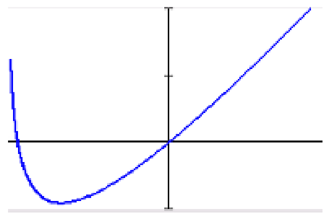
2023 Y5 FM Promotion Examination (Solutions)

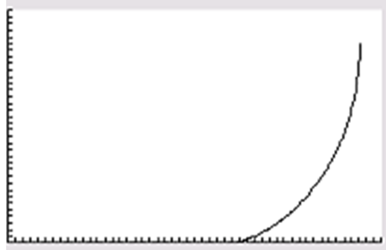
1	**Solutions**	
	Given $u_1 = 79$, $79 = 5^A + 6B$ ....(1) Given $u_2 = 949$, $949 = 5^{2A} + 36B$ ....(2) Sub (1) into (2), $949 = 5^{2A} + 6(79 - 5^A)$ $5^{2A} - 6\left(5^A\right) - 475 = 0$ $(5^A + 19)(5^A - 25) = 0$ $5^A = -19$ (NA) or $5^A = 25 \Rightarrow A = 2$ $\therefore B = 9$.	
	Let P_n be the statement $\mathrm{f}(n) = 10u_n - 2^n\left(3^{n+4}\right) = 10\left(5^{2n}\right) + 9\left(6^n\right)$ is a multiple of 19, where $n \in \mathbb{Z}_0^+$. For $n = 0$: $\mathrm{f}(0) = 10\left(5^{2(0)}\right) + 9\left(6^0\right) = 19$ which is a multiple of 19. Therefore P_0 is true. Assume that P_k is true for some $k \in \mathbb{Z}_0^+$, i.e. $\mathrm{f}(k) = 10\left(5^{2k}\right) + 9\left(6^k\right) = 19p$ for some $p \in \mathbb{Z}^+$. To prove P_{k+1} is true, i.e. $\mathrm{f}(k+1) = 10\left(5^{2(k+1)}\right) + 9\left(6^{k+1}\right)$ is a multiple of 19. $\mathrm{f}(k+1) = 10\left(5^{2(k+1)}\right) + 9\left(6^{k+1}\right)$ $= 250\left(5^{2k}\right) + 54\left(6^k\right)$ $= 25\left(10\left(5^{2k}\right) + 9\left(6^k\right)\right) - 171\left(6^k\right)$ $= 25(19p) - 19\left(9\left(6^k\right)\right)$ $= 19\left(25p - 9\left(6^k\right)\right)$ Thus $\mathrm{f}(k+1)$ is divisible by 19. Therefore P_k is true $\Rightarrow P_{k+1}$ is true. Since P_0 is true, by Mathematical Induction, P_n is true for all non-negative integers *n*.	

2	**Solutions**	
(a)	$PF_1 = \sqrt{(x-c)^2 + y^2}$ where $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$ and $b = \sqrt{a^2 - c^2}$. $= \sqrt{(x-c)^2 + \left(\frac{a^2 - x^2}{a^2}\right)b^2}$ $= \sqrt{(x-c)^2 + \left(\frac{a^2 - x^2}{a^2}\right)\left(a^2 - c^2\right)}$ $= \frac{1}{a}\sqrt{a^2(x-c)^2 + \left(a^2 - x^2\right)\left(a^2 - c^2\right)}$ $= \frac{1}{a}\sqrt{a^2x^2 - 2a^2cx + a^2c^2 + a^4 - a^2c^2 - a^2x^2 + x^2c^2}$ $= \frac{1}{a}\sqrt{+a^4 - 2a^2cx + x^2c^2}$ $= \frac{1}{a}\sqrt{\left(a^2 - cx\right)^2}$ $= \frac{a^2 - cx}{a}$ since $a > c$ The other focus will be at $F_2\left(-c,\, 0\right)$. Similarly, $PF_2 = \frac{a^2 + cx}{a}$. Hence $PF_1 + PF_2 = \frac{a^2 - cx}{a} + \frac{a^2 + cx}{a} = 2a$ (Shown)	
(b)	$QF_1 \times QF_2$ $= \left(\frac{a^2 - cx}{a}\right)\left(\frac{a^2 + cx}{a}\right)$ $= a^2 - \frac{c^2x^2}{a^2}$ Since $a^2 = c^2 + b^2$ and at the point $Q\left(4,\, 3\right)$, $QF_1 \times QF = k$, therefore, $QF_1 \times QF = a^2 - \frac{16c^2}{a^2} = a^2 - \frac{16\left(a^2 - b^2\right)}{a^2}$ $k = a^2 - 16 + \frac{16b^2}{a^2}$ (1) Also, $\frac{4^2}{a^2} + \frac{3^2}{b^2} = 1 \Rightarrow \frac{3^2}{b^2} = 1 - \frac{4^2}{a^2} = \frac{a^2 - 4^2}{a^2}$,	

