NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 3

MATHEMATICS 9820/01

Paper 1

20 September 2024
3 hours

Additional Materials: Answer Booklet
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and registration number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in the brackets [ ] at the end of each question or part question.

This document consists of **8** printed pages.

1 Let a, b, c, d, e and f be positive integers such that the sum $S = a+b+c+d+e+f$ divides both

$$abc + def \quad \text{and} \quad ab + bc + ca - de - ef - fd\ .$$

By considering the polynomial $\mathrm{p}(x) = (x+a)(x+b)(x+c) - (x-d)(x-e)(x-f)$, or otherwise, prove that S is composite. [6]

2 **(a)** Determine the number of solutions to the equation

$$x_1 + x_2 + x_3 = 28$$

where x_1, x_2 and x_3 are non-negative integers. [1]

(b) Determine the number of solutions to the equation

$$x_1 + x_2 + x_3 = 28$$

where x_1, x_2 and x_3 are non-negative integers and $x_1 \geq 12$. [2]

(c) Determine the number of solutions to the equation

$$x_1 + x_2 + x_3 = 28$$

where x_1, x_2 and x_3 are non-negative integers less than 12. [2]

(d) Let n, k and r be positive integers such that $k(r-1) \geq n$.

By considering the number of solutions to the equation

$$x_1 + x_2 + \ldots + x_k = n$$

where $x_1, x_2, \ldots, x_k$ are non-negative integers less than r, and using the Principle of Inclusion and Exclusion, evaluate

$$\sum_{m=0}^{\left\lfloor \frac{n}{r} \right\rfloor} (-1)^m \binom{k}{m} \binom{n-mr+k-1}{k-1}.$$ [6]

3 **(i)** Express $\dfrac{1}{u^4+1}$ in the form $\dfrac{Au+B}{u^2+Cu+1}-\dfrac{Au-B}{u^2-Cu+1}$, where A, B and C are positive constants to be determined. [5]

(ii) By substituting $u^2=\tan x$, or otherwise, evaluate $\displaystyle\int \frac{1}{\sqrt{\tan x}}\,\mathrm{d}x$. [6]

4 **(a)** A sequence of terms is defined by $x_n=\dfrac{n^2}{(2n-1)!}$.

(i) Prove that for all positive integers n, $\dfrac{x_{n+1}}{x_n}\leq\dfrac{2}{3}$. [2]

(ii) Show that $\displaystyle\sum_{n=1}^{\infty}x_n$ converges. [3]

A defining property of any convergent sequence of terms $u_1,u_2,u_3,\ldots,u_n,\ldots$ with limit L is as follows:

For any positive number k, we can always find a sufficiently large integer N such that

$$\left|u_n-L\right|<k \text{ for all } n>N.$$

In other words, the values of the terms $u_{N+1},u_{N+2},u_{N+3},\ldots$ are all bounded in the interval $(L-k,L+k)$.

(iii) Prove that if a sequence of positive terms $v_1,v_2,v_3,\ldots,v_n,\ldots$ is such that $\displaystyle\lim_{n\to\infty}\frac{v_{n+1}}{v_n}$ exists and the limit l is less than 1, then $\displaystyle\sum_{n=1}^{\infty}v_n$ converges. [4]

(b) Find exactly the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)!}=1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots$. [3]

5 **(i)** Show that if $m+1$ integers are chosen from the set $\{1, 2, 3, ..., 2m\}$, then there exist two integers among the chosen integers that satisfy the property that

(a) they are coprime, [2]

(b) one divides the other. [3]

It is also given that if $m+2$ integers are chosen from the set $\{1, 2, 3, ..., 2m+1\}$, then there exist two integers among the chosen integers that satisfy the property that one divides the other.

For integers *p* and *q* with $q \neq 0$, the fraction $\dfrac{p}{q}$ is said to be *irreducible* if $\gcd(p, q) = 1$.

Let *I* denote an open interval of length $\dfrac{1}{n}$ on the real line, where *n* is a positive integer.

(ii) Show that if $\dfrac{a}{b}$ is an irreducible fraction in *I* with $1 \leq b \leq n$, then *I* does not contain any other irreducible fraction with denominator *kb* such that $1 \leq kb \leq n$ where $k \in \mathbb{Z}$. [3]

(iii) Hence, show that *I* contains at most $\dfrac{n+1}{2}$ irreducible fractions with denominator between 1 and *n* inclusive. [3]

(iv) If the interval *I* was a closed interval instead of an open interval, would the result in part **(iii)** still hold? Justify your answer. [2]

6 Let $\mathrm{f}(x)$ be a polynomial of degree 4 with constant real coefficients.

(i) Show that an appropriate sequence of translations and scalings can transform the graph of $y = \mathrm{f}(x)$ into the graph of $y = \mathrm{g}(x)$, such that $\mathrm{g}(x) = x^4 + px^2 + qx$, where p and q are constants. [3]

(ii) Show that if the graph of $y = \mathrm{g}(x)$ has two points of inflexion, then p is negative. [2]

