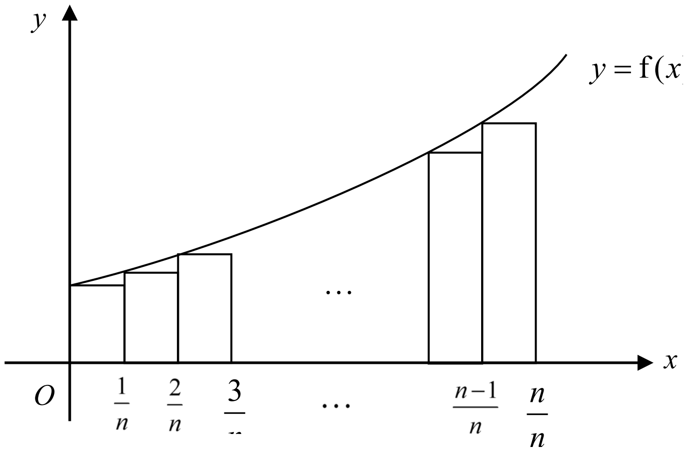
2023 H2 Math Prelim solutions

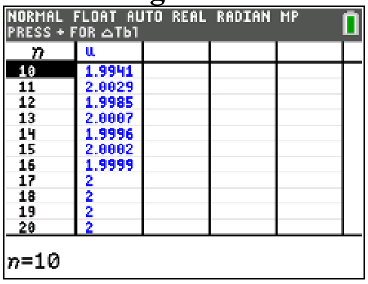
Qn	Suggested Solution	
1(a)	 Area of n rectangles under the above curve between $x=0$ and $x=1$, $=\frac{1}{n}\mathrm{f}(0)+\frac{1}{n}\mathrm{f}\left(\frac{1}{n}\right)+\frac{1}{n}\mathrm{f}\left(\frac{2}{n}\right)+...+\frac{1}{n}\mathrm{f}\left(\frac{n-1}{n}\right)$ $=\frac{1}{n}\sum_{r=0}^{n-1}\mathrm{f}\left(\frac{r}{n}\right)$ Since area of n rectangles is an **underestimate of / less than** the actual area under curve $=\int_0^1 \mathrm{f}(x)\ \mathrm{d}x$, $\therefore \frac{1}{n}\sum_{r=0}^{n-1}\mathrm{f}\left(\frac{r}{n}\right)<\int_0^1 \mathrm{f}(x)\ \mathrm{d}x$ (shown)	
(b)	Required expression $=\frac{1}{n}\sum_{r=1}^{n}\mathrm{f}\left(\frac{r}{n}\right)$	
(c)(i)	Lower limit $=\frac{1}{10}\sum_{r=0}^{9}\mathrm{f}\left(\frac{r}{10}\right)$ $=\frac{1}{10}\sum_{r=0}^{9}\left(\left(\frac{r}{10}\right)^2+1\right)$ $=1.285$ Upper limit $=\frac{1}{10}\sum_{r=1}^{10}\mathrm{f}\left(\frac{r}{10}\right)$ $=\frac{1}{10}\sum_{r=1}^{10}\left(\left(\frac{r}{10}\right)^2+1\right)$ $=1.385$	
(c)(ii)	From GC : $\int_0^1 x^2+1\ \mathrm{d}x=1.3333$ Difference from $\int_0^1 x^2+1\ \mathrm{d}x$ (a) lower limit = $\frac{1.3333-1.285}{1.3333}=\frac{0.0483}{1.3333}=3.62\%$ (b) upper limit = $\frac{1.385-1.3333}{1.3333}=\frac{0.0517}{1.3333}=3.88\%$ The lower limit is a better estimate since it has a smaller % / absolute difference from the exact value.	

Qn	Suggested Solution	

2a(i)

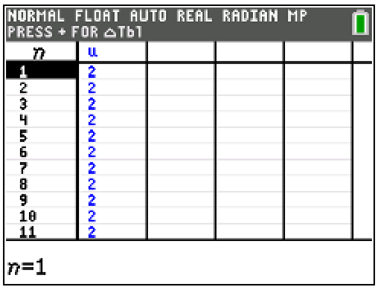
(A) $\underline{c=5}$

The sequence initially **decreases** and subsequently **alternates**, and **converges to 2**.



