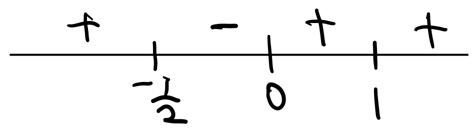
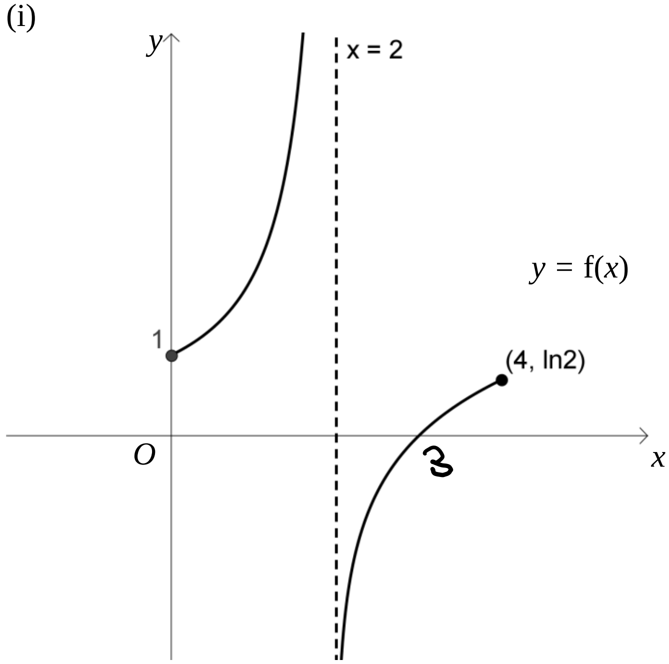
JC1 H2 Mathematics 2019 Common Test Solutions

1	Solution [5]	Markers' Comments
	Based on the given information, we have the following: $y = x + z \Rightarrow x - y + z = 0$ - - - - - (1) $29x + 21y + 124z = 295$ - - - - - - (2) $24x + 27y + 118z = 295 + 3 \Rightarrow 24x + 27y + 118z = 298$ - - (3) Using GC to solve the above system of equation, we have: $x = 3,\ y = 4$ and $z = 1$. Thus, the scores for the various units are as follow: Unit A: 298 points Unit B: 295 points Unit C: $26 \times 3 + 25 \times 4 + 121 \times 1 = 299$ points Thus, Unit C is the champion of the competition.	Generally quite well done across the level; some students tend to solve the system of linear equations manually instead of using the GC.

2	Solution [7]	
i	Method 1 $2x^2 - 2x + 1 = 2\left(x^2 - x\right) + 1$ $= 2\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}$ Since $\left(x - \frac{1}{2}\right)^2 \geq 0$, $2\left(x - \frac{1}{2}\right)^2 + \frac{1}{2} > 0$ for all real *x*. $\therefore 2x^2 - 2x + 1$ is always positive for all real *x*. ——— Method 2 Since discriminant = $(-2)^2 - 4(2)(1) = -4 < 0$ and coefficient of x^2 is positive, $\therefore 2x^2 - 2x + 1$ is always positive for all real *x*.	For method 1, many students made mistakes in their argument, e.g. writing $\left(x - \frac{1}{2}\right)^2 > 0$ instead of $\left(x - \frac{1}{2}\right)^2 \geq 0$ etc. Some completed the square wrongly. For method 2, a complete argument needs to have both "discriminant < 0" and "coefficient of x^2 > 0". Having just the former might imply the quadratic expression is always negative instead of positive.

