ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

FURTHER MATHEMATICS 9649/01

Paper 1 **12 September 2023**

3 hours

Additional Materials: Cover Sheet
Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 A curve has parametric equations

$$x = t^4 - 4\ln t,$$
$$y = 4t^2$$

where $t > 0$.

The surface area generated when the arc of the curve from $t = 1$ to $t = k$, where $k > 1$, is rotated through one complete revolution about the x-axis is 384π square units. Find the value of k. [4]

2 The sequence $\{x_n\}$ is given by $x_1 = 1$ and

$$nx_n = 4\left(1 - \frac{1}{n}\right)^{n-1} x_{n-1} + 2n^{1-n} \text{ for } n \geq 2.$$

By multiplying the recurrence relation throughout by n^{n-1}, use a suitable substitution to determine x_n as a function of n, simplifying your answer. [7]

3 The variables x and y are related by the differential equation

$$x^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + x^2 y - y^3, \qquad (*)$$

and it is given that $x > y > 0$.

(a) Use the substitution $y = ux$ to find the general solution of (*). [5]

A particular solution of (*) has $y = 1$ when $x = 2$.

(b) Use one step of the Euler method to calculate an approximate value of y when $x = 2.5$. [2]

(c) Given that the solution curve is concave downwards near the point $(2,1)$, determine with a sketch whether the approximate value of y found in **(b)** is an overestimate or underestimate. [1]

4 **(i)** A 3×3 square matrix $\mathbf{A}$ is said to be skew symmetric if $\mathbf{A}^{\mathrm{T}}=-\mathbf{A}$. Prove that for a given skew symmetric $\mathbf{A}$, $\det(\mathbf{A})$ is equal to zero. [3]

(ii) Let T be the linear transformation such that

$$\mathrm{T}:\mathbb{R}^3\to\mathbb{R}^3 \text{ and } \mathrm{T}(\mathbf{x})=\mathbf{a}\times\mathbf{x}$$

where $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and the multiplication is the usual vector product in $\mathbb{R}^3$.

Let $\mathbf{M}$ be the matrix representation of T.

(a) Find $\mathbf{M}$ in terms of a_1, a_2 and a_3. [2]

(b) Determine if $\mathbf{M}$ is invertible. [1]

(c) State $\ker(\mathrm{T})$ and $\mathrm{R}(\mathrm{T})$, the null space and range space of T respectively. Give a geometrical interpretation of your answers. [4]

5 Let $\mathbf{T}_\theta$ denote the matrix

$$\begin{pmatrix}\cos2\theta & \sin2\theta\\ \sin2\theta & -\cos2\theta\end{pmatrix}.$$

(i) By finding $\mathbf{T}_\theta\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ and $\mathbf{T}_\theta\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$, show that $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ and $\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ are eigenvectors of $\mathbf{T}_\theta$ and write down their corresponding eigenvalues. [4]

(ii) Express $\mathbf{T}_\theta$ in the form $\mathbf{RDR}^{-1}$ where $\mathbf{R}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. [1]

The matrix $\mathbf{T}_\theta$ represents the reflection of a position vector in $\mathbb{R}^2$ about the line through the origin that makes an angle θ with the positive x-axis.

(iii) Find the reflection of $\begin{pmatrix}2\\1\end{pmatrix}$ about the line $y=\sqrt{3}x$. [3]

(iv) The matrix $\mathbf{R}$ in **(ii)** represents the rotation of a position vector in $\mathbb{R}^2$ through an angle θ about the origin. By finding $\mathbf{T}_\alpha\mathbf{T}_\beta$, show that the product of two reflection matrices is a matrix that represents the rotation of a position vector in $\mathbb{R}^2$. [2]

6 **(i)** Show that the equation $\mathrm{f}(x)=0$, where

$$\mathrm{f}(x)=\frac{1}{x^2}-\sqrt{x-2},$$

has a root α in the interval $[k, k+1]$ where k is an integer to be determined. [2]

(ii) In order to find an approximation β to α, two stages of the linear interpolation process is used on the interval $[k, k+1]$. Find the value of β, correct to 3 significant figures. [2]

(iii) Show how the consideration of $\mathrm{f}''(x)$ in the interval $[k, k+1]$ enables you to determine whether β is an under-estimate or an over-estimate of α. [3]

(iv) Explain why the use of the Newton-Raphson iteration with initial approximation $x_0=k+\frac{1}{2}$ to find α would not work. [1]

(v) The root α for the equation $\mathrm{f}(x)=0$ satisfies the equation $x=\mathrm{g}(x)$ where

$$\mathrm{g}(x)=2+\frac{1}{x^4}.$$

Use an iterative method based on the form $x_{n+1}=\mathrm{g}(x_n)$ with $x_0=k+1$ to find an approximation, γ to α, correct to 2 decimal places. Justify whether γ is a sufficiently accurate approximation. [3]

7 The curve C has parametric equations

$$x=a(t-\sin t), \qquad y=a(1-\cos t),$$

where $0 \le t \le \frac{3}{2}\pi$ and a is a constant.