(B) $\underline{c=2}$

It is **a constant** with a value of **2**.



a(ii)

$u_1 = c$

$u_2 = 3 - 0.5u_1 = 3 - 0.5c$

$u_3 = 3 - 0.5u_2 = 3 - 0.5(3 - 0.5c) = 1.5 + 0.25c$

Given $2u_3 = -5u_2$

$2(1.5 + 0.25c) = -5(3 - 0.5c)$

$3 + 0.5c = -15 + 2.5c$

$2c = 18$

$c = 9$

b(i)

Given $v_3 + 1 = 2u_2 - u_1$,

$2v_1 + v_2 - 1 + 1 = 2u_2 - u_1$,

$2p + 2 = 2(3 - 0.5c) - c$

$2p + 2 = 6 - 2c$

$c + p = 2$

b(ii)

Given $v_1 = p,\ v_2 = 2$,

$v_3 = 2v_1 + v_2 - 1 = 2p + 2 - 1 = 2p + 1$

$v_4 = 2v_2 + v_3 - 1 = 2(2) + (2p + 1) - 1 = 2p + 4$

$v_5 = 2v_3 + v_4 - 1 = 2(2p + 1) + (2p + 4) - 1 = 6p + 5$

$\because 6p + 5 = 77$

$\therefore p = 12$

Qn	Suggested Solution	

3(a)	$y^2 = h^2 + \left(\frac{x}{2}\right)^2$ $= h^2 + \left(\frac{20-2y}{2}\right)^2$ $= h^2 + \left(100 - 20y + y^2\right)$ $20y = 100 + h^2$ $y = 5 + \frac{h^2}{20}$ (shown) $x + 2y = 20$ $x = 20 - 2y$ $x = 20 - 2y$ $= 20 - 2\left(5 + \frac{h^2}{20}\right)$ $= 10 - \frac{h^2}{10}$	
(b)	Volume of prism, $V = \frac{1}{2}hxz$ $= \frac{1}{2}h\left(10 - \frac{h^2}{10}\right)(20-2h)$ $= \frac{h}{10}\left(100-h^2\right)(10-h)$ $= \frac{h}{10}\left(1000 - 100h - 10h^2 + h^3\right)$ $= \frac{1}{10}\left(h^4 - 10h^3 - 100h^2 + 1000h\right)$ (shown) $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{10}\left(4h^3 - 30h^2 - 200h + 1000\right)$ For max.volume, $\frac{\mathrm{d}V}{\mathrm{d}h} = 0 \Rightarrow 4h^3 - 30h^2 - 200h + 1000 = 0$ From GC : $h = -6.40$ or 10 (reject $\because 0 < h < 10$) $\therefore h = 3.9039$ $\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = \frac{1}{10}(12h^2 - 60h - 200)$ When $h = 3.9039$, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = \frac{1}{10}(12(3.9039)^2 - 60(3.9039) - 200) = -25.1 < 0$ $\therefore h = 3.9039$ gives maximum volume	

	Max volume $=\frac{1}{10}\left(3.9039^4-10(3.9039)^3-100(3.9039)^2+1000(3.9039)\right)$ $=201.71=202\text{cm}^3$ (3 sf)	

Qn	Suggested Solution	
4(a)	Since the lines AB and AC meet, $\begin{pmatrix}1+2\lambda\\4+a\lambda\\6+\lambda\end{pmatrix}=\begin{pmatrix}4+\mu\\4+b\mu\\9+2\mu\end{pmatrix}$. $2\lambda-\mu=3$ --- (1) $a\lambda=b\mu$ --- (2) $\lambda-2\mu=3$ --- (3) Solving (1) & (3) gives $\lambda=1$ & $\mu=-1$ Hence (2) gives $a=-b \Rightarrow a+b=0$ (shown)	
(b)	$\begin{pmatrix}2\\a\\1\end{pmatrix}\times\begin{pmatrix}1\\b\\2\end{pmatrix}=\begin{pmatrix}2a-b\\-3\\2b-a\end{pmatrix}$ Since the normal of plane ABC is parallel to $\begin{pmatrix}-3\\1\\3\end{pmatrix}$, $2a-b=9$ --- (1) $2b-a=-9$ --- (2) Solving (1) & (2) gives $a=3$ & $b=-3$ Equation of plane ABC: $\mathbf{r}\bullet\begin{pmatrix}-3\\1\\3\end{pmatrix}=\begin{pmatrix}1\\4\\6\end{pmatrix}\bullet\begin{pmatrix}-3\\1\\3\end{pmatrix}=19 \Rightarrow -3x+y+3z=19$	
(c)	Since $\lambda=1$ & $\mu=-1$ and $a=3$ & $b=-3$, the coordinates of A are (3, 7, 7). Let the acute angle between lines AB and AC be θ. $\cos\theta=\dfrac{\left\|\begin{pmatrix}2\\3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-3\\2\end{pmatrix}\right\|}{\sqrt{4+9+1}\sqrt{1+9+4}}=\dfrac{5}{14}$ $\Rightarrow \theta=69.1°$ or 1.21 rad	