	$b^2 = \dfrac{9a^2}{a^2 - 16}$ (2) Substitute (2) into (1): $k = a^2 - \dfrac{16\left(a^2 - \dfrac{9a^2}{a^2-16}\right)}{a^2}$ $\Rightarrow k = a^2 - 16\left(1 - \dfrac{9}{a^2-16}\right)$ $\Rightarrow k = a^2 - 16 + \dfrac{144}{a^2-16}$ (Shown) Therefore $A = 144$, $B = 16$	

3	**Solutions**	
(i)	$x_2 = -0.536$, $x_5 = -0.0346$, $x_8 = -0.00132$. From GC, the sequence (appears to) converge to the larger root, 0.	
(ii)	Sketch a graph of $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = 3x - \ln(x+1)$  Note that around the origin, $\mathrm{f}(x) = 3x - \ln(x+1)$ closely "resembles a straight line". We use a tangent approximation to $y = 3x - \ln(x+1) - \varepsilon$ at $(0, -\varepsilon)$ to find root of the equation $0 = 3x - \ln(x+1) - \varepsilon$. $\frac{dy}{dx}\|_{x=0} = 3 - \frac{1}{1+0} = 2$ $\frac{0-(-\varepsilon)}{x-0} \approx 2 \Rightarrow \beta \approx \frac{\varepsilon}{2}$ Since ε is very close to zero, the root β is also close to 0. OR Alternatively, consider a graphical explanation. The equation $3x - \ln(x+1) = 0$ has a larger root $x = 0$. Since the graph of $y = \mathrm{f}(x)$ is a translation of the graph of $y = 3x - \ln(x+1)$ by a very small amount $-\varepsilon$ in the *y*-direction, (and f is strictly increasing around the point where *x*=0), the root β is a small change from 0 in the *x*-direction. OR Alternatively, show that there is a sign change over the interval 0 to ε. $\mathrm{f}(x) = 3x - \ln(x+1) - \varepsilon$ $\mathrm{f}(0) = 3(0) - \ln(0+1) - \varepsilon = -\varepsilon$ $\mathrm{f}(\varepsilon) = 3(\varepsilon) - \ln(\varepsilon+1) - \varepsilon = 3\varepsilon - \left(\varepsilon - \frac{\varepsilon^2}{2} + \ldots\right) - \varepsilon = \varepsilon + \frac{\varepsilon^2}{2} - \ldots$ $\mathrm{f}(0)\mathrm{f}(\varepsilon) = -\varepsilon^2 + \ldots < 0$ Root lies between 0 and ε. Since ε very close to 0, so β is close to 0.	
(iii)	$\mathrm{g}(x) = \frac{3x - \ln(x+1) - \varepsilon}{3 - \frac{1}{x+1}} = \frac{(x+1)\left[3x - \ln(x+1) - \varepsilon\right]}{3x+2}$	

