**Tutorial 12 : Vectors II (Planes)**

**Practice Questions**

1. The plane is given by the Cartesian equation $3x + y - 2z = 3$.

   (a) Find the perpendicular distance of the plane from the origin.

   (b) Find the perpendicular distance of the point (1, 3, – 1) from the plane.

   [Ans: (a) $\frac{3}{\sqrt{14}}$ , (b) $\frac{5}{\sqrt{14}}$ ]

N2007/P1/Q8

2. The line $l$ passes through $A$ and $B$ with coordinates (1, 2, 4) and (-2, 3, 1) respectively. The plane $p$ has equation $3x - y + 2z = 17$. Find

   (i) the coordinates of the point of intersection of $l$ and $p$,

   (ii) acute angle between $l$ and $p$,

   (iii) the perpendicular distance from $A$ to $p$.

   [Ans: (i) $(2.5, 1.5, 5.5)$ (ii) 78.8° (iii) $\frac{4\sqrt{14}}{7}$ ]

3. N2009/P1/Q10

   The planes $p_1$ and $p_2$ have equations $\mathbf{r} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 1$ and $\mathbf{r} \bullet \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 2$ respectively, and meet in a line $l$.

   (i) Find the acute angle between $p_1$ and $p_2$. [3]

   (ii) Find a vector equation of $l$. [4]

   (iii) The plane $p_3$ has equation $2x + y + 3z - 1 + k(-x + 2y + z - 2) = 0$. Explain why $l$ lies in $p_3$ for any constant $k$. Hence, or otherwise, find a Cartesian equation of the plane in which both $l$ and the point (2, 3, 4) lie. [5]

   [Ans: (i) 70.9° (ii) $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ (iii) $x - y = -1$]

4. N2013/P2/Q4

   The planes $p_1$ and $p_2$ have equations $\mathbf{r}.\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1$ and $\mathbf{r}.\begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} = -1$ respectively, and meet in the line $l$.

   (i) Find the acute angle between $p_1$ and $p_2$. [3]

   (ii) Find a vector equation for $l$. [4]

(iii) The point $A$ (4, 3, $c$) is equidistant from the planes $p_1$ and $p_2$. Calculate the two possible values of $c$. [6]

[Ans: (i) $40.4^{\circ}$ (ii) $\mathbf{r} = \begin{pmatrix} -1/6 \\ -2/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7/6 \\ 5/3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ (iii) $\dfrac{35}{13}$ or $-49$]

5. N2014/P1/Q9

Planes $p$ and $q$ are perpendicular. Plane $p$ has equation $x + 2y - 3z = 12$. Plane $q$ contains the line $l$ with equation $\dfrac{x-1}{2} = \dfrac{y+1}{-1} = \dfrac{z-3}{4}$. The point $A$ on $l$ has coordinates (1, -1, 3).

(i) Find a Cartesian equation of $q$.
(ii) Find a vector equation of the line $m$ where $p$ and $q$ meet.
(iii) $B$ is a general point on $m$. Find an expression for the square of the distance $AB$.

Hence, or otherwise, find the coordinates of the point on $m$ which is nearest to $A$.

[Ans: (i) $x - 2y - z = 0$ (ii) $\mathbf{r} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}$ (iii) $\left|\overrightarrow{AB}\right|^2 = 21\mu^2 + 36\mu + 50$; $\dfrac{3}{7}\begin{pmatrix} 6 \\ 5 \\ -4 \end{pmatrix}$ ]

6. The equations of the line $l_1$ and plane $\Pi_1$ are as follows:

$l_1$ : $\underset{\sim}{r} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$ ; $\Pi_1$ : $xa + z = 5a + 4$, where $a$ is a positive constant.

(i) If the angle between $l_1$ and $\Pi_1$ is $\dfrac{\pi}{6}$, show that $a = 1$.

(ii) Find the position vector of A, the point of intersection between $l_1$ and $\Pi_1$.

(iii) Given that $C(7,-3,4)$, find the position vector of $N$, the foot of perpendicular of $C$ on $\Pi_1$.

(iv) Point $C'$ is obtained by reflecting $C$ about $\Pi_1$. Determine the vector equation of the line $AC'$.

[Answers: (ii) $5\mathbf{i}-\mathbf{j}+4\mathbf{k}$ (iii) $6\mathbf{i}-3\mathbf{j}+3\mathbf{k}$ (iv) $\mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ ]

**7** [NYJC/2018/Prelims/P1/Q5]

The line $l$ passes through the points $A$ and $B$ with coordinates (5, 2, 4) and (4, −1, 3) respectively. The plane $p$ has equation $4x+7y+5z=24$.

**(i)** The point $C$ lies on $l$ such that the foot of perpendicular of $C$ onto $p$ has coordinates (3, 1, 1).

Find the coordinates of $C$. [4]

Plane $p_1$ has equation $3x-2y+\lambda z=\mu$.