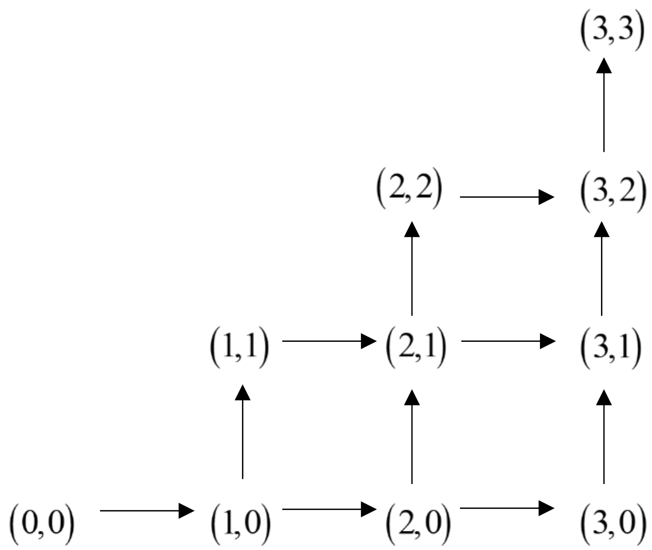
Suppose that p is negative. Let A and B be the points of inflection of $y = \mathrm{g}(x)$, with the positive and negative x-coordinates respectively.

(iii) Find the equation of the line ℓ that passes through A and B. [3]

(iv) Show that ℓ intersects the graph of $y = \mathrm{g}(x)$ in two more points, C and D, such that the order of points along ℓ is $DBAC$. [3]

(v) Show that $\dfrac{BA}{DB} = \dfrac{BA}{AC} = \dfrac{1+\sqrt{5}}{2}$. [3]

7 A network of roads consists of points with non-negative integer coordinates (i, j) where $i \geq j$. From the point (i, j), a person can travel only to the points $(i+1, j)$ and $(i, j+1)$, if they exist in the network. Part of the network is shown below.



A person starts at $(0,0)$. Let $a_{i,j}$ be the number of ways a person can travel from $(0,0)$ to (i, j), where i and j are non-negative integers and $i \geq j$. For all other values of i and j, let $a_{i,j}$ be 0.

(i) Explain why

(a) $a_{i,0} = 1$ if i is a non-negative integer ; [1]

(b) $a_{i,j} = a_{i-1,j} + a_{i,j-1}$ if $i \geq j > 0$. [2]

[continued on next page]

The outline of a variation of mathematical induction involving two variables used to prove statements of the form $\mathrm{P}_{i,j}$ for all non-negative integers *i* and *j*, where $i \geq j,$ is as follows:

- Prove that $\mathrm{P}_{i,0}$ is true for all non-negative integers *i*.
- Prove that if $P_{q,r}$ is true for some non-negative integer *r* and all integers *q* where $q \geq r$, then $P_{q,r+1}$ is true for all integers *q* where $q \geq r+1$ by doing the following:
  - Prove that $P_{r+1,r+1}$ is true.
  - Prove that if $P_{r+k,r+1}$ is true for some positive integer *k*, then $P_{r+k+1,r+1}$ is true.

(ii) Use the principle of mathematical induction to prove that

$$a_{i,j} = \frac{i-j+1}{i+1}\binom{i+j}{i}$$

for all non-negative integers *i* and *j* where $i \geq j$. [7]

(iii) Show that

(a) $a_{i,j} = \binom{i+j}{i} - \binom{i+j}{i+1}$ for $i \geq j \geq 1$; [2]

(b) $\sum_{r=0}^{n} a_{2n-r,r} = \binom{2n}{n}$. [2]

8 **(i)** Let $g, n \in \mathbb{Z}$. The integer d is called the *order* of g modulo n if d is the smallest positive integer such that $g^d \equiv 1 \pmod{n}$.

Find the

(a) order of 3 modulo 4, [1]

(b) order of 2 modulo 5. [1]

(ii) Fermat's Little Theorem states that:

If p is a prime number and a is an integer such that $\gcd(a, p) = 1$, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Let $g \in S_p = \{1, 2, \cdots, p-1\}$. By using Fermat's Little Theorem or otherwise, prove that if p is a prime number, the order of g modulo p divides $p-1$. [4]

(iii) Let p be a prime number. The number g is called a *primitive root modulo* p if for any $1 \le k \le p-1$, there exists a natural number n such that $g^n \equiv k \pmod{p}$.

(a) Verify that

A. 2 is a primitive root modulo 11, [2]

B. 2 is a primitive root modulo 13. [2]

(b) Let $2 \le g \le p-1$. Show that the set $\{g, g^2, \cdots, g^{p-1}\}$ consists of distinct elements modulo *p.* [4]

(c) Let p be a prime such that $q = \frac{1}{2}(p-1)$ is also a prime. Suppose that g is an integer that is coprime to p satisfying

$$g \not\equiv \pm 1 \pmod{p} \text{ and } g^q \not\equiv 1 \pmod{p}.$$

Prove that g is a primitive root modulo p. [5]