(iv)	$x_{n+1}=x_n-\dfrac{(x_n+1)\left[3x_n-\ln(x_n+1)-\varepsilon\right]}{3x_n+2}$ $x_2=0-\dfrac{1(-\varepsilon)}{2}=\dfrac{\varepsilon}{2}$ $x_3=\dfrac{\varepsilon}{2}-\dfrac{\left(\dfrac{\varepsilon}{2}+1\right)\left[\dfrac{3}{2}\varepsilon-\dfrac{\varepsilon}{2}+\dfrac{\left(\dfrac{\varepsilon}{2}\right)^2}{2}+\ldots-\varepsilon\right]}{\dfrac{3}{2}\varepsilon+2}$ $=\dfrac{\varepsilon}{2}-\dfrac{\dfrac{\varepsilon^2}{8}}{2\left(1+\dfrac{3}{4}\varepsilon\right)}+\ldots$ $=\dfrac{\varepsilon}{2}-\dfrac{\varepsilon^2}{8}\left(\dfrac{1}{2}\right)\left(1-\dfrac{3}{4}\varepsilon\right)+\ldots$ $\beta\approx\dfrac{\varepsilon}{2}-\dfrac{\varepsilon^2}{16}$ (shown)	

4	**Solutions**	
(i)	 $x = 3(\cos\theta + \theta\sin\theta) \Rightarrow \frac{dx}{d\theta} = 3(-\sin\theta + \theta\cos\theta + \sin\theta) = 3\theta\cos\theta$ $y = 3(\sin\theta - \theta\cos\theta)$ $\Rightarrow \frac{dy}{d\theta} = 3(\cos\theta + \theta\sin\theta - \cos\theta) = 3\theta\sin\theta$ Length of the arc PQ $= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$ $= \int_0^{\frac{\pi}{2}} \sqrt{9\theta^2\left[(\cos\theta)^2 + (\sin\theta)^2\right]}\, d\theta$ $= \int_0^{\frac{\pi}{2}} 3\lvert\theta\rvert\, d\theta$ $= \int_0^{\frac{\pi}{2}} 3\theta\, d\theta \qquad \lvert\theta\rvert = \theta \text{ when } \theta \in \left(0, \frac{\pi}{2}\right)$ $= \left[\frac{3\theta^2}{2}\right]_0^{\frac{\pi}{2}}$ $= \frac{3\pi^2}{8}$	
(ii)	Area of the surface formed $= 2\pi\int_0^{\frac{\pi}{2}} y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$ $= 2\pi\int_0^{\frac{\pi}{2}} 3(\sin\theta - \theta\cos\theta)(3\theta)\, d\theta$ $= 18\pi\int_0^{\frac{\pi}{2}} \theta\sin\theta - \theta^2\cos\theta\, d\theta$ Now, $\int_0^{\frac{\pi}{2}} \theta\sin\theta\, d\theta = [-\theta\cos\theta]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta$ $= [\sin\theta]_0^{\frac{\pi}{2}}$ $= 1$	

	Or Alternatively note that $\frac{dy}{d\theta}=\frac{d}{d\theta}\left[\ 3(\sin\theta-\theta\cos\theta)\right]=3\theta\sin\theta$, so $\int_0^{\frac{\pi}{2}}\theta\sin\theta\ d\theta=\left[\sin\theta-\theta\cos\theta\right]_0^{\frac{\pi}{2}}=1$ $\int_0^{\frac{\pi}{2}}\theta^2\cos\theta\ d\theta=\left[\theta^2\sin\theta\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}2\theta\sin\theta\ d\theta$ $=\left(\frac{\pi^2}{4}\right)-2(1)$ (from above) $=\frac{\pi^2}{4}-2$ Area of the surface formed $=18\pi\left[1-\frac{\pi^2}{4}+2\right]$ $=54\pi-\frac{9\pi^3}{2}$	