ii	$\dfrac{2x^3-2x^2+x}{(2x+1)(x-1)^2} \geq 0, \quad x \neq -\dfrac{1}{2},\ x \neq 1$ $\dfrac{x\left(2x^2-2x+1\right)}{(2x+1)(x-1)^2} \geq 0$ Since $2x^2-2x+1>0$ for all real x, $\dfrac{x}{(2x+1)(x-1)^2} \geq 0$ Using number line,  $x<-\dfrac{1}{2}$ or $x \geq 0$, $x \neq 1$	It would be good to note from the start that $x \neq -\dfrac{1}{2}$ and $x \neq 1$. Students need to show proper working and explain why $\left(2x^2-2x+1\right)$ need not be considered in the number-line investigation. Students need to take note of when "=" is included. Some students used other analytical approaches which considered cases how each sub-expression's sign should be etc. It is tedious to do that and students have to take care of many details in the argument.
iii	$\dfrac{2(\ln x)^3-2(\ln x)^2+\ln x}{(2\ln x+1)(1-\ln x)^2} \geq 0$ Replace x by $\ln x$: $\ln x<-\dfrac{1}{2}$ or $\ln x \geq 0$, $\ln x \neq 1$ $0<x<\mathrm{e}^{-\frac{1}{2}}$ or $x \geq 1$, $x \neq \mathrm{e}$	Almost all students know to make use of part (ii)'s answer and replace x with ln$x$. Quite many students thought that $\ln x<-\dfrac{1}{2}$ cannot be solved (that ln$x$ has to be >0). For those who attempted solving it, many did not realise that " $x>0$ ", else ln$x$ is undefined.

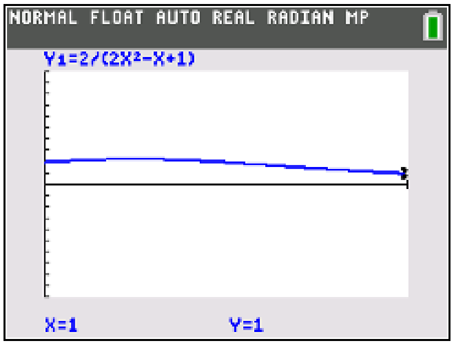
3	Solution [7]	
	(i)  Note: the point (0,1) should be above the point (4, ln2)	Generally quite badly done. Some common mistakes are: • Missing axial intercepts; • Missing ending points (need to be obvious) indicated and labeled; • Need to know that $\ln 2 < 1$ and hence there is a need to be indicated correctly on the graph
	(ii) $R_{\mathrm{g}} = \left(0,\ \frac{9}{8}\right]$ Since $R_{\mathrm{g}} = \left(0, \frac{9}{8}\right] \subset [0,4] \setminus \{2\} = D_{\mathrm{f}}$, The composite function fg exist.	Very badly attempted as students did not look at the graph of g to determine the range. The correct domain of f should also be $[0,2) \cup (2,4]$. A lot of students did not read the question carefully to determine the domain of f.
	(iii) $\mathrm{f}(0) = \frac{2}{2-0} = 1,\ \ \mathrm{f}\left(\frac{9}{8}\right) = \frac{2}{2-\frac{9}{8}} = \frac{16}{7}$ By mapping method, $[0,1) \xrightarrow{\mathrm{g}} \left(0, \frac{9}{8}\right] \xrightarrow{\mathrm{f}} \left(1, \frac{16}{7}\right]$ $R_{\mathrm{fg}} = \left(1, \frac{16}{7}\right]$ <u>Alternatively,</u>	Quite a number of students simply ignore all workings and simply write out the wrong answer and this resulted in a lot of marks lost. Students should show clear workings or thoughts. Some common errors include $R_{\mathrm{fg}} = R_{\mathrm{g}}$ or $R_{\mathrm{fg}} = R_{\mathrm{f}}$

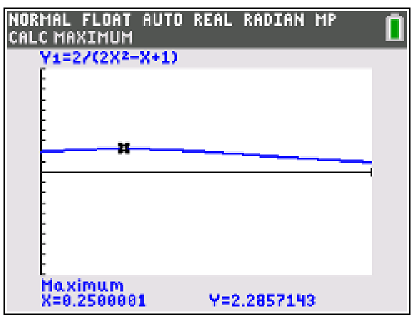
$$\mathrm{f\,g}(x) = \mathrm{f}\left(-2x^2 + x + 1\right)$$

$$= \frac{2}{2 - \left(-2x^2 + x + 1\right)} \quad \left(\text{since } R_{\mathrm{g}} = \left(0, \frac{9}{8}\right]\right)$$

