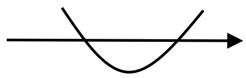
4E AM Prelim 2024 P2 Solutions

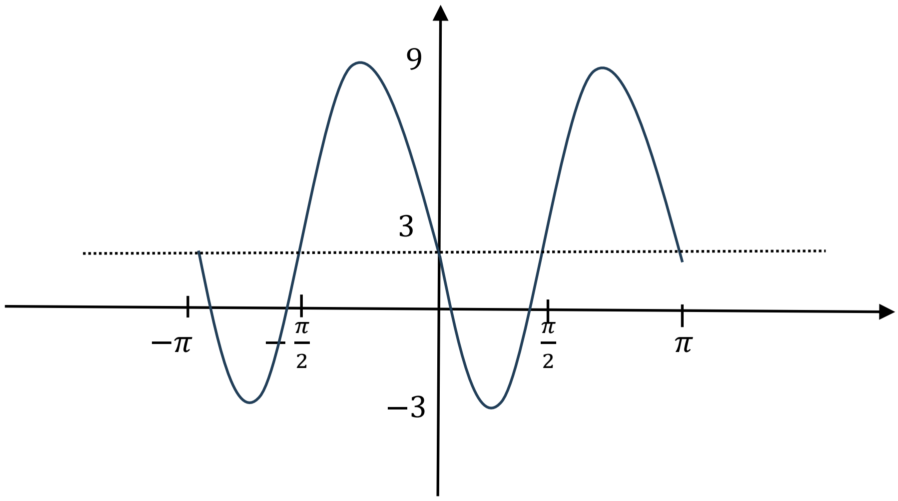
1.		The Singapore government issued a savings bond in January 2024 with a yield of 2.75% per year. Mr Tan invested \$15 000 in the bond. The total amount he will receive, after t years, is given by $A = 15000(1.0275)^t$.
	(a)	Calculate the total amount he will receive in January 2030, correct to the nearest dollar.
		$A = 15000(1.0275)^6$ $= 17651.52$ $Amount = \$\ 17652$
	(b)	In which year will the amount first exceed \$22 000?
		$22000 = 15000(1.0275)^t$ $1.0275^t = \frac{22}{15}$ $t = \ln\left(\frac{22}{15}\right) / \ln 1.0275$ $t = 14.117$ Year 2039.

2.	(a)	Without using a calculator, evaluate the value of 6^x given that $2^{2x+6} \times 3^{5x-1} = 27^{x+1}$
		$$2^{2x+6} \times 3^{5x-1} = 3^{3x+3}$$ $$2^{2x} \times 2^6 \times 3^{5x} \times 3^{-1} = 3^{3x} \times 3^3$$ $$\frac{2^{2x} \times 3^{5x}}{3^{3x}} = \frac{3^3}{2^6 \times 3^{-1}}$$ $$2^{2x} \times 3^{2x} = \frac{3^4}{2^6}$$ $$6^{2x} = \frac{81}{64}$$ $$6^x = \frac{9}{8}$$
	(b)	Solve the equation $log_x 9 = 5log_3 x$, giving your answers correct to 2 significant figures.
		$$log_x 9 = 5log_3 x$$ $$\frac{log_3 9}{log_3 x} = 5log_3 x$$ $$2log_3 3 = 5(log_3 x)^2$$ $$(log_3 x)^2 = \frac{2}{5}$$ $$log_3 x = \pm\sqrt{\frac{2}{5}}$$ $$x = 3^{\sqrt{\frac{2}{5}}}\ or\ 3^{-\sqrt{\frac{2}{5}}}$$ $$x = 2.0\ or\ 0.50$$

