$$s = \int 3t^2 - 12t + 9 \; dt$$
$$= \frac{3t^3}{3} - \frac{12t^2}{2} + 9t + d$$
$$= t^3 - 6t^2 + 9t + d$$ [M1]

when $t = 0, s = 0$: $d = 0$

$s = t^3 - 6t^2 + 9t$

when $v = 0$: $3t^2 - 12t + 9 = 0$ [M1]

$3(t-3)(t-1) = 0$

$t = 1$ or $t = 3$. [M1]

when $t = 1$, $s = 1 - 6 + 9$

$= 4$ m [A1]

Question 12 continues on the next page.

(iii) Find the total distance travelled by the particle during the first 5 seconds. [3]

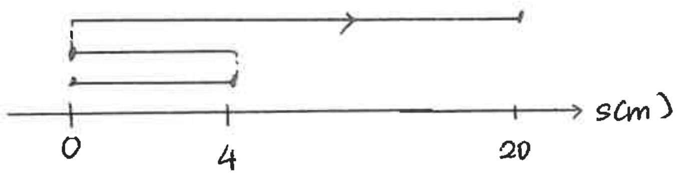
When $t=3$, $s = 3^3 - 6(3)^2 + 9(3)$ [M1]

$= 0\text{m}$

When $t=5$, $s = 5^3 - 6(5)^2 + 9(5)$

$= 20\text{m}$ [M1]

$\therefore$ Total distance travelled $= 4 + 4 + 20 = 28\text{m}$ [A1]



(iv) Explain why the particle never returns to O. after 3 seconds. [2]

$s = t^3 - 6t^2 + 9t$

$= t(t^2 - 6t + 9)$

$= t(t-3)^2$

Since the particle only turns when $t=1$ or $t=3$,

$\therefore$ when $t > 3$, $t(t-3)^2 > 0$ [M1]

$s > 0$

$\therefore$ The particle never returns to O. [A1]

Alternatively

$v = 3t^2 - 12t + 9$

$= 3[t^2 - 4t + (2)^2 - (2)^2] + 9$

$= 3(t-2)^2 - 3$

When $t > 3$, $3(t-2)^2 - 3 > 0$ [M1]

$v > 0$

Hence the particle is always moving in the positive direction [A1]

13 The equation of a curve is $y = 3\sin 2x - 1$ for $0 \le x \le 2\pi$.

(a) State the amplitude and period of y. [2]

Amplitude = 3 [B1]

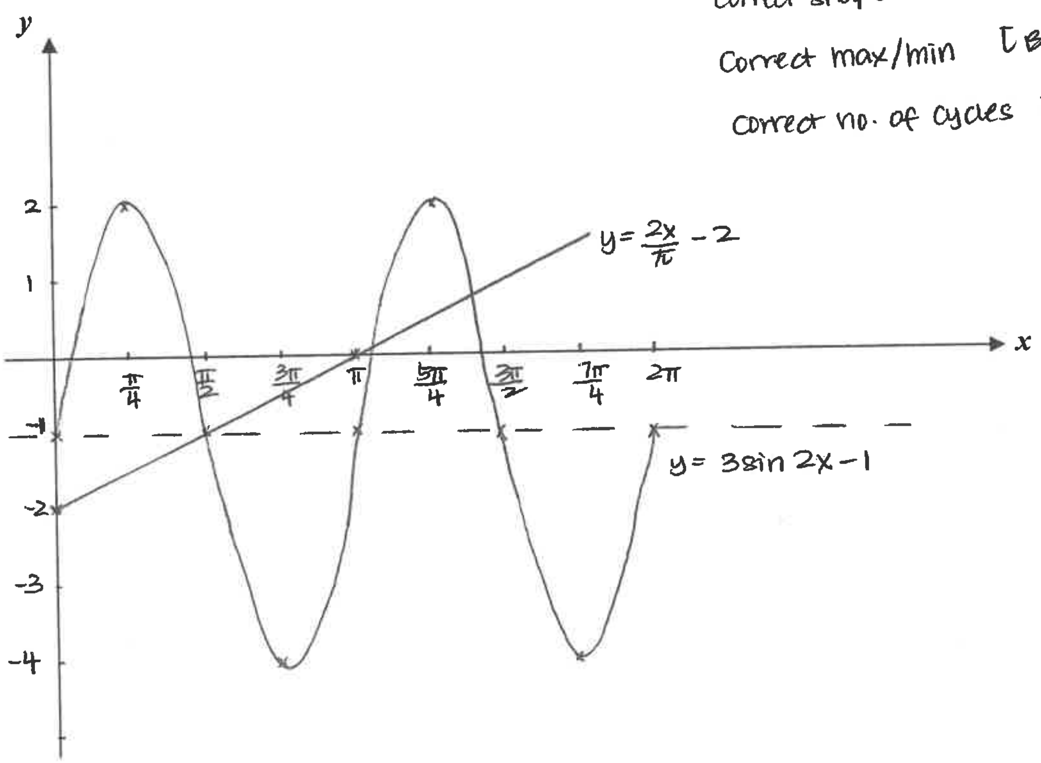
Period = π [B1]

(b) Sketch the graph of $y = 3\sin 2x - 1$, showing clearly the intercepts, turning points and end points. [3]

Correct shape [B1]

Correct max/min [B1]

Correct no. of cycles [B1]



(c) On the same diagram in part (b), sketch the graph of $y = \frac{2x}{\pi} - 2$ for $0 \le x \le 2\pi$.

Hence state the number of solutions to the equation $\sin 2x = \frac{2x}{3\pi} - \frac{1}{3}$. [3]

Drawing of $y = \frac{2x}{\pi} - 2$ [B1]

$\sin 2x = \frac{2x}{3\pi} - \frac{1}{3}$

$3\sin 2x = \frac{2x}{\pi} - 1$

$3\sin 2x - 1 = \frac{2x}{\pi} - 2$ [M1]

Number of solutions = 3 [A1]

End of Paper 1