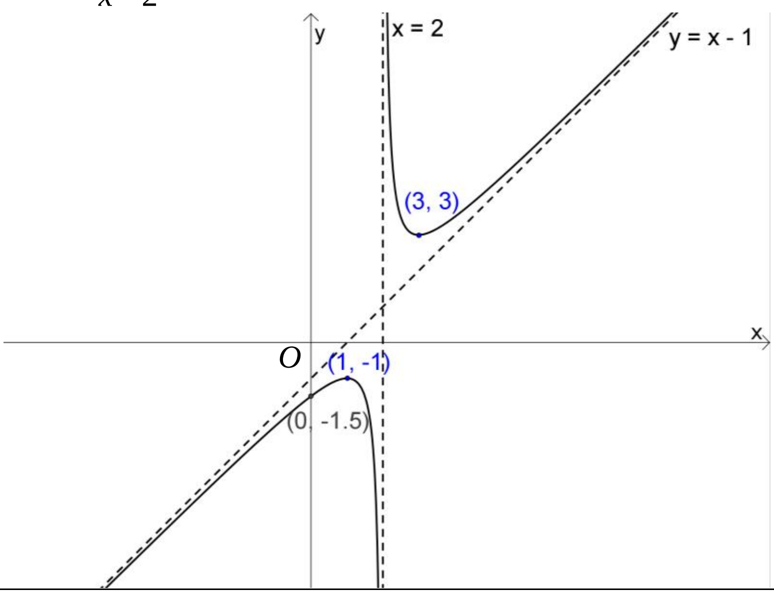
$$= \frac{2}{2x^2 - x + 1}$$

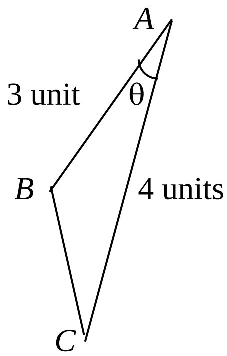
$D_{\mathrm{fg}} = D_{\mathrm{g}} = [0,1)$

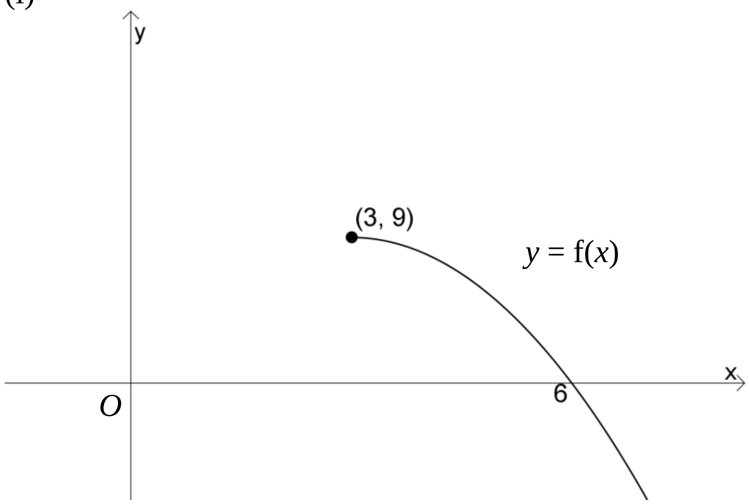




$R_{\mathrm{fg}} = (1, 2.29]$ (to 3 s.f.)

4	Solution [8]	
i	Since C has an oblique asymptote at $y=x-1$ with coefficient of *x* equals to 1, $a = 1$ Since C has a vertical asymptote at $x=2$, $c=2$ Since C passes through the point $P\left(0,-\frac{3}{2}\right)$, $-\frac{3}{2}=\frac{b}{-2}\Rightarrow b=3$ (shown)	Most students largely have an issue with explaining the reason why $a = 1$. Many left their answer as 'Because $y=x-1$ is the asymptote, therefore $a = 1$."
ii	$y=\frac{x^2-3x+3}{x-2}$ 	
iii	$y=\frac{x^2-3x+3}{x-2}=x-1+\frac{1}{x-2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{1}{(x-2)^2}$ When $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{4}$ The equation of normal is thus, $y-\left(-\frac{3}{2}\right)=-\frac{1}{\frac{3}{4}}(x-0)$ $\therefore y=-\frac{4}{3}x-\frac{3}{2}$	