3.	(a)	A curve has the equation $y = (p-1)x^2 + 2(p-3)$, where p is a constant. A line has the equation $y = 6x + 3$. Find the range of values of p if the curve lies completely above the line.
		$(p-1)x^2 + 2p - 6 = 6x + 3$ $(p-1)x^2 - 6x + 2p - 9 = 0$ $b^2 - 4ac < 0$ $(-6)^2 - 4(p-1)(2p-9) < 0$ $36 - (4p-4)(2p-9) < 0$ $36 - 8p^2 + 8p + 36p - 36 < 0$ $-8p^2 + 44p < 0$ $8p^2 - 44p > 0$ $4p(2p-11) > 0$  $p < 0$ *or* $p > 5.5$ (rejected)
	(b)	By expressing $y = -2x^2 + 10x - 5$ in the form $y = a(x+b)^2 + c$, where a, b and c are constants, find the maximum value of y.
		$y = -2(x^2 - 5x) - 5$ $= -2\left[x^2 - 5x + \left(-\frac{5}{2}\right)^2 - \left(-\frac{5}{2}\right)^2\right] - 5$ $= -2[(x-2.5)^2 - 2.5^2] - 5$ $= -2(x-2.5)^2 + 7.5$ Max value of y = 7.5

4.	(a)	In the binomial expansion of $\left(1-\frac{2}{7}x\right)^n$, the sum of the coefficient of the second and third term is zero. Calculate the value of n and hence, find the sixth term.
		$T_2 = \binom{n}{1}(1)^{n-1}\left(-\frac{2}{7}x\right)^1 = -\frac{2}{7}nx$ $T_{3=}\binom{n}{2}(1)^{n-2}\left(-\frac{2}{7}x\right)^2 = \frac{n(n-1)}{2}\left(\frac{4}{49}x^2\right)$ $-\frac{2}{7}n + \frac{2n(n-1)}{49} = 0$ $-14n + 2n^2 - 2n = 0$ $2n^2 - 16n = 0$ $2n(n-8) = 0$ $n = 0 \;\; or \; n = 8$ (rejected) $T_6 = \binom{8}{5}(1)^{8-5}\left(-\frac{2}{7}x\right)^5 = -\frac{256}{2401}x^5$

	(b)	Write down the general term in the binomial expansion of $\left(\frac{1}{x^3} - 2x\right)^8$. Hence, find the value of the constant term in the expansion of $\left(3 + \frac{x^2}{2}\right)^2 \left(\frac{1}{x^3} - 2x\right)^8$.
		$$T_{r+1} = \binom{8}{r}\left(\frac{1}{x^3}\right)^{8-r} (-2x)^r$$ $$= \binom{8}{r} x^{-24+3r}(-2)^r x^r$$ $$= \binom{8}{r} (-2)^r x^{4r-24}$$ $$\left(3 + \frac{x^2}{2}\right)^2 = 9 + 3x^2 + \frac{x^4}{4}$$ $$x^{4r-24} = x^0 \quad or \;\; x^{4r-24} = x^{-2} \qquad or \; x^{4r-24} = x^{-4}$$ $$r = 6 \quad or \; no \; integer \; value \quad or \quad r = 5$$ $$T_{7=} \binom{8}{6} (-2)^6 = 1792$$ $$T_{6=} \binom{8}{5} (-2)^5 = -1792$$ Constant term = $(1792 \times 9) + \left(\frac{1}{4} \times -1792\right) = 15680$

5.	(a)	The function f is defined as $f(x) = p - q\sin(rx)$, for $-\pi \le x \le \pi$, where p, q and r are positive integers. Given that the amplitude of the function is 6, the period is π and the maximum value of is 9.
		(i) State the values of p, q and r.
		$p = 3$ $q = 6$ $r = 2$
		(ii) Hence, sketch the graph of $f(x)$.
		 - Shape of curve and range - Points plotted correctly

	(b)	The acute angles A and B are such that $\cot(A-B)=\frac{1}{3}$ and $\cot A=\frac{1}{5}$. Without using a calculator, find the exact value of $\cos B$.

$$\cot A = \frac{1}{5}$$

$$\tan A = 5$$

$$\cot(A-B) = \frac{1}{3}$$

$$\tan(A-B) = 3$$

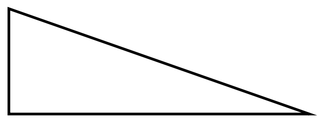
$$\frac{\tan A - \tan B}{1+\tan A \tan B} = 3$$