5	**Solutions**	
(i)	$I_1=\int_0^a x^1 e^{-x^2}\,dx=-\frac{1}{2}\int_0^a -2xe^{-x^2}\,dx=-\frac{1}{2}\left[e^{-x^2}\right]_0^a=\frac{1}{2}\left(1-e^{-a^2}\right)$	
(ii)	$I_{n+2}=\int_0^a x^{n+2}e^{-x^2}\ dx$ $=-\frac{1}{2}\int_0^a x^{n+1}(-2xe^{-x^2})\ dx$ $=-\frac{1}{2}\left[e^{-x^2}x^{n+1}\right]_0^a+\frac{1}{2}\int_0^a e^{-x^2}(n+1)x^n\ dx$ $=-\frac{1}{2}\left[e^{-a^2}a^{n+1}-0\right]+\frac{n+1}{2}\int_0^a x^n e^{-x^2}dx$ $\therefore I_{n+2}=\frac{n+1}{2}I_n-\frac{1}{2}\left(e^{-a^2}a^{n+1}\right)$ (shown)	
(iii)	$I_5=I_{3+2}=\frac{3+1}{2}I_3-\frac{1}{2}\left(e^{-a^2}a^{3+1}\right)=2I_3-\frac{1}{2}\left(e^{-a^2}a^4\right)$ $I_3=\frac{1+1}{2}I_1-\frac{1}{2}\left(e^{-a^2}a^{1+1}\right)=I_1-\frac{1}{2}\left(e^{-a^2}a^2\right)=\frac{1}{2}\left(1-e^{-a^2}\right)-\frac{1}{2}\left(e^{-a^2}a^2\right)$ $I_5=2\left[\frac{1}{2}\left(1-e^{-a^2}\right)-\frac{1}{2}\left(e^{-a^2}a^2\right)\right]-\frac{1}{2}\left(e^{-a^2}a^4\right)$ $=1-e^{-a^2}\left(1+a^2+\frac{1}{2}a^4\right)$	
(iv)	$\int_0^a (2\pi xy)\ dx=\int_0^a (2\pi x^5e^{-x^2})\ dx$ $=2\pi I_5$ $=2\pi\left(1-e^{-a^2}\left(1+a^2+\frac{1}{2}a^4\right)\right)$ As $a\to\infty$, $e^{-a^2}\left(1+a^2+\frac{1}{2}a^4\right)\to 0$, $1-e^{-a^2}\left(1+a^2+\frac{1}{2}a^4\right)\to 1$. Therefore volume required $=2\pi$ units^3.	

6 Solutions

(i)

$5+5\mathrm{i}=2+\mathrm{i}+3+4\mathrm{i}$

The required vertex is

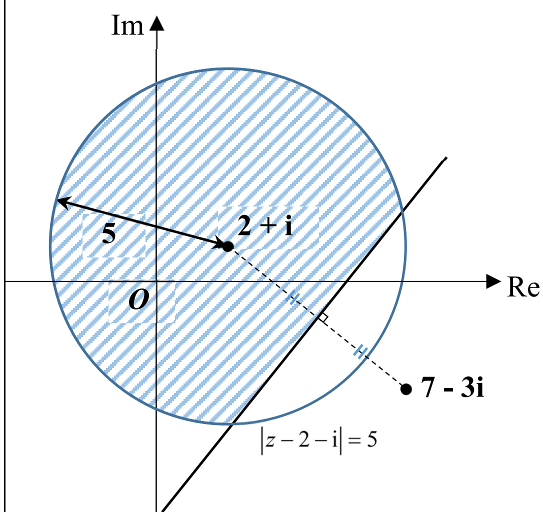
$$2+\mathrm{i}+(3+4\mathrm{i})\mathrm{e}^{\frac{-2}{3}\pi\mathrm{i}}=2+\mathrm{i}+(3+4\mathrm{i})\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$$

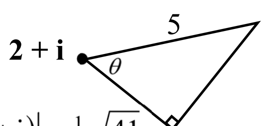
$$=2+\mathrm{i}+\left(-\frac{3}{2}+2\sqrt{3}-2\mathrm{i}-\frac{3\sqrt{3}}{2}\mathrm{i}\right)$$

$$=\left(\frac{1}{2}+2\sqrt{3}\right)-\left(\frac{3\sqrt{3}}{2}+1\right)\mathrm{i}$$