(d)	$$\frac{\left\lvert \mathbf{r}\bullet\begin{pmatrix}-3\\1\\3\end{pmatrix}\right\rvert}{\sqrt{9+1+9}}=\frac{19}{\sqrt{9+1+9}}=\sqrt{19}$$ Distance of plane ABC from the origin is $\sqrt{19}$ units. Hence the two required planes: $$\mathbf{r}\bullet\begin{pmatrix}-3\\1\\3\end{pmatrix}=0 \quad \& \quad \mathbf{r}\bullet\begin{pmatrix}-3\\1\\3\end{pmatrix}=2\sqrt{19}\sqrt{19}=38$$	

5(a)	$k=\dfrac{1}{7}$	
(b)	Let G be absolute difference of two scores. Probability Distribution of G:	

g	0	1	3	4
$\mathrm{P}(G=g)$	$\left(\frac{1}{7}\right)\left(\frac{1}{7}\right)+$ $+\left(\frac{2}{7}\right)\left(\frac{2}{7}\right)$ $+\left(\frac{4}{7}\right)\left(\frac{4}{7}\right)$ $=\frac{21}{49}$	$2\left(\frac{2}{7}\right)\left(\frac{4}{7}\right)$ $=\frac{16}{49}$	$2\left(\frac{1}{7}\right)\left(\frac{2}{7}\right)$ $=\frac{4}{49}$	$2\left(\frac{1}{7}\right)\left(\frac{4}{7}\right)$ $=\frac{8}{49}$

$$\mathrm{E}(G)=1\left(\frac{16}{49}\right)+3\left(\frac{4}{49}\right)+4\left(\frac{8}{49}\right)$$
$$=\frac{60}{49}$$
$$\mathrm{E}(2G-m)>0$$
$$2\mathrm{E}(\mathrm{G})-m>0$$
$$2\left(\frac{60}{49}\right)-m>0$$
$$\therefore 0<m<\frac{120}{49}=2.45$$

(c)	Tim's winnings is based on $\mathrm{E}(G)$, which is the **long term** average score. He may still lose for some of the 2 games, but in the long run, he makes a profit.	

Qn	Suggested Solution	
6(a) (i)	Number of ways $= {}^{5}C_1 \times {}^{12}P_5 \times {}^{9}C_1 = 4276800$	
(ii)	Case (1) : 3 letters + 4 digits $= {}^{5}P_3 \times {}^{9}P_4 = 181440$ Case (2): 4 letters + 3 digits $= {}^{5}P_4 \times {}^{9}P_3 = 60480$ Total Number of ways $= 181440 + 60480$ $= 241920$	
(b)	All possible ways with 3 identical letters $= {}^{2}C_1 \times {}^{13}C_4 \times \dfrac{7!}{3!} = 1201200$ "e" or "s" Choose from 4 letters & 9 digits <u>All letters and no digit</u> Number of ways $= {}^{2}C_1 \times {}^{4}C_4 \times \dfrac{7!}{3!} = 1680$ "e" or "s" All other 4 letters chosen Since alphanumeric requires at least 1 digit, $\therefore$ Using complement, # ways with 3 identical letters and at least 1 digit $= {}^{2}C_1 \times \dfrac{7!}{3!} \times \left({}^{13}C_4 - {}^{4}C_4\right)$ $= 1199520$	

Qn	Suggested Solution	
7(a)	Probability $=\frac{1}{9}.\frac{2}{6}+\frac{3}{9}.\frac{3}{6}+\frac{5}{9}.\frac{1}{6}$ $=\frac{16}{54}=\frac{8}{27}$	
(b)	Probability $=\frac{\frac{1}{9}.\frac{2}{6}}{\frac{8}{27}}=\frac{1}{8}$	
(c)	To win \$6 in total for 3 games, each game he must win \$2. Probability $=\frac{1}{9}.\frac{2}{6}\times\frac{3}{9}.\frac{3}{6}\times\frac{5}{9}.\frac{1}{6}\times 3!$ $=\frac{540}{157464}$ $=\frac{5}{1458}$ or 0.00343 (3 s.f.)	
(d)	The participant should end the game by taking the first option because if he proceeds to throw the die, there is only a one-sixth chance that he will take home a higher amount. OR $\frac{2}{6}(0.02)+\frac{3}{6}(0.5)+\frac{1}{6}(2)=0.59$ For the throw of die, the expected factor is 0.59 which is less than 1. This means that the participant is unlikely to take home a higher amount if he were to proceed to throw the die.	

