H2 MATHEMATICS 9758
Topic 12: INTEGRATION TECHNIQUES
TUTORIAL WORKSHEET

Section 1: Discussion Questions (Students are to attempt all these questions.)

1 Integrands in the form $f'(x)\left[f(x)\right]^n$

(a) $\displaystyle\int \frac{x^2-1}{\sqrt{x^3-3x}}\,dx$ $\boxed{\frac{2}{3}\left(x^3-3x\right)^{\frac{1}{2}}+c}$

$$\int \frac{x^2-1}{\sqrt{x^3-3x}}\,dx = \frac{1}{3}\int \frac{3(x^2-1)}{\sqrt{x^3-3x}}\,dx = \frac{1}{3}\frac{\left(x^3-3x\right)^{\frac{1}{2}}}{\frac{1}{2}}+c = \frac{2}{3}\left(x^3-3x\right)^{\frac{1}{2}}+c$$

(b) $\displaystyle\int \left(x^2-1\right)\left(x^3-3x\right)^5\,dx$ $\boxed{\frac{1}{18}\left(x^3-3x\right)^6+c}$

$$\int \left(x^2-1\right)\left(x^3-3x\right)^5\,dx = \frac{1}{3}\int 3\left(x^2-1\right)\left(x^3-3x\right)^5\,dx = \frac{1}{18}\left(x^3-3x\right)^6+c$$

(c) $\displaystyle\int \sin x\cos^5 x\,dx$ $\boxed{-\frac{\cos^6 x}{6}+c}$

$$\int \sin x\cos^5 x\,dx = -\int (-\sin x)\cos^5 x\,dx = -\frac{\cos^6 x}{6}+c$$

(d) $\displaystyle\int \sec^5 x\tan x\,dx$ $\boxed{\frac{\sec^5 x}{5}+c}$

$$\int \sec^5 x\tan x\,dx = \int (\sec x\tan x)\sec^4 x\,dx = \frac{\sec^5 x}{5}+c$$

(e) $\displaystyle\int \cos(3x)\sin^3(3x)\,dx$ $\boxed{\frac{\sin^4(3x)}{12}+c}$

$$\int \cos(3x)\sin^3(3x)\,dx = \frac{1}{3}\int 3\cos(3x)\sin^3(3x)\,dx = \frac{1}{3}\frac{\sin^4(3x)}{4}+c = \frac{1}{12}\sin^4(3x)+c$$

2 Integrands in the form $\dfrac{f'(x)}{f(x)}$

(a) $\displaystyle\int \frac{1}{\sqrt{x}(1-\sqrt{x})}\,dx$ $\boxed{-2\ln\left|1-\sqrt{x}\right|+c}$

$$\int \frac{1}{\sqrt{x}(1-\sqrt{x})}\ \mathrm{d}x = -2\int \frac{1}{-2\sqrt{x}}\frac{1}{(1-\sqrt{x})}\ \mathrm{d}x = -2\ln\left|1-\sqrt{x}\right| + c$$

(b) $\int \frac{x}{x+3}\,\mathrm{d}x$ $\boxed{x-3\ln\left|x+3\right|+c}$

$$\int \frac{x}{x+3}\mathrm{d}x = \int \frac{x+3-3}{x+3}\mathrm{d}x = \int\left(1-\frac{3}{x+3}\right)\mathrm{d}x = x-3\ln\left|x+3\right|+c$$

(c) $\int \cot\theta\,\mathrm{d}\theta$ $\boxed{\ln\left|\sin\theta\right|+c}$

$$\int \cot\theta\,\mathrm{d}\theta = \int \frac{\cos\theta}{\sin\theta}\,\mathrm{d}\theta = \ln\left|\sin\theta\right|+c$$

3 Integrands requiring use of trigonometric identities

(a) $\int \sin^2 3x \, dx$ $\quad \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + c$

$$\int \sin^2 3x \, dx = \frac{1}{2}\int (1 - \cos 6x) \, dx = \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + c$$

(b) $\int \sin\frac{x}{2}\cos\frac{x}{2} \, dx$ $\quad -\frac{1}{2}\cos x + c$

$$\int \sin\frac{x}{2}\cos\frac{x}{2} \, dx = \frac{1}{2}\int \sin x \, dx = -\frac{1}{2}\cos x + c$$

(c) $\int \frac{1}{1-\cos 2x} \, dx$ $\quad -\frac{1}{2}\cot x + c$

$$\int \frac{1}{1-\cos 2x} \, dx = \int \frac{1}{2\sin^2 x} \, dx = \frac{1}{2}\int \operatorname{cosec}^2 x \, dx = -\frac{1}{2}\cot x + c$$

(d) $\int \frac{1}{1+\cos x} \, dx$ $\quad \tan\frac{1}{2}x + c$

$$\int \frac{1}{1+\cos x} \, dx = \int \frac{1}{2\cos^2 \frac{x}{2}} \, dx = \int \frac{1}{2}\sec^2 \tfrac{x}{2} \, dx = \tan\frac{x}{2} + c$$

(e) $\int \sin\frac{5x}{2}\cos\frac{3x}{2} \, dx$ $\quad -\frac{1}{8}\cos 4x - \frac{1}{2}\cos x + c$

$$\int \sin\frac{5x}{2}\cos\frac{3x}{2} \, dx$$
$$= \int \frac{1}{2}(\sin 4x + \sin x) \, dx = \frac{1}{2}\left(-\frac{\cos 4x}{4} - \cos x\right) + c = -\frac{1}{8}(\cos 4x + 4\cos x) + c$$

(f) $\int \cos^3 x \, dx$ $\quad \sin x - \frac{\sin^3 x}{3} + c$

$$\int \cos^3 x \, dx = \int \cos x(1 - \sin^2 x) \, dx = \int \cos x - \cos x \sin^2 x) \, dx = \sin x - \frac{\sin^3 x}{3} + c$$