5	Solution [7]	
	 Applying cosine rule: $BC^2 = 3^2 + 4^2 - 2\times 3\times 4\cos\theta$ $\approx 25 - 24\left(1 - \frac{\theta^2}{2}\right)$, since θ is small $= 1 + 12\theta^2$ $\therefore BC \approx \sqrt{1+12\theta^2}$ (shown) Then $BC \approx \left(1+12\theta^2\right)^{\frac{1}{2}}$ $= 1 + \frac{1}{2}\left(12\theta^2\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(12\theta^2\right)^2 + ...$ $\approx 1 + 6\theta^2 - 18\theta^4$ (up to θ^4 term) where $p = 6,\ \ q = -18$. (shown)	Most students are aware that they need to use cosine rule and small angle approximation, and were able to show the correct result, except for some who fumbled with applying binomial expansion to obtain the correct final answer. Overall a huge majority lost presentation mark in this question due to the careless use of "=" and "$\approx$" interchangeably.
	Sub $\theta = \frac{1}{4}$ into the above result: $\sqrt{1+12\left(\frac{1}{4}\right)^2} \approx 1 + 6\left(\frac{1}{4}\right)^2 - 18\left(\frac{1}{4}\right)^4$ $\sqrt{\frac{7}{4}} \approx 1 + 6\left(\frac{1}{4}\right)^2 - 18\left(\frac{1}{4}\right)^4$ Thus, $\sqrt{7} \approx \sqrt{4}\times\left(1 + 6\left(\frac{1}{4}\right)^2 - 18\left(\frac{1}{4}\right)^4\right) = \frac{167}{64}$	This part was not well done with a significant number of students only substituting ¼ to the LHS of the expression, some did not obtain the correct answer due to the error in the previous part.
	As $\frac{1}{10}$ is closer to 0 compared to $\frac{1}{4}$, the substitution $\theta = \frac{1}{10}$ will give a better estimate of the value of $\sqrt{7}$.	There are many who did not provide proper explanation for this part e.g. some used "closer to 1" and some others gave the correct reason but stated that ¼ is better

<table>
<tr><td>6</td><td>Solution [10]</td><td></td></tr>
<tr><td></td><td>(i)

When $a \geq 3$, any horizontal line $y = k$ where $k \in \mathbb{R}$ will intersect the graph of $y = \mathrm{f}\left(x\right)$, where $x \in \left[a, \infty\right)$, at most once. Thus f is one-to-one and f^{-1} exist.
Also, the turning point $y = 6x - x^2$ occurs when $x = 3$.
$\therefore$ Least value of a is 3.</td><td>Most students were able to sketch the curve with the domain [3, ∞), with some not labelling the max pt and x-intercept.

For the 2nd part, students need to mention about the max pt (3, 9). Otherwise it is not sufficient to explain that the function is not 1-1 for $a < 3$ as we still need to ascertain that the function is 1-1 for $a \geq 3$. Also, attention should be paid to the correct use of term like ‘at most 1 point’ or ‘more than 1 point’ in the explanation of horizontal line test for 1-1 property.</td></tr>
<tr><td></td><td>(ii)
Let $y = 6x - x^2$
$y = -\left(x^2 - 6x\right)$
$= -\left(x^2 - 6x + 3^2 - 3^2\right)$
$= -\left(x - 3\right)^2 + 9$
$\left(x - 3\right)^2 = 9 - y$
$x - 3 = \pm\sqrt{9 - y}$
$x = 3 \pm \sqrt{9 - y}$
Since $x \geq 3$, $x = 3 + \sqrt{9 - y}$
$D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = \left(-\infty, 9\right]$
$\therefore \mathrm{f}^{-1} : x \mapsto 3 + \sqrt{9 - x}, x \leq 9$</td><td>Some students were not able to perform completing square or use the quadratic root formula in trying to express x in terms of y.

Some students also did not explain on the rejection of the expression $x = 3 - \sqrt{9 - y}$.