(ii)

$$|z-2-\mathrm{i}|\le|z^*-7-3\mathrm{i}| \quad\Rightarrow\quad |z-2-\mathrm{i}|\le|z-7+3\mathrm{i}|$$





$$\tfrac{1}{2}\left|(7-3\mathrm{i})-(2+\mathrm{i})\right|=\tfrac{1}{2}\sqrt{41}$$

$$\cos\theta=\frac{\sqrt{41}}{10}$$

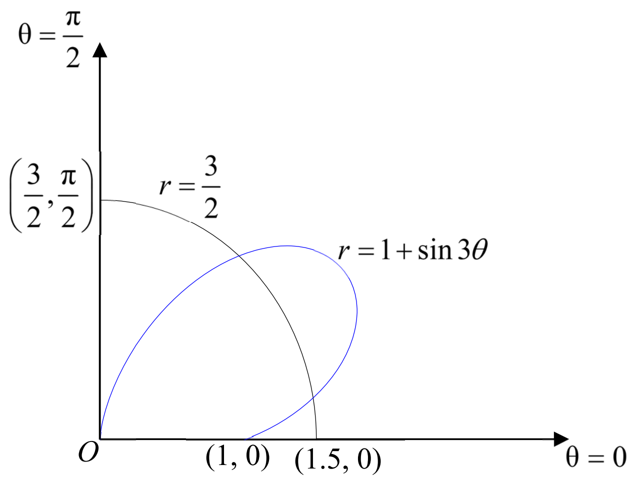
Required area is

$$\frac{1}{2}\cdot5^2(2\pi-2\theta)+\frac{1}{2}\cdot5^2\sin2\theta=25\left(\pi-\cos^{-1}\frac{\sqrt{41}}{10}\right)+25\sin\theta\cos\theta$$

$$=25\left(\pi-\cos^{-1}\frac{\sqrt{41}}{10}\right)+25\frac{\sqrt{59}}{10}\frac{\sqrt{41}}{10}$$

$$=25\left(\pi-\cos^{-1}\frac{\sqrt{41}}{10}+\frac{\sqrt{2419}}{100}\right)$$

$A=25,\ B=41$ and $C=\sqrt{2419}$

7	Solutions	
1(a) **[4]**	 At point of intersection, $$1+\sin 3\theta = \frac{3}{2} \Rightarrow 3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$ $$\Rightarrow \theta = \frac{\pi}{18}, \frac{5\pi}{18}$$ Polar coordinates of the points of intersection are $\left(\frac{3}{2}, \frac{\pi}{18}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{18}\right)$.	
(b) **[3]**	$r = 1+\sin 3\theta$ and $\frac{dr}{d\theta} = 3\cos 3\theta$ Perimeter of S $$= \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \sqrt{(1+\sin 3\theta)^2 + (3\cos 3\theta)^2}\, d\theta + \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \frac{3}{2}\, d\theta = 1.695032 + \frac{\pi}{3} = 2.742$$	
(c) **[4]**	Area of $S =$ $$\frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} (1+\sin 3\theta)^2 - \left(\frac{3}{2}\right)^2 d\theta$$ $$= \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} -\frac{5}{4} + 2\sin 3\theta + \sin^2 3\theta\, d\theta$$ $$= \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} -\frac{3}{4} + 2\sin 3\theta - \frac{\cos 6\theta}{2}\, d\theta$$ $$= \frac{1}{2}\left[-\frac{3}{4}\theta - \frac{2\cos 3\theta}{3} - \frac{\sin 6\theta}{12}\right]_{\frac{\pi}{18}}^{\frac{5\pi}{18}}$$ $$= -\frac{\pi}{12} + \frac{3}{8}\sqrt{3}$$	