OR

$$= \int \cos x(1 - \sin^2 x) \, dx = \int \cos x(\cos 2x + \sin^2 x) \, dx = \frac{1}{2}\int 2\cos x \cos 2x \, dx + \int \cos x \sin^2 x \, dx$$
$$= \frac{1}{2}\int (\cos 3x + \cos x) \, dx + \int \cos x \sin^2 x \, dx = \frac{1}{6}\sin 3x + \frac{1}{2}\sin x + \frac{1}{2}\sin^3 x + c$$

(g) $\int_{\frac{\pi}{4a}}^{\frac{\pi}{2a}} \cos\left(\frac{a\theta}{2}\right)\cos\left(\frac{3a\theta}{2}\right) d\theta$ $\quad \frac{1}{4a}\left(1 - \sqrt{2}\right)$

$$\int_{\frac{\pi}{4a}}^{\frac{\pi}{2a}} \cos\left(\frac{a\theta}{2}\right)\cos\left(\frac{3a\theta}{2}\right)\mathrm{d}\theta = \frac{1}{2}\int_{\frac{\pi}{4a}}^{\frac{\pi}{2a}} \cos(2a\theta)+\cos(a\theta)\,\mathrm{d}\theta = \frac{1}{2}\left[\frac{\sin 2a\theta}{2a}+\frac{\sin a\theta}{a}\right]_{\frac{\pi}{4a}}^{\frac{\pi}{2a}}$$

$$= \frac{1}{2a}\left[(0+1)-(\frac{1}{2}+\frac{1}{\sqrt{2}})\right] = \frac{1}{4a}\left[1-\sqrt{2}\right]$$

(h) $\int_0^1 \sin(\pi+1)\theta\sin(\pi-1)\theta\,\mathrm{d}\theta$ $\boxed{\frac{\sin 2}{4}}$

$$\int_0^1 \sin(\pi+1)\theta\sin(\pi-1)\theta\,\mathrm{d}\theta = -\frac{1}{2}\int_0^1 \cos(2\pi\theta)-\cos(2\theta)\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\left[\frac{\sin 2\pi\theta}{2\pi}-\frac{\sin 2\theta}{2}\right]_0^1 = -\frac{1}{2}\left[\left(0-\frac{\sin 2}{2}\right)-0\right] = \frac{\sin 2}{4}$$

4 Integrands requiring long division and/or partial fractions

(a) $\int \frac{1}{x^2+2x}\,\mathrm{d}x$ $\boxed{\frac{1}{2}\ln|x|-\frac{1}{2}\ln|x+2|+c}$

$$\int \frac{1}{x^2+2x}\,\mathrm{d}x = \int \frac{1}{x(x+2)}\,\mathrm{d}x = \frac{1}{2}\int\left(\frac{1}{x}-\frac{1}{x+2}\right)\mathrm{d}x = \frac{1}{2}\left(\ln|x|-\ln|x+2|\right)+c$$

(b) $\int \frac{1}{2x^2-12x-14}\,\mathrm{d}x$ $\boxed{\frac{1}{16}\left[\ln|x-7|-\ln|x+1|\right]+c}$

$$\int \frac{1}{2x^2-12x-14}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{x^2-6x-7}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{(x-7)(x+1)}\,\mathrm{d}x = \frac{1}{16}\int\left(\frac{1}{x-7}-\frac{1}{x+1}\right)\mathrm{d}x$$

$$= \frac{1}{16}\left(\ln|x-7|-\ln|x+1|\right)+c$$

(c) $\int \frac{x^3+2}{x^2-1}\,\mathrm{d}x$ $\boxed{\frac{1}{2}x^2-\frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x-1|+c}$

$$\int \frac{x^3+2}{x^2-1}\,\mathrm{d}x = \int \frac{x(x^2-1)+x+2}{x^2-1}\,\mathrm{d}x = \int x+\frac{x+2}{(x+1)(x-1)}\,\mathrm{d}x = \int x+\frac{-\frac{1}{2}}{x+1}+\frac{\frac{3}{2}}{x-1}\,\mathrm{d}x$$

$$= \frac{x^2}{2}-\frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x-1|+c$$

(d) $\int \frac{2x^2-x+9}{(x+1)(x-3)^2}\,\mathrm{d}x$ $\boxed{\frac{3}{4}\ln|x+1|+\frac{5}{4}\ln|x-3|-\frac{6}{x-3}+c}$

$$\int \frac{2x^2-x+9}{(x+1)(x-3)^2}\,\mathrm{d}x = \int \frac{\frac{3}{4}}{x+1}+\frac{\frac{5}{4}}{x-3}+\frac{6}{(x-3)^2}\,\mathrm{d}x = \frac{3}{4}\ln|x+1|+\frac{5}{4}\ln|x-3|-\frac{6}{x-3}+c$$

5 Integrands in the form $\dfrac{1}{\sqrt{a^2-x^2}}, \dfrac{1}{a^2+x^2}, \dfrac{1}{a^2-x^2}$

(a) $\displaystyle\int \frac{1}{\sqrt{3-4x^2}}\ \mathrm{d}x$ $\boxed{\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)+c}$

$$\int \frac{1}{\sqrt{3-4x^2}}\ \mathrm{d}x = \frac{1}{2}\int \frac{2}{\sqrt{\left(\sqrt{3}\right)^2-(2x)^2}}\ \mathrm{d}x = \frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{3}}+c$$

(b) $\displaystyle\int \frac{1}{4+81x^2}\ \mathrm{d}x$ $\boxed{\frac{1}{18}\tan^{-1}\left(\frac{9x}{2}\right)+c}$

$$\int \frac{1}{4+81x^2}\ \mathrm{d}x = \frac{1}{9}\int \frac{9}{2^2+(9x)^2}\ \mathrm{d}x = \frac{1}{9}\cdot\frac{1}{2}\tan^{-1}\frac{9x}{2}+c = \frac{1}{18}\tan^{-1}\frac{9x}{2}+c$$