Other errors include not expressing the answer for f^{-1} in similar form and not stating the domain for f^{-1}.</td></tr>
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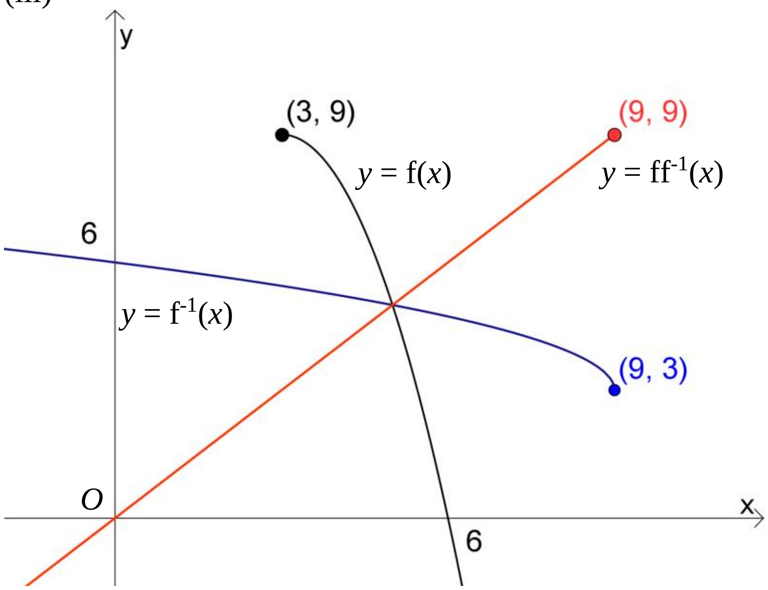
<td>(iii)

</td>
<td>Many students did not take note of the domain of all 3 graphs for this part while sketching. There were many instances where students did not label the end points and relevant axial intercepts

Also there should be equal scaling on both axes so as to exhibit the symmetrical effect of the both $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ about the line $y = \mathrm{ff}^{-1}(x) = x$.

Very few commendable students displayed good understanding of the fact that domain of $y = \mathrm{ff}^{-1}(x)$ = domain of $y = \mathrm{f}^{-1}(x)$ = $(-\infty, 9]$ and marked the end point (9, 9) for $y = \mathrm{ff}^{-1}(x)$.</td>
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<tr>
<td>7</td>
<td>Solution [9]</td>
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$$u_n - u_{n+1}$$
$$= \frac{2n-1}{n(n-1)} - \frac{2n+1}{(n+1)(n)}$$
$$= \frac{(2n-1)(n+1)-(2n+1)(n-1)}{n(n-1)(n+1)}$$
$$= \frac{2n^2+n-1-(2n^2-n-1)}{n(n-1)(n+1)}$$
$$= \frac{2n}{n(n-1)(n+1)}$$
$$= \frac{2}{(n-1)(n+1)} \quad \text{(shown)}$$