8(a)	For a player to reach point B, there must be 5 right steps and 3 up steps in total. Let R_1 be the number of right steps taken by a player out of 5 to move from X to Y. $R_1 \sim \mathrm{B}(5, p)$ $\mathrm{P}(R_1 = 3)$ $= {}^5C_3 \times p^3q^2$ $=10p^3q^2$ Let R_2 be the number of right steps taken by a player out of 3 to move from Y to B. $R_2 \sim \mathrm{B}(3, p)$	

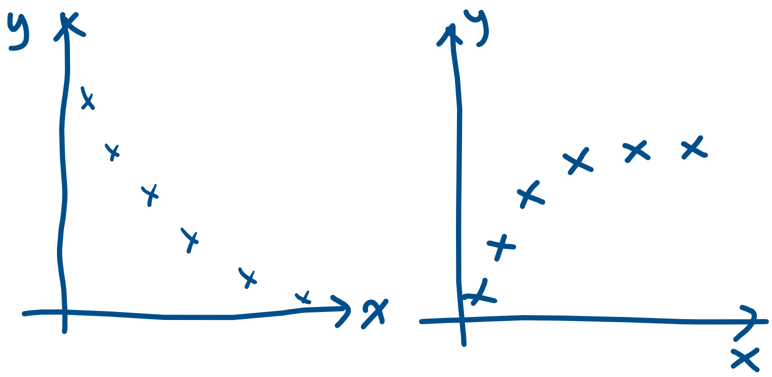
	$\mathrm{P}(R_2=2)$ $={}^3C_2\times p^2q$ $=3p^2q$ Required probability $=10p^3q^2\times 3p^2q$ $=30p^5q^3$ (shown)	
(b)	$W\sim \mathrm{B}\left(15,30\left(\frac{4}{5}\right)^5\left(\frac{1}{5}\right)^3\right)\Rightarrow W\sim \mathrm{B}(15,0.078643)$ $\mathrm{P}(W\geq 5)$ $=1-\mathrm{P}(W\leq 4)$ $=0.0046167$ $=0.00462$ (3 s.f.)	
(c)	$W\sim \mathrm{B}(15,0.078643)$ $\mathrm{E}(W)=15(0.078643)=1.1796$ $\mathrm{Var}(W)=1.1796(1-0.078643)=1.0869$ Since sample size = 40 is large, by Central Limit Theorem, $\overline{W}=\frac{W_1+...+W_{40}}{40}\sim N\left(1.1796,\ \frac{1.0869}{40}\right)$ approximately. $\mathrm{P}(\overline{W}\leq 1)=0.138$ (3 s.f.)	

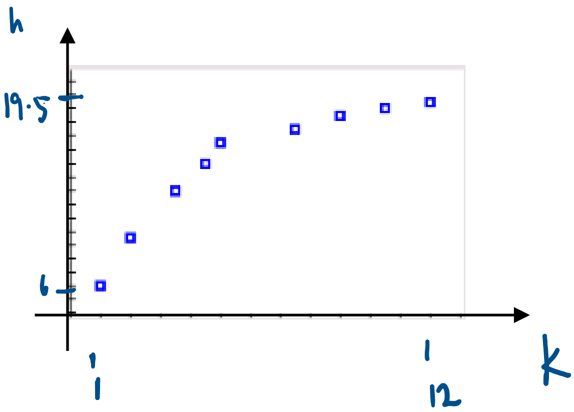
9(a)	Let F and G be the mass of a Fuji and Gala apple respectively $F\sim \mathrm{N}(205,9^2),\ \ G\sim \mathrm{N}(180,6^2)$ $F-G\sim \mathrm{N}(205-180,\ 9^2+6^2)\Rightarrow F-G\sim \mathrm{N}(25,\ 117)$ $\mathrm{P}(F>G)=\mathrm{P}(F-G>0)$ $=0.98960$ $=0.990$ (3 s.f.)	
(b)	Required probability $=\left[\mathrm{P}(F>203)\right]^2\times \mathrm{P}(F<185)\times\frac{3!}{2!}$ $=(0.58793)^2\times 0.013134\times 3$ $=0.013620$ $=0.0136$ (3 s.f.)	
(c)	Let A denotes the mass of an assorted packet of ten apples. $A=(F_1+F_2+...+F_n)+(G_1+G_2+...+G_{10-n})$ $\mathrm{E}(A)=n\mathrm{E}(F)+(10-n)\mathrm{E}(G)$	