(c) $\displaystyle\int \frac{1}{\sqrt{4-(x-3)^2}}\ \mathrm{d}x$ $\boxed{\sin^{-1}\left(\frac{x-3}{2}\right)+c}$

$$\int \frac{1}{\sqrt{4-(x-3)^2}}\ \mathrm{d}x = \sin^{-1}\frac{x-3}{2}+c$$

(d) $\displaystyle\int \frac{1}{x^2+2x+3}\ \mathrm{d}x$ $\boxed{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c}$

$$\int \frac{1}{x^2+2x+3}\ \mathrm{d}x = \int \frac{1}{(x+1)^2+2}\ \mathrm{d}x = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}+c$$

(e) $\displaystyle\int \frac{1}{\sqrt{6-4x-x^2}}\mathrm{d}x$ $\boxed{\sin^{-1}\left(\frac{x+2}{\sqrt{10}}\right)+c}$

$$\int \frac{1}{\sqrt{6-4x-x^2}}\mathrm{d}x$$
$$= \int \frac{1}{\sqrt{-(x^2+4x-6)}}\mathrm{d}x = \int \frac{1}{\sqrt{-\left[(x-2)^2-10\right]}}\mathrm{d}x = \int \frac{1}{\sqrt{10-(x+2)^2}}\mathrm{d}x = \sin^{-1}\frac{x+2}{\sqrt{10}}+c$$

6 Integrands that can be split into different standard forms

(a) $\displaystyle\int \frac{x+2}{x^2+2x+5}\mathrm{d}x$ $\boxed{\frac{1}{2}\ln\left|x^2+2x+5\right|+\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C}$

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$$\int \frac{x+2}{x^2+2x+5}\,\mathrm{d}x$$
$$=\frac{1}{2}\int\left(\frac{2x+2}{x^2+2x+5}+\frac{2}{x^2+2x+5}\right)\mathrm{d}x$$
$$=\frac{1}{2}\int\left(\frac{2x+2}{x^2+2x+5}+\frac{2}{(x+1)^2+4}\right)\mathrm{d}x$$
$$=\frac{1}{2}\ln\left|x^2+2x+5\right|+\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

(b) $\displaystyle\int \frac{x^2}{x^2+2x+5}\,\mathrm{d}x$ $\boxed{x-\ln\left|x^2+2x+5\right|-\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C}$

$$\int \frac{x^2}{x^2+2x+5}\,\mathrm{d}x=\int\left(1-\frac{2x+2}{x^2+2x+5}-\frac{3}{x^2+2x+5}\right)\mathrm{d}x$$
$$=\int\left(1-\frac{2x+2}{x^2+2x+5}-\frac{3}{(x+1)^2+2^2}\right)\mathrm{d}x$$
$$=x-\ln\left|x^2+2x+5\right|-\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

(c) $\displaystyle\int \frac{3x+4}{\sqrt{8-6x-9x^2}}\,\mathrm{d}x$ $\boxed{-\frac{1}{3}\sqrt{8-6x-9x^2}+\sin^{-1}\left(x+\tfrac{1}{3}\right)+C}$

$$\int \frac{3x+4}{\sqrt{8-6x-9x^2}}\,\mathrm{d}x$$
$$=\int \frac{-\frac{1}{6}(-6-18x)+3}{\sqrt{8-6x-9x^2}}\,\mathrm{d}x$$
$$=-\frac{1}{6}\int \frac{-6-18x}{\sqrt{8-6x-9x^2}}\,\mathrm{d}x+\int \frac{3}{\sqrt{8-6x-9x^2}}\,\mathrm{d}x \quad =-\frac{1}{3}\sqrt{8-6x-9x^2}+\sin^{-1}\left(x+\tfrac{1}{3}\right)+C$$
$$=-\frac{1}{6}\left[2\sqrt{8-6x-9x^2}\right]+\int \frac{3}{3\sqrt{1-\left(x+\frac{1}{3}\right)^2}}\,\mathrm{d}x$$
$$=-\frac{1}{3}\sqrt{8-6x-9x^2}+\sin^{-1}\left(x+\tfrac{1}{3}\right)+C$$

7. Integration by Substitution

By using the given substitution, determine the following integrals, leaving your answers in an exact form whenever applicable.

(a) $\displaystyle\int \frac{e^{2x}}{e^x+2}\,dx \qquad u=e^x+2$ $\boxed{e^x+2-2\ln\left|e^x+2\right|+c}$

$\displaystyle\int \frac{e^{2x}}{e^x+2}\,dx = \int \frac{(u-2)^2}{u}\,\frac{du}{u-2} = \int \frac{u-2}{u}\,du = \int 1-\frac{2}{u}\,du$ $= u-2\ln\lvert u\rvert+c$ $= e^x+2-2\ln\left\lvert e^x+2\right\rvert+c$ $= e^x-2\ln\left\lvert e^x+2\right\rvert+c'$	$u=e^x+2$ $\dfrac{du}{dx}=e^x=u-2$ $dx=\dfrac{du}{u-2}$

(b) $\displaystyle\int x\sqrt{x-3}\,dx \qquad x=u^2+3$ $\boxed{\frac{2}{5}(x-3)^{\frac{5}{2}}+2(x-3)^{\frac{3}{2}}+c}$

$\displaystyle\int x\sqrt{x-3}\,dx = \int (u^2+3)\sqrt{u^2}\,(2u\,du) = \int (u^2+3)\,2u^2\,du = 2\int u^4+3u^2\,du$ $= 2\left(\dfrac{u^5}{5}+u^3\right)+c = 2\left(\dfrac{1}{5}(x-2)^{\frac{5}{2}}+(x-2)^{\frac{3}{2}}\right)+c$	$x=u^2+2$ $\Rightarrow u=\sqrt{x-2}$ $\dfrac{dx}{du}=2u$ $dx=2u\,du$