</td>
<td>No problem for this part except for very few algebraic manipulation errors.</td>
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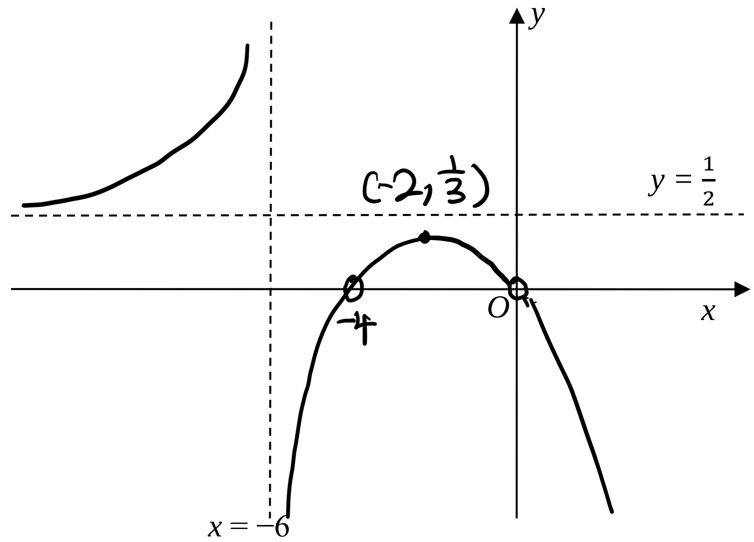
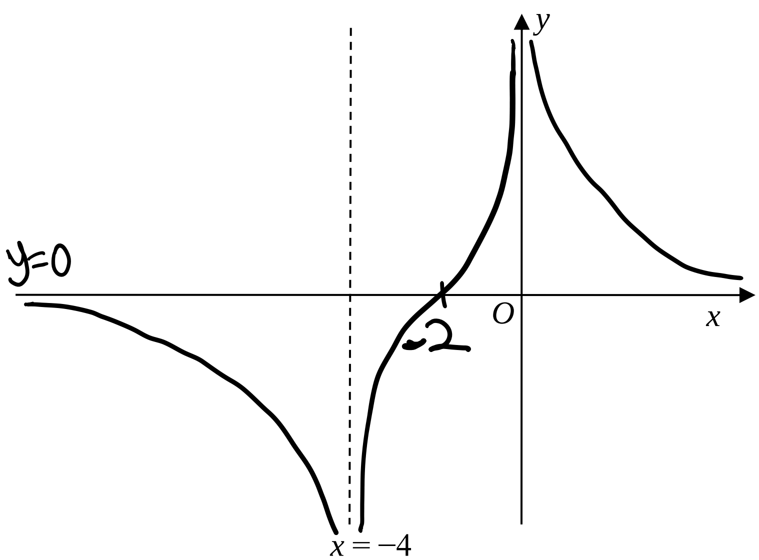
i	$\sum_{r=2}^{N}\frac{1}{(r+1)(r-1)}$ $=\frac{1}{2}\sum_{r=2}^{N}\frac{2}{(r+1)(r-1)}$ $=\frac{1}{2}\sum_{r=2}^{N}(u_r-u_{r+1})$ $=\frac{1}{2}\begin{bmatrix}u_2-\cancel{u_3}+\\ \cancel{u_3}-\cancel{u_4}+\\ \vdots\\ \cancel{u_N}-u_{N+1}\end{bmatrix}=\frac{1}{2}(u_2-u_{N+1})$ $=\frac{1}{2}\left(\frac{2(2)-1}{2(2-1)}-\frac{2(N+1)-1}{(N+1)(N+1-1)}\right)$ $=\frac{1}{2}\left(\frac{3}{2}-\frac{2N+1}{N(N+1)}\right)$	Most students were able to answer this part perfectly well. However some students did not follow the requirement of question ie to use the fact $\sum_{r=2}^{N}\frac{1}{(r+1)(r-1)}$ $==\frac{1}{2}\sum_{r=2}^{N}(u_r-u_{r+1})$ from the earlier part and perform method of difference. Instead they attempted to express $\frac{1}{(r+1)(r-1)}$ $=\frac{\frac{1}{2}}{r-1}-\frac{\frac{1}{2}}{r+1}$ and applied MOD which is not correct.
ii	Since $\frac{2N+1}{N(N+1)}=\frac{\frac{2}{N}+\frac{1}{N^2}}{1+\frac{1}{N}}\to 0$ as $N\to\infty$, the series converges and $\sum_{r=2}^{\infty}\frac{1}{(r+1)(r-1)}=\frac{1}{2}\left(\frac{3}{2}-0\right)=\frac{3}{4}$	Many students failed to explain that $\frac{2N+1}{N(N+1)}\to 0$ as $N\to\infty$. Instead arguing that term $\frac{1}{(r+1)(r-1)}\to 0$ is NOT sufficient. Recall the non- convergent series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+....$
iii	$\sum_{r=2}^{N-1}\frac{1}{(r+2)(r)}$ (replace r by $r-1$) $=\sum_{r-1=2}^{r-1=N-1}\frac{1}{(r-1+2)(r-1)}$ -----------(*) $=\sum_{r=3}^{N}\frac{1}{(r+1)(r-1)}=\sum_{r=2}^{N}\frac{1}{(r+1)(r-1)}-\frac{1}{(3)(1)}$ $=\frac{1}{2}\left(\frac{3}{2}-\frac{2N+1}{N(N+1)}\right)-\frac{1}{3}$ $=\frac{5}{12}-\frac{2N+1}{2N(N+1)}$	Students need to show the manipulation steps involving summation sign clearly beyond just mentioning ‘replace replace *r* by $r-1$’ or ‘replace *r* by $r+1$’ Students are supposed to use the FINAL result of (i) instead of the MOD intermediate steps or even to repeat the whole MOD process in trying to find answer to expression (*).