$$\tan A - \tan B = 3 + 3\tan A \tan B$$

$$5 - \tan B = 3 + 15\tan B$$

$$16\tan B = 2$$

$$\tan B = \frac{1}{8}$$



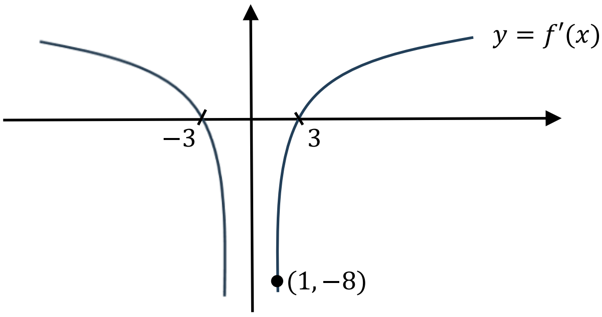
$$\cos B = \frac{8}{\sqrt{65}} = \frac{8\sqrt{65}}{65}$$

6.		A circle passes through the points $(-5, 12)$ and $(9, 14)$. The centre of the circle lies on the line $2y + x = 15$.
	(a)	Find the equation of the circle.
		$$m = \frac{14-12}{9-(-5)} = \frac{1}{7}$$ $$m_{\perp} = -7$$ $$midpoint = \left(\frac{-5+9}{2}, \frac{12+14}{2}\right) = (2, 13)$$ $$13 = -7(2) + c$$ $$c = 27$$ Eqn of perpendicular bisector $y = -7x + 27$ $$2(-7x + 27) + x = 15$$ $$-14x + 54 + x = 15$$ $$-13x = -39$$ $$x = 3$$ $$y = 6$$ Centre of circle $(3,6)$ Radius = $\sqrt{(9-3)^2 + (14-6)^2} = 10\ units$ Equation of circle $$(x-3)^2 + (y-6)^2 = 100$$ Or $x^2 - 6x + y^2 - 12y - 55 = 0$

	(b)	Explain why the line $y = mx + 6$ intersects the circle at 2 distinct points for all values of m.
		$$(x-3)^2 + (mx+6-6)^2 = 100$$ $$x^2 - 6x + 9 + m^2x^2 - 100 = 0$$ $$(1+m^2)x^2 - 6x - 91 = 0$$ $$b^2 - 4ac = (-6)^2 - 4(1+m^2)(-91)$$ $$= 400 + 364m^2$$ $400 + 364m^2 > 0$ for all values of m ∴line cuts circle at 2 distinct points
7.	(a)	A curve has the equation $y = \frac{3x-5}{4x+1}$ for $x > 0$. Explain, with working, why the curve has no stationary points.
		$$\frac{dy}{dx} = \frac{3(4x+1) - 4(3x-5)}{(4x+1)^2}$$ $$= \frac{12x + 3 - 12x + 20}{(4x+1)^2}$$ $$= \frac{23}{(4x+1)^2}$$ Since $(4x+1)^2 \geq 0$ for all x, $\frac{dy}{dx} > 0$ Curve has no stationary point as $\frac{dy}{dx} \neq 0$.

	(b)	It is given that $f(x)$ is such that $f'(x) = \cos 4x - 3\sin 2x$. Given also that $f(\pi) = 0$, show that $f''(x) + 4f(x) = -3(\sin 4x + 2)$.
		$$f(x) = \frac{1}{4}sin4x + \frac{3}{2}cos2x + c$$ $$f(\pi) = 0$$ $$\frac{3}{2} + c = 0$$ $$c = -\frac{3}{2}$$ $$f(x) = \frac{1}{4}\sin 4x + \frac{3}{2}\cos 2x - \frac{3}{2}$$ $$f''(x) = -4\sin 4x - 6\cos 2x$$ $$f''(x) + 4f(x) = -4\sin 4x - 6\cos 2x + 4\left(\frac{1}{4}\sin 4x + \frac{3}{2}\cos 2x - \frac{3}{2}\right)$$ $$= -4\sin 4x - 6\cos 2x + \sin 4x + 6\cos 2x - 6$$ $$= -3\sin 4x - 6$$ $$= -3(\sin 4x + 2)\ \ (shown)$$
	(c)	Given that $\frac{d}{dx}\left(\frac{2-x}{\sqrt{1-2x}}\right) = \frac{ax+b}{\sqrt{(1-2x)^3}}$, find the value of a and b.
		$$\frac{d}{dx}\left(\frac{2-x}{\sqrt{1-2x}}\right) = \frac{(1-2x)^{\frac{1}{2}}(-1) - (2-x)\left(\frac{1}{2}\right)(1-2x)^{-\frac{1}{2}}(-2)}{(1-2x)}$$ $$= \frac{(1-2x)^{-\frac{1}{2}}[(-1)(1-2x) - (2-x)\left(\frac{1}{2}\right)(-2)}{(1-2x)}$$ $$= \frac{(1-2x)^{-\frac{1}{2}}[-1 + 2x + 2 - x]}{(1-2x)}$$ $$= \frac{1+x}{(1-2x)^{\frac{3}{2}}}$$ $$a = 1,\ \ b = 1$$