(c) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx \qquad u=x^2$ $\boxed{\frac{1}{2}\sin^{-1}\left(x^2\right)+c}$

$\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-x^4}}\,2x\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\sin^{-1}u+c = \frac{1}{2}\sin^{-1}x^2+c$	$u=x^2$ $\dfrac{du}{dx}=2x$ $2x\,dx=du$

(d) $\displaystyle\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx \qquad x=\tan\theta$ $\boxed{\frac{1}{2}}$

$\int_0^1 \frac{1-x^2}{(1+x^2)^2}\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{1-\tan^2\theta}{(1+\tan^2\theta)^2}\ \sec^2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2\theta}{(\sec^2\theta)^2}\ \sec^2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2\theta}{\sec^2\theta}\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} (\cos^2\theta - \sin^2\theta)\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \cos 2\theta\ \mathrm{d}\theta = \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{2}$	$x = \tan\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$ $\mathrm{d}x = \sec^2\theta\ \mathrm{d}\theta$ When $x=1,\quad \theta = \frac{\pi}{4}$ When $x=0,\quad \theta = 0$

(e) $\int_0^3 \sqrt{9-x^2}\ \mathrm{d}x \qquad x = 3\sin\theta$ $\qquad \boxed{\frac{9\pi}{4}}$

$\int_0^3 \sqrt{9-x^2}\ \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\ 3\cos\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{2}} \sqrt{9\cos^2\theta}\ 3\cos\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{2}} 9\cos^2\theta\ \mathrm{d}\theta$ $= \frac{9}{2}\int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)\ \mathrm{d}\theta$ $= \frac{9}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}} = \frac{9}{2}\left[\frac{\sin\pi}{2} + \frac{\pi}{2}\right] = \frac{9\pi}{4}$	$x = 3\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\cos\theta$ $\mathrm{d}x = 3\cos\theta\ \mathrm{d}\theta$ When $x=3,\quad \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$ When $x=0,\quad \sin\theta = 0 \Rightarrow \theta = 0$

(f) $\int \frac{\mathrm{d}x}{\left(a^2-x^2\right)^{\frac{3}{2}}} \qquad x = a\sin\theta$ $\qquad \boxed{\frac{1}{a^2}\left(\frac{x}{\sqrt{a^2-x^2}}\right) + C}$

$\int \frac{\mathrm{d}x}{\left(a^2-x^2\right)^{\frac{3}{2}}}$ Let $x = a\sin\theta$. Then $\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\cos\theta$

$= \int \frac{a\cos\theta}{\left(a^2 - a^2\sin^2\theta\right)^{\frac{3}{2}}}\ \mathrm{d}\theta$

$= \int \frac{a\cos\theta}{a^3\left(1-\sin^2\theta\right)^{\frac{3}{2}}}\ \mathrm{d}\theta \quad = \frac{1}{a^2}\int \frac{\cos\theta}{\cos^3\theta}\ \mathrm{d}\theta$

$= \frac{1}{a^2}\int \sec^2\theta\ \mathrm{d}\theta$

$= \frac{1}{a^2}\tan\theta + C$

$$=\frac{1}{a^2}\left(\frac{x}{\sqrt{a^2-x^2}}\right)+C$$

8 Integration by Parts

(a) $\int x\mathrm{e}^{4x}\ \mathrm{d}x$ $\qquad \frac{1}{4}x\mathrm{e}^{4x}-\frac{1}{16}\mathrm{e}^{4x}+c$

$$\int x\mathrm{e}^{4x}\ \mathrm{d}x=x\left(\frac{1}{4}\mathrm{e}^{4x}\right)-\int\frac{1}{4}\mathrm{e}^{4x}\ \mathrm{d}x=\frac{1}{4}x\mathrm{e}^{4x}-\frac{1}{16}\mathrm{e}^{4x}+c$$

$u=x$	$\frac{\mathrm{d}v}{\mathrm{d}x}=\mathrm{e}^{4x}$
$\frac{\mathrm{d}u}{\mathrm{d}x}=1$	$v=\frac{1}{4}\mathrm{e}^{4x}$

(b) $\int\sqrt{x}\ln x\ \mathrm{d}x$ $\qquad \frac{2}{3}x^{\frac{3}{2}}\ln x-\frac{4}{9}x^{\frac{3}{2}}+c$

$$\int\sqrt{x}\ln x\ \mathrm{d}x=\frac{2}{3}x^{\frac{3}{2}}\ln x-\int\frac{2}{3}x^{\frac{1}{2}}\ \mathrm{d}x=\frac{2}{3}x^{\frac{3}{2}}\ln x-\frac{4}{9}x^{\frac{3}{2}}+c$$
$$=\frac{2}{3}x^{\frac{3}{2}}\left(\ln x-\frac{2}{3}\right)+c$$

$u=\ln x$	$\frac{\mathrm{d}v}{\mathrm{d}x}=\sqrt{x}$
$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$	$v=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2}{3}x^{\frac{3}{2}}$

(c) $\int_0^1 x\tan^{-1}(x^2)\ \mathrm{d}x$ $\qquad \frac{\pi}{8}-\frac{1}{4}\ln 2$

$$\int_0^1 x\tan^{-1}(x^2)\ \mathrm{d}x$$
$$=\left[\frac{x^2}{2}\tan^{-1}(x^2)\right]_0^1-\int_0^1\frac{x^2}{2}\frac{2x}{1+x^4}\ \mathrm{d}x$$
$$=\frac{1}{2}\left(\frac{\pi}{4}\right)-\frac{1}{4}\int_0^1\frac{4x^3}{1+x^4}\ \mathrm{d}x=\frac{\pi}{8}-\left[\frac{1}{4}\ln\left|1+x^4\right|\right]_0^1=\frac{\pi}{8}-\frac{1}{4}\ln 2$$