8	Solution [8]	
	(a)(i) $y=\frac{1}{\mathrm{f}(x)}$ 	Some common mistakes include: • Turning point is $(-2,3)$ or even $\left(-2,\frac{3}{2}\right)$ Students should not draw vertical or horizontal dotted lines but not using them as asymptotes.
	(a)(ii) $y=\mathrm{f}'(x)$ 	
	(b) $x=t,\quad y=2t^2$ Scale parallel to x-axis by factor 2, (i.e. $x=2t,\quad y=2t^2$) Translate 1 unit in the positive x-direction, (i.e. $x=2t+1,\quad y=2t^2$) Scale parallel to y-axis by factor $\frac{1}{2}$. (i.e. $x=2t+1,\ y=t^2$)	

<table>
<tr><td>9</td><td>Solution [12]</td><td></td></tr>
<tr><td>i</td><td>$u_1 = 1.005(20000) - 450 = 19650$
$u_2 = 1.005(19650) - 450 = 19298.25$
$u_3 = 1.005(19298.25) - 450 = 18944.74$</td><td>Some students do not know how to make use of the recurrence relation.
Many did not give answers to nearest cents (2 d.p.).</td></tr>
<tr><td>ii</td><td>$$\begin{aligned} u_n &= 1.005u_{n-1} - 450 \\ &= 1.005\left(1.005u_{n-2} - 450\right) - 450 \\ &= 1.005^2 u_{n-2} - 1.005\left(450\right) - 450 \\ &= 1.005^2\left(1.005u_{n-3} - 450\right) - 1.005\left(450\right) - 450 \\ &= 1.005^3 u_{n-3} - 1.005^2(450) - 1.005(450) - 450 \\ &\ \vdots \\ &= 1.005^n u_0 - 1.005^{n-1}(450) - 1.005^{n-2}(450) - \ldots - 450 \\ &= 1.005^n\left(20000\right) - 450\left(1 + 1.005 + \ldots + 1.005^{n-1}\right) \\ &= 1.005^n\left(20000\right) - 450\left(\frac{1.005^n - 1}{1.005 - 1}\right) \\ &= 90000 - 1.005^n\left(70000\right) \quad \text{(shown)} \end{aligned}$$</td><td>Quite many students do not know how to attempt this question.
Some students managed to get the answers (the solutions are mathematically sound) but the approach of formulating the series did not meet the requirement of question. Students need to apply recurrence relation to the expression already given in the box.

Those who can manage the question come up with very nice and succinct solutions.</td></tr>
<tr><td>iii</td><td>The loan will eventually by paid off when $90000 - 1.005^n\left(70000\right) \le 0$
$9 \le 1.005^n\left(7\right)$
$1.005^n \ge \frac{9}{7} \Rightarrow n \ge \frac{\ln\frac{9}{7}}{\ln 1.005} = 50.388$

<u>Alternatively,</u>
<table><tr><td>n</td><td>$u_n = 90000 - 1.005^n\left(70000\right)$</td></tr><tr><td>50</td><td>174.19 > 0</td></tr><tr><td>51</td><td>-274.90 < 0</td></tr></table>
$\therefore 51$ months were required to pay off the loan.

$u_{50} = 90000 - 1.005^{50}\left(70000\right)$
$= \$174.19$
Hence, the final repayment amount
$= \$174.19 \times 1.005 = \$175.06$</td><td>Students need to realise that even if they are stuck in part (ii), they can use its result to continue solving for this question part.

If students did not use inequality sign, they need to <u>explain</u> why answer is 51 and not 50 (look at the alternative approach).