8. A curve $y = f(x)$ passes through the point $(1, 10)$.



The graph shown above is $y = f'(x)$.

(a) State the x-coordinates of the stationary points of the curve $y = f(x)$ and hence, determine their nature.

$x = -3$ $\quad$ *and* $\quad$ $x = 3$

x	-3.1	-3	-2.9		2.9	3	3.1
$f'(x)$	$+$	0	$-$		$-$	0	$+$

Maximum point $\qquad$ Minimum point

(b) Find the equation of the normal to the curve at the point $(1, 10)$.

$$x = 1, f'(x) = -8$$

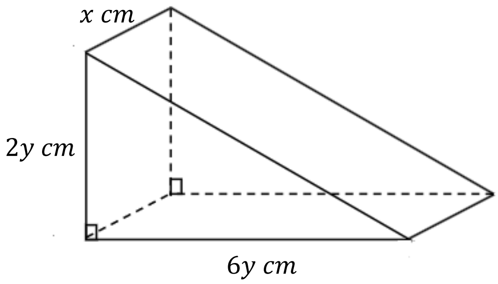
Gradient of tangent = -8

Gradient of normal = $\frac{1}{8}$

$$10 = \frac{1}{8}(1) + c$$

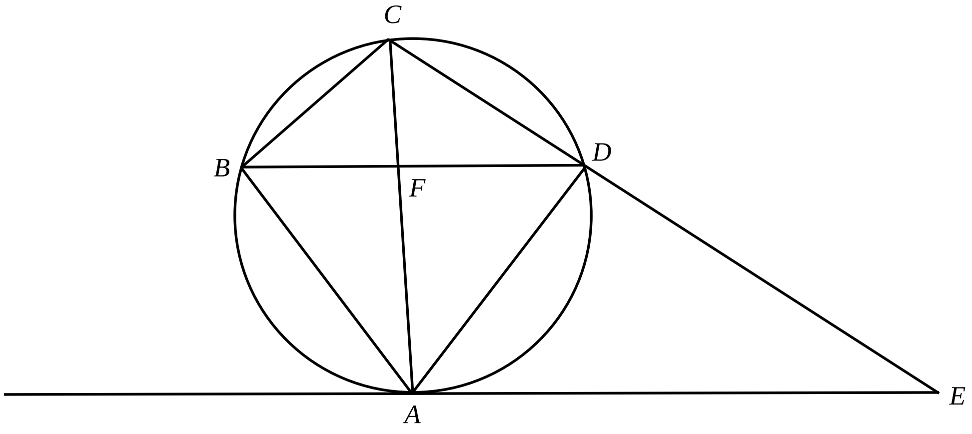
$$c = \frac{79}{8}$$

$$y = \frac{1}{8}x + \frac{79}{8} \quad or\ 8y = x + 79$$

9.		The diagram below shows a wooden door stopper in the shape of a right prism with a volume of 60 cm^3. The cross-section of the prism is a triangle, with side lengths of $2y$ and $6y$ cm respectively, with a width of x cm. 
	(a)	Express x in terms of y and show that the total surface area of the door stopper A, is given as $A = 12y^2 + \frac{20}{y}\left(\sqrt{10} + 4\right)\ cm^2$.
		$\frac{1}{2} \times 2y \times 6y \times x = 60$ $x = \frac{10}{y^2}$ Total S.A = $2\left(\frac{1}{2} \times 2y \times 6y\right) + 2xy + 6xy + \left(\sqrt{(2y)^2 + (6y)^2}\right)x$ $= 12y^2 + 8xy + \left(\sqrt{40y^2}\right)x$ $= 12y^2 + 8y\left(\frac{10}{y^2}\right) + 2y\sqrt{10}\left(\frac{10}{y^2}\right)$ $= 12y^2 + \frac{80}{y} + \frac{20\sqrt{10}}{y}$ $= 12y^2 + \frac{20}{y}\left(\sqrt{10} + 4\right)\ cm^2$