$u=\tan^{-1}(x^2)$	$\frac{\mathrm{d}v}{\mathrm{d}x}=x$
$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2x}{1+x^4}$	$v=\frac{x^2}{2}$

(d) $\int x\sec x\tan x\ \mathrm{d}x$ $\qquad x\sec x-\ln\left|\sec x+\tan x\right|+c$

$$\int x\sec x\tan x\ \mathrm{d}x=x\sec x-\int\sec x\ \mathrm{d}x$$
$$=x\sec x-\ln\left|\sec x\tan x\right|+c$$

$u=x$	$\frac{\mathrm{d}v}{\mathrm{d}x}=\sec x\tan x$
$\frac{\mathrm{d}u}{\mathrm{d}x}=1$	$v=\sec x$

(e) $\int x^2\cos x\ \mathrm{d}x$ $\qquad x^2\sin x+2x\cos x-2\sin x+c$

$$\int x^2\cos x\,\mathrm{d}x = x^2\sin x - \int 2x\sin x\,\mathrm{d}x$$
$$= x^2\sin x - \left[-2x\cos x - \int -2\cos x\,\mathrm{d}x\right]$$
$$= x^2\sin x + 2x\cos x - 2\int \cos x\,\mathrm{d}x$$
$$= x^2\sin x + 2x\cos x - 2\sin x + c$$

$u = x^2$	$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos x$
$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$	$v = \sin x$

$u = 2x$	$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sin x$
$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2$	$v = -\cos x$

(f) $\int \mathrm{e}^x \sin 2x\,\mathrm{d}x$ $\qquad \dfrac{1}{5}\mathrm{e}^x(\sin 2x - 2\cos 2x) + c$

$$\int \mathrm{e}^x\sin 2x\,\mathrm{d}x = \mathrm{e}^x\sin 2x - \int 2\mathrm{e}^x\cos 2x\,\mathrm{d}x$$
$$= \mathrm{e}^x\sin 2x - 2\left[\mathrm{e}^x\cos 2x - \int -2\mathrm{e}^x\sin 2x\,\mathrm{d}x\right]$$
$$= \mathrm{e}^x\sin 2x - 2\mathrm{e}^x\cos 2x - \int 4\mathrm{e}^x\sin 2x\,\mathrm{d}x$$

Then

$$5\int \mathrm{e}^x\sin 2x\,\mathrm{d}x = \mathrm{e}^x\sin 2x - 2\mathrm{e}^x\cos 2x$$
$$\therefore \int \mathrm{e}^x\sin 2x\,\mathrm{d}x = \frac{\mathrm{e}^x}{5}\left(\sin 2x - 2\cos 2x\right) + c$$

$u = \sin 2x$	$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$
$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2\cos 2x$	$v = \mathrm{e}^x$

$u = \cos 2x$	$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$
$\dfrac{\mathrm{d}u}{\mathrm{d}x} = -2\sin 2x$	$v = \mathrm{e}^x$

(g) $\int x^3\cos x^2\,\mathrm{d}x$ $\qquad \dfrac{1}{2}x^2\sin x^2 + \dfrac{1}{2}\cos x^2 + c$

$$\int x^3\cos x^2\,\mathrm{d}x = \int x^2(x\cos x^2)\,\mathrm{d}x$$
$$= \frac{1}{2}x^2\sin x^2 - \int x\sin x^2\,\mathrm{d}x = \frac{1}{2}x^2\sin x^2 + \frac{1}{2}\cos x^2 + c$$

$u = x^2$	$\dfrac{\mathrm{d}v}{\mathrm{d}x} = x\cos x^2$
$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$	$v = \dfrac{1}{2}\sin x^2$

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Find the exact value of p such that $\displaystyle\int_0^1 \frac{1}{4-x^2}\,\mathrm{d}x = \int_0^{\frac{1}{2p}} \frac{1}{\sqrt{1-p^2x^2}}\,\mathrm{d}x$. $\qquad \boxed{\dfrac{2\pi}{3\ln 3}}$

$$\int_0^1 \frac{1}{4-x^2}\,\mathrm{d}x = \int_0^{\frac{1}{2p}} \frac{1}{\sqrt{1-p^2x^2}}\,\mathrm{d}x.$$

$$\left[\frac{1}{4}\ln\left|\frac{x+2}{x-2}\right|\right]_0^1 = \frac{1}{p}\left[\sin^{-1}(px)\right]_0^{\frac{1}{2p}}$$

$$\frac{1}{4}\ln 3 = \frac{1}{p}.\frac{\pi}{6}$$

$$p = \frac{2\pi}{3\ln 3}$$

10 Find the positive values of a, such that $\int_a^{a^2} \frac{1}{1+x^2}\,\mathrm{d}x = 0.22$. Give your answer to three significant figures. $\boxed{2.04, 2.62}$

By plotting the graphs of $y = \int_x^{x^2} \frac{1}{1+x^2}\,\mathrm{d}x$ and $y = 0.22$ on the GC, and finding the intersections of the graphs, $a = 2.04$ or $a = 2.62$