8	**Solutions**	
(a)	Let $c = \cos\,\theta$ and $s = \sin\,\theta$ $(c+is)^5 = c^5 + 5(c)^4(\mathrm{i}s) + 10(c)^3(\mathrm{i}s)^2 + 10(c)^2(\mathrm{i}s)^3$ $+5(c)(\mathrm{i}s)^4 + (\mathrm{i}s)^5$ $= c^5 + 5\mathrm{i}c^4s - 10c^3s^2 - 10\mathrm{i}c^2s^3 + 5cs^4 + \mathrm{i}s^5$ By de Moivre's theorem,	

	$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ $= 5\left(1-\sin^2\theta\right)^2\sin\theta - 10\left(1-\sin^2\theta\right)\sin^3\theta + \sin^5\theta$ $= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$ $= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ Let $\theta = \dfrac{\pi}{5}$ Then $\sin\left[5\left(\dfrac{\pi}{5}\right)\right] = \sin\pi = 0 = 16\sin^5\left(\dfrac{\pi}{5}\right) - 20\sin^3\left(\dfrac{\pi}{5}\right) + 5\sin\left(\dfrac{\pi}{5}\right)$ $\Rightarrow 16\sin^5\left(\dfrac{\pi}{5}\right) - 20\sin^3\left(\dfrac{\pi}{5}\right) + 5\sin\left(\dfrac{\pi}{5}\right) = 0$ $\Rightarrow \sin\left(\dfrac{\pi}{5}\right)\left[16\sin^4\left(\dfrac{\pi}{5}\right) - 20\sin^2\left(\dfrac{\pi}{5}\right) + 5\right] = 0$ $\Rightarrow 16\sin^4\left(\dfrac{\pi}{5}\right) - 20\sin^2\left(\dfrac{\pi}{5}\right) + 5 = 0 \quad$ since $\sin\left(\dfrac{\pi}{5}\right) \neq 0$ $\Rightarrow \sin^2\left(\dfrac{\pi}{5}\right) = \dfrac{20 \pm \sqrt{400 - 4(16)(5)}}{2(16)}$ $= \dfrac{20 \pm \sqrt{80}}{32}$ $= \dfrac{5}{8} - \dfrac{\sqrt{5}}{8} \quad$ since $\sin^2\dfrac{\pi}{5} < \sin^2\dfrac{\pi}{3} = \dfrac{3}{4}$ Since $\sin 2\pi = 0 = 16\sin^5\left(\dfrac{2\pi}{5}\right) - 20\sin^3\left(\dfrac{2\pi}{5}\right) + 5\sin\left(\dfrac{2\pi}{5}\right)$ Therefore, $\sin^2\dfrac{2\pi}{5} = \dfrac{5}{8} + \dfrac{\sqrt{5}}{8}$	
(b)	Let $z = \cos\theta + \mathrm{i}\sin\theta$ then $\dfrac{1}{z} = \cos\theta - \mathrm{i}\sin\theta$ Therefore $\quad z + \dfrac{1}{z} = 2\cos\theta$ then $(2\cos\theta)^6 = \left(z + \dfrac{1}{z}\right)^6$ $= z^6 + 6z^4 + 15z^2 + 20 + \dfrac{15}{z^2} + \dfrac{6}{z^4} + \dfrac{1}{z^6}$ Using de Moivre’s theorem,	

	$64\cos^6\theta = \left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20$ $64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$ $\int_0^{\frac{\pi}{4}} \cos^6\theta \, \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16} \, \mathrm{d}\theta$ $= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{192}\sin\frac{6\pi}{4} + \frac{3}{64}\sin\frac{4\pi}{4} + \frac{15}{64}\sin\frac{2\pi}{4} + \frac{5}{16}\left(\frac{\pi}{4}\right) - 0$ $= \frac{1}{192}\sin\frac{3\pi}{2} + 0 + \frac{15}{64}\sin\frac{\pi}{2} + \frac{5\pi}{64}$ $= \frac{11}{48} + \frac{5\pi}{64}$	