Among the students who attempted the last part, only a handful realised the need to multiply u_{50} by 1.005 to get \$175.06.</td></tr>
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<tr><td>10</td><td>Solution [12]</td><td></td></tr>
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(i) Given $x = t - \frac{1}{t} + 2$; $y = t + \frac{1}{t}$,

$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \frac{1}{t^2}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - \frac{1}{t^2}$

Then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{1-\frac{1}{t^2}}{1+\frac{1}{t^2}}$$

$$= \frac{t^2 - 1}{t^2 + 1}$$

</td><td>This part was mostly quite well done.</td></tr>
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(ii) For stationary point on curve, we let $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$

Then, $\frac{t^2 - 1}{t^2 + 1} = 0 \Rightarrow t^2 = 1 \Rightarrow t = 1$ (since $t \geq 0.5$)

When $t = 1$,
$x = 1 - 1 + 2 = 2$ and

$y = 1 + 1 = 2$
Thus the stationary point is (2, 2)

Next, we perform sign test to determine its nature:

<table>
<tr><td>x</td><td>1.995</td><td>2</td><td>2.005</td></tr>
<tr><td>t</td><td>0.9975031</td><td>1</td><td>1.0025031</td></tr>
<tr><td>$\frac{dy}{dx}$</td><td>$-0.00250 < 0$</td><td>0</td><td>$0.00250 > 0$</td></tr>
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OR

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$$

$$= \frac{\left(t^2+1\right)(2t) - \left(t^2-1\right)(2t)}{\left(t^2+1\right)^2} \cdot \frac{t^2}{1+t^2}$$

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{t=1} = \frac{1}{2} > 0$$

Thus, we conclude that (2, 2) is a minimum point.

</td><td>

Most students were able to obtain $t = 1$.

Some students incorrectly solved $t^2 + 1 = 0$ as well which should not be the case.

Some students also forgot to find the coordinates of the point.

The major error in this part was in the determining of nature, majority of students did not perform the first/second derivative test correctly. For first derivative test, there is a need to consider small increments in ***x***, find the corresponding *t* and then find the corresponding dy/dx. For second derivative test, many assumed that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ which is not true.

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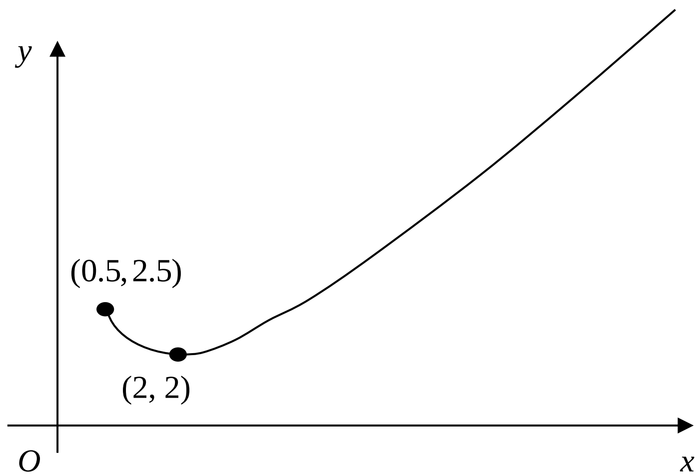
<td>(iii) We first note that for $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{t^2-1}{t^2+1}=1-\dfrac{2}{t^2+1}$,
As $t\to+\infty$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}\to 1$.
Thus, for large value of t, the gradient of the path tends to 1.

Hence a detailed sketch of the curve is as follow:
(Note: when $t\to+\infty$, $x\to+\infty$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}\to 1$)

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<td>While many gave the correct conclusion that gradient tends to 1, there were a number who used incorrect mathematical arguments to explain why.

Many students failed to realize that the previous part was meant to be a hint for sketching of the correct graph, in that the long term behavior of the graph involved gradient tending to 1. Thus many students failed to sketch the correct shape of the graph.
Others did not label the starting point and min point on the graph.
Note: The asymptote of this graph is $y = x + 2$ and not $y = x$ (even though it is not required in this question)</td>
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<td>(iv)</td>
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