11 Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}\right)$, and hence or otherwise show that $\int_0^{\sqrt{\ln 2}} x^3\mathrm{e}^{x^2}\,dx = \ln 2 - \frac{1}{2}$. $\boxed{2x\mathrm{e}^{x^2}}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}\right) = 2x\mathrm{e}^{x^2}$$

$$\int_0^{\sqrt{\ln 2}} x^3\mathrm{e}^{x^2}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\sqrt{\ln 2}} x^2\left(2x\mathrm{e}^{x^2}\right)\,\mathrm{d}x$$

$$= \frac{1}{2}\left\{\left[x^2\mathrm{e}^{x^2}\right]_0^{\sqrt{\ln 2}} - \int_0^{\sqrt{\ln 2}} 2x\mathrm{e}^{x^2}\,\mathrm{d}x\right\}$$

$$= \frac{1}{2}\left(\sqrt{\ln 2}\right)^2 2 - \frac{1}{2}\left[\mathrm{e}^{x^2}\right]_0^{\sqrt{\ln 2}}$$

$$= \ln 2 - \frac{1}{2}(2-1)$$

$$= \ln 2 - \frac{1}{2} \quad \text{(shown)}$$

$u = x^2$	$\frac{\mathrm{d}v}{\mathrm{d}x} = 2x\mathrm{e}^{x^2}$
$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$	$v = \mathrm{e}^{x^2}$

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(i) Prove that $\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x + \tan x) = \sec x$.

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(ii) Find $\int x\sin x\,\mathrm{d}x$. $\boxed{-x\cos x+\sin x+c}$

(iii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} x\sin x\ln\left(\sec x+\tan x\right)\,\mathrm{d}x$.

$\boxed{\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)\ln\left(\sqrt{2}+1\right)+\frac{\pi^2}{32}-\ln\sqrt{2}}$

(ii)

$$\int x\sin x\,\mathrm{d}x$$
$$=-x\cos x-\int -\cos x\,\mathrm{d}x$$
$$=-x\cos x+\int \cos x\,\mathrm{d}x$$
$$=-x\cos x+\sin x+c$$

$u=x$	$\frac{\mathrm{d}v}{\mathrm{d}x}=\sin x$
$\frac{\mathrm{d}u}{\mathrm{d}x}=1$	$v=-\cos x$

(iii)

$$\int_0^{\frac{\pi}{4}} x\sin x\ln\left(\sec x+\tan x\right)\,\mathrm{d}x$$
$$=\left[(-x\cos x+\sin x)\ln(\sec x+\tan x)\right]_0^{\frac{\pi}{4}}$$
$$-\int_0^{\frac{\pi}{4}} (-x\cos x+\sin x)\sec x\ \ \mathrm{d}x$$
$$=\left[-\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\right]\ln\left(\sqrt{2}+1\right)$$
$$+\int_0^{\frac{\pi}{4}} (x-\tan x)\ \ \mathrm{d}x$$
$$=\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)\ln\left(\sqrt{2}+1\right)+\left[\frac{x^2}{2}-\ln\sec x\right]_0^{\frac{\pi}{4}}$$
$$=\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)\ln\left(\sqrt{2}+1\right)+\frac{\pi^2}{32}-\ln\sqrt{2}$$

$u=\ln(\sec x+\tan x)$	$\frac{\mathrm{d}v}{\mathrm{d}x}=x\sin x$
$\frac{\mathrm{d}u}{\mathrm{d}x}=\sec x$	$v=-x\cos x+\sin x$

Section 2: Supplementary Questions (For Students to practice after going through tutorial for extra practice)

1 (a) $\int \tan 3x \, dx$ $\boxed{-\frac{1}{3}\ln|\cos 3x|+c}$

$$\int \tan 3x \, dx = -\frac{1}{3}\int \frac{-3\sin 3x}{\cos 3x} \, dx = -\frac{1}{3}\ln|\cos 3x|+c$$

(b) $\int \frac{x+2}{x^2-x} \, dx$ $\boxed{-2\ln|x|+3\ln|x-1|+c}$

$$\frac{x+2}{x(x-1)} = \frac{A}{x}+\frac{B}{x-1}$$

Then $x+2 = A(x-1)+Bx$

Let $x=0$, $2=-A$

Let $x=1$, $3=B$

$$\int \frac{x+2}{x^2-x} \, dx = \int\left(\frac{-2}{x}+\frac{3}{x-1}\right) dx = -2\ln|x|+3\ln|x-1|+c$$

(c) $\int \frac{\sec^2 x}{(1+\tan x)^3} \, dx$ $\boxed{-\frac{1}{2(1+\tan x)^2}+c}$

$$\int \frac{\sec^2 x}{(1+\tan x)^3} \, dx = \int \sec^2 x(1+\tan x)^{-3} \, dx = \frac{(1+\tan x)^{-2}}{-2}+c = \frac{1}{-2(1+\tan x)^2}+c$$

(d) $\int \frac{x^2}{x-2} \, dx$ $\boxed{\frac{1}{2}x^2+2x+4\ln|x-2|+c}$

$$\int \frac{x^2}{x-2} \, dx$$

$$= \int \frac{x(x-2)+2x}{x-2} \, dx = \int x+\frac{2(x-2)+4}{x-2} \, dx = \int x+2+\frac{4}{x-2} \, dx = \frac{x^2}{2}+2x+4\ln|x-2|+c$$

(e) $\int \frac{2}{x^2-6x+8} \, dx$ $\boxed{\ln\left|\frac{x-4}{x-2}\right|+c}$

$$\int \frac{2}{x^2-6x+8} \, dx = \int \frac{2}{(x-3)^2-1} \, dx = 2\left(\frac{1}{2}\right)\ln\left|\frac{(x-3)-1}{(x-3)+1}\right|+c = \ln\left|\frac{x-4}{x-2}\right|+c$$

(f) $\int x\cos 5x \, dx$ $\boxed{\frac{1}{5}x\sin 5x+\frac{1}{25}\cos 5x+c}$

$$\int x\cos 5x \, dx = \frac{1}{5}x\sin 5x-\frac{1}{5}\int \sin 5x \, dx = \frac{1}{5}x\sin 5x+\frac{1}{25}\cos 5x+c$$