9	**Solutions**	
(i)	Let $\$ v_n$ be the amount in the account at the end of the n th year, and the yearly management fee be $\$ k$ Then $v_n = 1.06v_{n-1} - k$ General solution: $v_n = 1.06^n C + D$ $v_0 = 30\,000 = C + D$ ---(1)	

	$v_1 = 1.06 \times 30\,000 - k = 1.06C + D$ ---(2) Solve (1) and (2) for C and D, then $v_n = 1.06^n\left(30\,000 - \frac{50}{3}k\right) + \frac{50}{3}k$ OR $v_n = 1.06^n(30\,000) - \frac{50}{3}k\left(1.06^n - 1\right)$ Using GC, for $v_{10} = 1.7 \times 30\,000$, the total fees is \$$10k =$ \$2067.73	
(ii) (a)	$u_n = bu_{n-1} - \frac{1}{50}u_{n-2}$ The characteristic equation is $\lambda^2 - b\lambda + \frac{1}{50} = 0$, giving $\lambda = \frac{b \pm \sqrt{b^2 - \frac{4}{50}}}{2}$ $u_n = A\left(\frac{b + \sqrt{b^2 - \frac{2}{25}}}{2}\right)^n + B\left(\frac{b - \sqrt{b^2 - \frac{2}{25}}}{2}\right)^n$ $u_0 = 30\,000 = A + B$ ---(1) $u_1 = 30\,000b = \frac{b + \sqrt{b^2 - \frac{2}{25}}}{2}A + \frac{b - \sqrt{b^2 - \frac{2}{25}}}{2}B$ ---(2) Solve (1) and (2): $A = \frac{30\,000\left(b + \sqrt{b^2 - \frac{2}{25}}\right)}{2\sqrt{b^2 - \frac{2}{25}}}$, $B = \frac{-30\,000\left(b - \sqrt{b^2 - \frac{2}{25}}\right)}{2\sqrt{b^2 - \frac{2}{25}}}$ OR any equivalent form such as $A = 15000 + \frac{15000b}{\sqrt{b^2 - \frac{2}{25}}}$, $B = 15000 - \frac{15000b}{\sqrt{b^2 - \frac{2}{25}}}$ etc. $u_n = \frac{30\,000}{\sqrt{b^2 - \frac{2}{25}}}\left(\left(\frac{b + \sqrt{b^2 - \frac{2}{25}}}{2}\right)^{n+1} - \left(\frac{b - \sqrt{b^2 - \frac{2}{25}}}{2}\right)^{n+1}\right)$	
(b)	Using GC, for $u_{10} = 1.7 \times 30\,000$, $b = 1.0716$ (5 sf)	

10	**Solutions**	
(a) **[2]**	Since (16, 8) is a point on the curve $x^2 = 4ay$, $a = 8$. Thus, coordinates of $M = (0,8)$. Equation of the directrix is $y = -8$.	
(b) **(i)** **[5]**	Since parabola has equation $x^2 = 4ay$, polar equation of the parabola is $(r\cos\theta)^2 = 32(r\sin\theta)$. Hence polar equation of the rotated parabola is $(r\cos(\theta-\alpha))^2 = 32(r\sin(\theta-\alpha))$ where $\tan\alpha = \frac{300}{400} = \frac{3}{4}$. Thus, $\sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$ and $(r\cos(\theta-\alpha))^2 = 32(r\sin(\theta-\alpha))$ $r^2\left(\cos\theta\cos\alpha + \sin\theta\sin\alpha\right)^2 = 32r\left(\sin\theta\cos\alpha - \cos\theta\sin\alpha\right)$ $x^2\left(\frac{4}{5}\right)^2 + y^2\left(\frac{3}{5}\right)^2 + 2xy\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = 32y\left(\frac{4}{5}\right) - 32x\left(\frac{3}{5}\right)$ $16x^2 + 24xy + 9y^2 + 480x - 640y = 0$ $x^2 + \frac{3xy}{2} + \frac{9y^2}{16} + 30x - 40y = 0$ Cartesian equation of the rotated parabola is $x^2 + \frac{3xy}{2} + \frac{9y^2}{16} + 30x - 40y = 0$.	
	Alternatively we use standard polar form of parabola with focus at pole, $r = \frac{16}{1-\sin(\theta-\alpha)}$, $\tan\alpha = \frac{3}{4}$. $r = 16 + r\sin\theta\cos\alpha - r\cos\theta\sin\alpha$ $\sqrt{x^2+y^2} = 16 + \frac{4y}{5} - \frac{3x}{5}$ $25\left(x^2+y^2\right) = 80^2 + 16y^2 + 9x^2 + 2\left(320y - 240x - 12xy\right)$ $16x^2 + 24xy + 9y^2 - 640y + 480x - 6400 = 0$ We then translate the graph so that focus is at $\left(-\frac{24}{5}, \frac{32}{5}\right)$. Replace x by $x + \frac{24}{5}$ and y by $y - \frac{32}{5}$	