(i) The region R is bounded by the curve C, the x-axis and the line $x=a\left(\frac{3}{2}\pi+1\right)$. Using the shell method, show that V, the volume of solid generated when R is rotated through 2π radians about the y-axis is given by

$$2\pi a^3\int_0^b (t-\sin t)(1-\cos t)^2 \,\mathrm{d}t,$$

where b is a constant to be determined. [2]

(ii) Find the exact value of V. [6]

8 Consider a body moving horizontally in one dimension through a liquid medium. The body experiences a force $\mathrm{F}(t)$ in the direction of motion and a frictional force in the opposite direction. At lower speeds of motion, the frictional force is directly proportional to the velocity $v(t)$ of the motion.

According to Newton's second law of motion, for a body with constant mass *m*, the product of the body's mass and acceleration is equal to the net force on the body in the direction of motion. This basic model leads to the differential equation

$$m\frac{\mathrm{d}v}{\mathrm{d}t}=\mathrm{F}(t)-kv(t)$$

where *k* is a positive constant.

(a) Suppose $\mathrm{F}(t)=C$ where *C* is a positive constant. Given that the initial velocity of the body is v_0, find an expression for *v*. [4]

(b) Show that, after a long time, the velocity of the body tends to a limiting value that is independent of the initial velocity. [2]

(c) Now suppose $\mathrm{F}(t)=C-qt$ where *C* and *q* are positive constants.

(i) Given that the body is initially at rest, show that

$$v=\left(\frac{C}{k}+\frac{qm}{k^2}\right)\left(1-\mathrm{e}^{-\frac{kt}{m}}\right)-\frac{qt}{k}.$$ [5]

(ii) Show that at the instant when $\mathrm{F}(t)=0$, the velocity of the body is positive. [2]

(You may use the fact that $\mathrm{e}^{-x}<\frac{1}{1+x}$ when $x>0$.)

9 **(a)** The terms in the sequence $\{u_n\}$ satisfy the recurrence relation

$$4u_r = 2u_{r-1} - u_{r-2}.$$

(i) Find the general solution of this recurrence relation. [3]

(ii) Show that $u_{k+3} = \alpha u_k$ for all $k \geq 0$ where α is a constant to be determined. Find $\sum_{r=1}^{\infty} u_{3r-2}$ in terms of u_1. [4]

(b) Five different signals are available for transmitting a particular message. Two of these signals require 1 microsecond each for transmission, while the other three signals require 2 microseconds each. It is assumed that the signals in a message are sent immediately one after another. Signals can be repeated within a message.

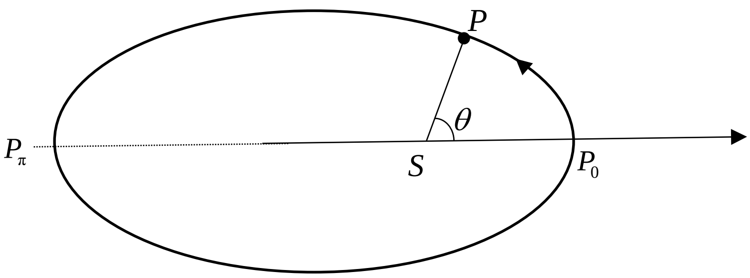
Let a_n be the number of messages that can be transmitted in exactly *n* microseconds using these five different signals, where *n* is a positive integer.

(i) Explain why the sequence $a_1, a_2, a_3, \ldots$ satisfies the recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2}.$$ [2]

(ii) Solve the recurrence relation. [5]

10 The diagram shows the elliptical orbit of a Planet P and its sun, S, which is at a focus of the ellipse. The point P_0 is called the *perihelion* and is the point on the orbit when P is closest to S. The point P_π is called the *aphelion* and is the point on the orbit when P is furthest from S.



Taking S as the pole and the line SP_0 as the initial line, the equation of the orbit of P is

$$r = \frac{a}{1 + e\cos\theta} \text{ where } \theta \geq 0,$$

where e is the eccentricity of the ellipse, a is a positive constant and θ is in radians.

(i) Given that the distance between P_0 and P_π is 5.9 astronomical units (AU) and the distance between S and P_π is 2.15 times that of between S and P_0, determine the values of a and e. [5]

Kepler's Second Law of Planetary Motion states that the line segment SP sweeps out equal areas in equal times.

(ii) Given that the orbital period of Planet P is 130 days, find the time P takes to travel from the point where $\theta = \frac{1}{2}\pi$ to the point where $\theta = 3$. [4]

(iii) Find also the additional time P takes to travel from the point where $\theta = 3$ to P_π. [2]

Kepler's Third Law of Planetary Motion states that the orbital period of a planet, T, is related to the length of the semi-major axis of its orbit, l, by the equation

$$T^2 = kl^3,$$

where k is constant for planets orbiting the same sun.

Planet Q orbits around the same sun as Planet P with the sun at one of its foci. The equation of the orbit of Q is given by

$$25x^2 + 4y^2 - 50x_0x - 8y_0y + 25x_0^2 + 4y_0^2 - 100 = 0$$

where (x_0, y_0) are the coordinates of the centre of the orbit.

Use Kepler's Third Law to determine the orbital period of Q. [4]