$u = x$ $\quad \frac{dv}{dx} = \cos 5x$

$\frac{du}{dx} = 1$ $\quad v = \frac{1}{5}\sin 5x$

(g) $\int \frac{x}{\sqrt{x-1}}\,\mathrm{d}x$ $\boxed{\frac{2}{3}(x-1)^{\frac{1}{2}}(x+2)+c}$

$$\int \frac{x}{\sqrt{x-1}}\,\mathrm{d}x=\int \frac{x-1+1}{\sqrt{x-1}}\,\mathrm{d}x=\int\left(\sqrt{x-1}+\frac{1}{\sqrt{x-1}}\right)\mathrm{d}x=\frac{(x-1)^{\frac{3}{2}}}{\frac{3}{2}}+\frac{(x-1)^{\frac{1}{2}}}{\frac{1}{2}}+c$$
$$=\tfrac{2}{3}(x-1)^{\frac{3}{2}}+2(x-1)^{\frac{1}{2}}+c=2(x-1)^{\frac{1}{2}}\left[\tfrac{1}{3}(x-1)+1\right]+c=2(x-1)^{\frac{1}{2}}\left(\tfrac{1}{3}x+\tfrac{2}{3}\right)+c$$

OR

$u=x$ $\quad \frac{dv}{dx}=(x-1)^{-\frac{1}{2}}$

$\frac{\mathrm{d}u}{\mathrm{d}x}=1$ $\quad v=2(x-1)^{\frac{1}{2}}$

$$\int \frac{x}{\sqrt{x-1}}\,\mathrm{d}x=2x(x-1)^{\frac{1}{2}}-\int 2(x-1)^{\frac{1}{2}}\,\mathrm{d}x=2x(x-1)^{\frac{1}{2}}-\frac{2(x-1)^{\frac{3}{2}}}{\frac{3}{2}}+c$$
$$=2x(x-1)^{\frac{1}{2}}-\tfrac{4}{3}(x-1)^{\frac{3}{2}}+c=2(x-1)^{\frac{1}{2}}\left[x-\tfrac{2}{3}(x-1)\right]+c$$
$$=2(x-1)^{\frac{1}{2}}\left(\tfrac{1}{3}x+\tfrac{2}{3}\right)+c$$

OR

Let $y^2=x-1$

$2y\frac{\mathrm{d}y}{\mathrm{d}x}=1$

$$\int \frac{x}{\sqrt{x-1}}\,\mathrm{d}x=\int \frac{y^2+1}{y}\,2y\,\mathrm{d}y=2\int\left(y^2+1\right)\mathrm{d}y=2\left(\frac{y^3}{3}+y\right)+c$$
$$=2\left[\tfrac{1}{3}(x-1)^{\frac{3}{2}}+(x-1)^{\frac{1}{2}}\right]+c$$

2 Express $\frac{x^2+x+28}{(1-x)(x^2+9)}$ in partial fractions. Hence, show that

$$\int_0^3 \frac{x^2+x+28}{(1-x)(x^2+9)}\,\mathrm{d}x=\frac{\pi}{12}-2\ln 2.$$

$\boxed{\frac{3}{1-x}+\frac{2x+1}{x^2+9}_{\#}}$

$$\frac{x^2+x+28}{(1-x)(x^2+9)}=\frac{3}{1-x}+\frac{Ax+B}{x^2+9}$$
$$x^2+x+28=3(x^2+9)+(Ax+B)(1-x)$$

When $x=0$, $28=27+B\Rightarrow B=1$

Comparing coefficient of x^2, $1=3-A\Rightarrow A=2$

$$\therefore\ \frac{x^2+x+28}{(1-x)(x^2+9)}=\frac{3}{1-x}+\frac{2x+1}{x^2+9}_{\#}$$

Topic 12: Integration Techniques

$$\int_0^3 \frac{x^2+x+28}{(1-x)(x^2+9)}\,\mathrm{d}x$$
$$=\int_0^3 \frac{3}{1-x}+\frac{2x+1}{x^2+9}\,\mathrm{d}x$$
$$=\left[-3\ln|1-x|+\ln\left|x^2+9\right|+\frac{1}{3}\tan^{-1}(\frac{x}{3})\right]_0^3$$
$$=-3\ln 2+\ln 18+\frac{\pi}{12}-\ln 9$$
$$=\ln\left(\frac{18}{9\times 8}\right)+\frac{\pi}{12}$$
$$=\frac{\pi}{12}+\ln\frac{1}{4}$$
$$=\frac{\pi}{12}-2\ln 2_{\#}$$

3 By using the substitution $u=\sqrt{x+2}$, or otherwise, show that $\int_2^7 \frac{1}{(x+1)\sqrt{x+2}}\,\mathrm{d}x=\ln\frac{3}{2}$.

Hence, find the exact value of $\int_2^7 \frac{(x-1)}{(x+1)\sqrt{x+2}}\,\mathrm{d}x$. $\boxed{2-2\ln\frac{3}{2}}$

$$\int_2^7 \frac{1}{(x+1)\sqrt{x+2}}\,\mathrm{d}x$$
$$=\int_2^3 \frac{1}{(u^2-1)u}\,2u\,\mathrm{d}u$$
$$=\left[\ln\left|\frac{u-1}{u+1}\right|\right]_2^3$$
$$=\ln\frac{2}{4}-\ln\frac{1}{3}$$
$$=\ln\frac{3}{2}_{\#}$$

Let $u=\sqrt{x+2}$

$u^2=x+2$

$\frac{\mathrm{d}x}{\mathrm{d}u}=2u$

When $x=2,\ u=2$

When $x=7,\ u=3$

$$\int_2^7 \frac{(x-1)}{(x+1)\sqrt{x+2}}\,\mathrm{d}x=\int_2^7 \frac{(x+1)-2}{(x+1)\sqrt{x+2}}\,\mathrm{d}x$$
$$=\int_2^7 \frac{1}{\sqrt{x+2}}-\frac{2}{(x+1)\sqrt{x+2}}\,\mathrm{d}x$$
$$=\left[2\sqrt{x+2}\right]_2^7-2\ln\frac{3}{2}$$
$$=2-2\ln\frac{3}{2}_{\#}$$