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$$16\left(x+\frac{24}{5}\right)^2+24\left(x+\frac{24}{5}\right)\left(y-\frac{32}{5}\right)+9\left(y-\frac{32}{5}\right)^2$$
$$-640\left(y-\frac{32}{5}\right)+480\left(x+\frac{24}{5}\right)-6400=0$$

$$16x^2+24xy+9y^2+480x-640y$$
$$+\left[16\left(\frac{48}{5}\right)-24\left(\frac{32}{5}\right)\right]x+\left[24\left(\frac{24}{5}\right)-9\left(\frac{64}{5}\right)\right]y$$
$$+16\left(\frac{24}{5}\right)^2-24\left(\frac{24}{5}\right)\left(\frac{32}{5}\right)+9\left(\frac{32}{5}\right)^2+640\left(\frac{32}{5}\right)+480\left(\frac{24}{5}\right)-6400=0$$

$$16x^2+24xy+9y^2+480x-640y=0$$

Cartesian equation of the rotated parabola is $x^2+\frac{3xy}{2}+\frac{9y^2}{16}+30x-40y=0$.

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(b)
(ii)
[5]

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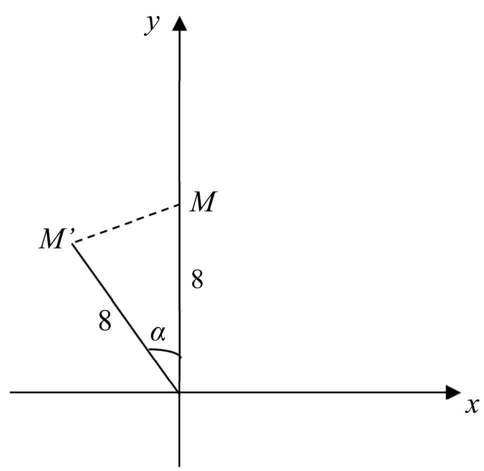

Coordinates of the focus of the rotated parabola is $(-8\sin\alpha, 8\cos\alpha)=\left(-\frac{24}{5},\frac{32}{5}\right)$.

By symmetry, $\left(\frac{24}{5},-\frac{32}{5}\right)$ is a point on the directrix.

Gradient of OC = $-\frac{4}{3}$

Gradient of the directrix = $\frac{3}{4}$

Thus the Cartesian equation of the directrix of the rotated parabola is

$$\frac{y+\frac{32}{5}}{x-\frac{24}{5}}=\frac{3}{4}\Rightarrow y=\frac{3}{4}x-10.$$

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	OR Alternatively Polar equation of the directrix of the rotated parabola is $r\sin(\theta-\alpha)=-8$. $r\sin(\theta-\alpha)=-8$ $r\sin\theta\cos\alpha - r\cos\theta\sin\alpha = -8$ $y\left(\frac{4}{5}\right)-x\left(\frac{3}{5}\right)=-8$ $y=\frac{3}{4}x-10$ Thus the Cartesian equation of the directrix of the rotated parabola is $y=\frac{3}{4}x-10$.	