	$= 205n + (10-n)180 = 25n + 1800$ $\text{Var}(A) = n\text{Var}(F) + (10-n)\text{Var}(G)$ $= 9^2 n + (10-n)(6^2) = 45n + 360$ $A \sim \text{N}(25n+1800,\ 45n+360)$ $\therefore A - 9F \sim \text{N}\left(25n+1800-9(205),\ 45n+360+9^2(9^2)\right)$ $\Rightarrow A - 9F \sim \text{N}(25n-45,\ 45n+6921)$ $\text{P}(A-9F > 28) \geq 0.5$ **Method 1: Standardisation** $\text{P}\left(Z > \frac{28-(25n-45)}{\sqrt{45n+6921}}\right) \geq 0.5$ $28-(25n-45) \leq 0 \quad \Rightarrow n \geq 2.93$ Least n in an assorted packet is 3. **Method 2: Graphic Calculator** From GC, $\text{P}(A-9F > 28) \geq 0.5$ When $n = 2$, $\text{P}(A-9F > 28) = 0.3918 < 0.5$ When $n = 3$, $\text{P}(A-9F > 28) = 0.5095 > 0.5$ Least n in an assorted packet is 3.	
(d)	The mass of **every** apple is independent of one another.	

Qn	Suggested Solution	
10(a)	Let X be the waiting time for a customer, in minutes. Using GC, unbiased estimate of population mean, $\bar{x} = 3.0556 = 3.06 \quad$ (3sf) $s^2 = 0.46667^2 = 0.21778 = 0.218$ (3sf)	
(b)(i)	$\text{H}_0 : \mu = 3.3$ vs $\text{H}_1 : \mu < 3.3$ Level of significance: 5% Under H_0, $Z = \frac{\overline{X} - 3.3}{\sqrt{\frac{0.22}{9}}} \sim N(0,1)$	

	$z = \frac{3.0556-3.3}{\sqrt{\frac{0.22}{9}}} = -1.56319 = -1.56$ (3sf) or p-value = **0.059004 = 0.0590** Since p-value is **0.0590** > 0.05, we **do not reject H$_0$** and conclude that there is **insufficient evidence** at 5% significance level to claim that the mean waiting time has improved. **<u>Assumption</u>** The customers' waiting time follows a **normal distribution**	
(ii)	The p-value is the lowest level of significance for which the null hypothesis that **mean waiting time being 3.3 min**, is **rejected**.	
(c)	$H_0 : \mu = k$ vs $H_1 : \mu < k$ Level of significance: 10% To reject H_0 , $z \le -1.28155$ $\frac{3.0556-k}{\sqrt{\frac{0.22}{9}}} \le -1.28155$ Solving: $k \ge 3.2559$ ie, $k \ge 3.26$ (3 sf)	

Qn	Suggested Solution	
11(a)		
(b)(i)	$r = 0.91099 = 0.911$ (3 s.f.)	

	LinReg y=a+bx a=8.215598492 b=1.101083883 r²=0.8298975784 r=0.910987145 The value of r will not change because it is not affected by scaling / r is independent of units.	
(ii)		
(iii)	A linear model is not possible as the coconut cannot grow infinitely huge. A quadratic model is not possible as the size of coconut may reach a maximum but **not decrease in size infinitely** as it matures.	
(iv)	**LinReg** y=a+bx a=6.14535613 b=5.61332019 r²=0.9841668828 r=0.9920518549 $r = 0.99205 = 0.992$ (3 s.f.) $h = 6.1454 + 5.6133 \ln k$ $h = 6.15 + 5.61 \ln k$ (3 s.f.)	
(v)	$13.7 = 6.1454 + 5.6133 \ln k$ $\ln k = 1.3458$ $k = 3.8414 = 3.8$ (1 d.p.) A coconut with diameter 13.7cm will be about 3.8 months old. The estimate is reliable since ***r* = 0.992 is close to 1** which means there is a **strong positive linear relationship between diameter and ln(age of coconut)**, and furthermore, the estimate is an **interpolation.**	
(vi)	$2.54h = 6.1454 + 5.6133 \ln k$ $h = 2.4194 + 2.20996 \ln k$ $h = 2.42 + 2.21 \ln k$ (3 s.f.)	