4 **(a)** $\int \frac{x+2}{\sqrt{1-3x^2}}\mathrm{d}x$ $\boxed{-\frac{1}{3}\sqrt{1-3x^2}+\frac{2}{\sqrt{3}}\sin^{-1}(\sqrt{3}x)+c}$

$$\int \frac{x+2}{\sqrt{1-3x^2}}\,\mathrm{d}x = -\frac{1}{6}\int -6x(1-3x^2)^{-\frac{1}{2}}\,\mathrm{d}x + 2\int \frac{1}{\sqrt{1-3x^2}}\,\mathrm{d}x$$
$$= -\frac{1}{6}\frac{\sqrt{1-3x^2}}{\frac{1}{2}} + \frac{2}{\sqrt{3}}\sin^{-1}(\sqrt{3}x)+c = -\frac{1}{3}\sqrt{1-3x^2}+\frac{2}{\sqrt{3}}\sin^{-1}(\sqrt{3}x)+c$$

(b) $\int \frac{2}{x\ln x^2}\mathrm{d}x$ $\boxed{\ln|\ln x^2|+c}$

$$\int \frac{2}{x\ln x^2}\,\mathrm{d}x = \ln|\ln x^2|+c$$

(c) $\int \frac{1}{1+\mathrm{e}^x}\,\mathrm{d}x$ $\boxed{x-\ln\left|1+\mathrm{e}^x\right|+c}$

$$\int \frac{1}{1+\mathrm{e}^x}\,\mathrm{d}x = \int \frac{1+\mathrm{e}^x-\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x = \int 1-\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x = x-\ln\left|1+\mathrm{e}^x\right|+c$$

(d) $\int \frac{1}{x\left[4+(\ln x)^2\right]}\,\mathrm{d}x$ $\boxed{\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{2}\right)+c}$

$$\int \frac{1}{x\left[4+(\ln x)^2\right]}\,\mathrm{d}x = \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{2}\right)+c$$

(e) $\int \ln(2x+1)\,\mathrm{d}x$ $\boxed{x\ln(2x+1)-x+\frac{1}{2}\ln(2x+1)+c}$

$$\int \ln(2x+1)\,\mathrm{d}x = x\ln(2x+1)-\int\frac{2x}{2x+1}\,\mathrm{d}x = x\ln(2x+1)-\int 1-\frac{1}{2x+1}\,\mathrm{d}x$$
$$= x\ln(2x+1)-x+\frac{1}{2}\ln(2x+1)+c$$

(f) $\int \frac{6x+4}{(1-2x)(1+3x^2)}\mathrm{d}x$ $\boxed{-2\ln\left|1-2x\right|+\ln\left|1+3x^2\right|+c}$

$$\int \frac{6x+4}{(1-2x)(1+3x^2)}\,\mathrm{d}x = \int \frac{4}{1-2x}+\frac{6x}{1+3x^2}\,\mathrm{d}x = -2\ln\left|1-2x\right|+\ln\left|1+3x^2\right|+c$$

5 By using the substitution $u = x^4+1$, find $\int_1^3 \frac{1}{x(x^4+1)}\,\mathrm{d}x$, leaving your answers in an exact form. $\boxed{\frac{1}{4}\ln\left(\frac{81}{41}\right)}$

$$\int_1^3 \frac{1}{x\left(x^4+1\right)}\,\mathrm{d}x$$
$$=\int_2^{82} \frac{1}{(u-1)^{1/4}u}\cdot\frac{1}{4(u-1)^{3/4}}\,\mathrm{d}u$$
$$=\int_2^{82} \frac{1}{4u(u-1)}\,\mathrm{d}u$$
$$=\frac{1}{4}\int_2^{82} \frac{-1}{u}+\frac{1}{(u-1)}\,\mathrm{d}u$$
$$=\frac{1}{4}\Big[-\ln|u|+\ln|u-1|\Big]_2^{82}$$
$$=\frac{1}{4}\left(-\ln 82+\ln 81+\ln 2\right)$$
$$=\frac{1}{4}\ln\left(\frac{81}{41}\right)_{\#}$$

6 Determine the exact value of $\int_0^{\frac{\pi}{4}} x\cos 2x\,\mathrm{d}x$. $\boxed{\frac{\pi}{8}-\frac{1}{4}}$

Hence, evaluate $\int_0^{\frac{\pi}{4}} x\cos^2 x\,\mathrm{d}x$, leaving your answer in an exact form. $\boxed{\frac{\pi^2}{64}+\frac{\pi}{16}-\frac{1}{8}}$

$$\int_0^{\frac{\pi}{4}} x\cos 2x\,\mathrm{d}x$$
$$=\left[\frac{x\sin 2x}{2}\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{\sin 2x}{2}\,\mathrm{d}x$$
$$=\left[\frac{x\sin 2x}{2}+\frac{\cos 2x}{4}\right]_0^{\frac{\pi}{4}}$$
$$=\frac{\pi}{8}-\frac{1}{4}_{\#}$$

$$\int_0^{\frac{\pi}{4}} x\cos^2 x\,\mathrm{d}x$$
$$=\int_0^{\frac{\pi}{4}} x\cdot\frac{\cos 2x+1}{2}\,\mathrm{d}x$$
$$=\frac{1}{2}\left(\frac{\pi}{8}-\frac{1}{4}\right)+\left[\frac{x^2}{4}\right]_0^{\frac{\pi}{4}}$$
$$=\frac{\pi}{16}-\frac{1}{8}+\frac{\pi^2}{64}_{\#}$$