	(b)	Given that *y* can vary, find the value of *y* for which A has a minimum value.
		$$\frac{dA}{dy} = 24y - \left(\frac{20}{y^2}\right)\left(\sqrt{10}+4\right)$$ $$24y - \left(\frac{20}{y^2}\right)\left(\sqrt{10}+4\right) = 0$$ $$24y^3 = 20\left(\sqrt{10}+4\right)$$ $$y^3 = \frac{5}{6}\left(\sqrt{10}+4\right)$$ $$y = 1.8139$$ $$\frac{d^2A}{dy^2} = 24 + \frac{40}{y^3}\left(\sqrt{10}+4\right)$$ $$= 72 > 0$$ Value of A is at a minimum.
10.	(a)	Find all angles between 0 and 2π which satisfy $3\cos 2x + 4\sin x = 3$.
		$$3\cos 2x + 4\sin x = 3$$ $$3(1 - 2\sin^2 x) + 4sinx - 3 = 0$$ $$3 - 6\sin^2 x + 4sinx - 3 = 0$$ $$-6\sin^2 x + 4sinx = 0$$ $$-2\sin x(3\sin x - 2) = 0$$ $$-2\sin x = 0 \quad or \quad 3\sin x - 2 = 0$$ $$sinx = 0 \quad or \quad sinx = \frac{2}{3}$$ $$Basic\ angle = 0 \quad or\ 0.72972$$ $$x = \pi \quad or \quad x = 0.72972\ or\ 2.4118$$ $$x = 0.730, 2.41, \pi$$ (minus 1 mark for each wrong value)

	(b)	Without using a calculator,
	(i)	Show that $\cos\frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4}$.
		$\cos\frac{7\pi}{12} = \cos\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right)$ $= \cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$ $= \left(\cos\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3}\right) - \left(\sin\frac{\pi}{4}\right)\left(\sin\frac{\pi}{3}\right)$ $= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)$ $= \frac{1-\sqrt{3}}{2\sqrt{2}}$ $= \frac{\sqrt{2}-\sqrt{6}}{4} \quad (shown)$
	(ii)	Hence, find the exact value of $\sin^2\frac{7\pi}{12}$.
		$\sin^2\frac{7\pi}{12} = 1-\cos^2\frac{7\pi}{12}$ $= 1-\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2$ $= 1-\frac{2-2\sqrt{2}\sqrt{6}+6}{16}$ $= \frac{16-(2-2\sqrt{12}+6)}{16}$ $= \frac{16-2+4\sqrt{3}-6}{16}$ $= \frac{8+4\sqrt{3}}{16}$ $= \frac{2+\sqrt{3}}{4}$

12.		
		Points *A*, *B*, *C* and *D* is inscribed in a circle such that *AB* = *AD*. A tangent to the circle at point *A* meets the line *CD* produced at E. The lines *AC* and *BD* intersects at point *F*.
	(i)	Prove that the line *BFD* is parallel to line *AE*.
		$\angle ABD = \angle DAE (alt. seg. thm)$ $\angle ABD = \angle ADB\ (base\ angle\ of\ iso\ \Delta)$ $\angle ABD = \angle DAE$ BFD is parallel to AE as $\angle ABD\ and\ \angle DAE$ are equal, alternate angles in parallel lines.
	(ii)	Show that triangle *ABC* is similar to triangle *EDA*.
		$\angle BCA = \angle BDA\ (\measuredangle in\ same\ segment)$ $\angle BCA = \angle DAE\ (part\ (i))$ $\angle ABC + \angle CDA = 180\ (\measuredangle in\ opp\ segment)$ $\angle CDA + \angle EDA = 180\ (adj. \measuredangle on\ a\ str. line)$ $\angle ABC = \angle EDA$ $\Delta ABC\ is\ similar\ to\ \Delta EDA$ (angle-angle theorem).