**(ii)** What can be said about the values of $\lambda$ and $\mu$ if $l$ does not intersect $p_1$? [2]

**(iii)** Hence find the exact values of $\mu$ if the distance between $p_1$ and $l$ is 2 units. [3]

[Ans: (i) (7, 8, 6) (ii) $\lambda=3, \mu\neq 23$ (iii) $\mu=23+2\sqrt{22}$ or $23-2\sqrt{22}$ ]

8. RHVS/2018/Prelim/P1/Q4

Relative to the origin $O$, the points $A$, $B$ and $C$ are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OC}=\mathbf{c}$ respectively, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors which are mutually non-parallel.

The plane $\Pi$ and the line $l$ have the following equations

$$\Pi : \mathbf{r}=\lambda\mathbf{a}+\mu\mathbf{b} \quad \text{where } \lambda, \mu \in \mathbb{R} \text{ and}$$

$$l : \mathbf{r}=\mathbf{a}+\beta\mathbf{c} \quad \text{where } \beta \in \mathbb{R}$$

respectively.

**(i)** Find the point(s) of intersection between $\Pi$ and $l$ given that the points $O$, $A$, $B$ and $C$ are

(a) coplanar,

(b) not coplanar. [3]

**(ii)** State the geometrical meaning of $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. [1]

In the rest of the question, $O$, $A$, $B$ and $C$ are not coplanar.

**(iii)** The vector $\mathbf{p}$ is a unit vector in the direction of $\mathbf{a}\times\mathbf{b}$. State the geometrical meaning of $|\mathbf{c}\cdot\mathbf{p}|$. [1]

**(iv)** It is given that $\overrightarrow{OA}=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}0\\0\\2\end{pmatrix}$. Find the volume of the pyramid $OABC$. [3]

[Ans: (iv) $\dfrac{1}{3}$ units$^2$ ]

9. TPJC Prelim 9758/2018/02/Q3

The line $l$ has equation $\dfrac{x+9}{3}=\dfrac{y+5}{1}$, $z=1$, and the plane $p_1$ has equation $-x+2y+z=6$.

**(i)** Find the acute angle between $l$ and $p_1$. [3]

Referred to the origin $O$, the point $A$ has position vector $2\mathbf{i}+\mathbf{j}-6\mathbf{k}$.

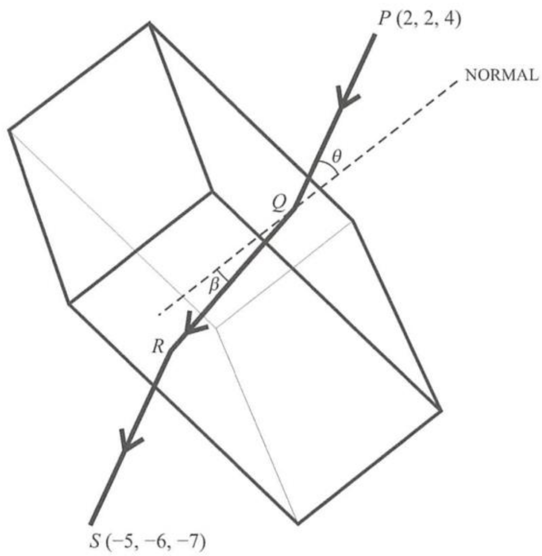
**(ii)** Find the position vector of $F$, the foot of the perpendicular from $A$ to $p_1$. [3]

**(iii)** Find the perpendicular distance from $A$ to $p_1$, in exact form. [2]

**(iv)** Given that $l$ is the line of intersection of the planes $p_2$ and $p_3$ with equation $x-3y-z=a$ and $x+by+z=7$ respectively, where $a$ and $b$ are real constants, find the values of $a$ and $b$. [4]

[Ans: (i) $7.4°$ (ii) $\overrightarrow{OF}=\begin{pmatrix}0\\5\\-4\end{pmatrix}$ (iii) $2\sqrt{6}$ (iv) $a=5, b=-3$ ]

10. 2019 A levels Exams/P1/Q12




A ray of light passes from air into a material made into a rectangular prism. The ray of light is sent in direction $\begin{pmatrix}-2\\-3\\-6\end{pmatrix}$ from a light source at the point $P$ with coordinates $(2,2,4)$. The prism is placed so that the ray of light passes through the prism, entering at the point $Q$ and emerging at the point $R$ and is picked up by a sensor at point $S$ with coordinates $(-5,-6,-7)$. The acute angle between $PQ$ and the normal to the top of the prism at $Q$ is $\theta$ and the acute angle between QR and the same normal is $\beta$ (see diagram).

It is given that the top of the prism is a part of the plane $x+y+z=1$, and that the base of the prism is a part of the plane $x+y+z=-9$. It is also given that the ray of light along $PQ$ is parallel to the ray of light along $RS$ so that $P$, $Q$, $R$ *and* $S$ lie in the same plane.

**(i)** Find the exact coordinates of $Q$ and $R$. [5]

**(ii)** Find the values of $\cos\theta$ and $\cos\beta$. [3]

**(iii)** Find the thickness of the prism measured in the direction of the normal at $Q$. [3]

Snell's law states that $\sin\theta = k\sin\beta$, where $k$ is a constant called the refractive index.

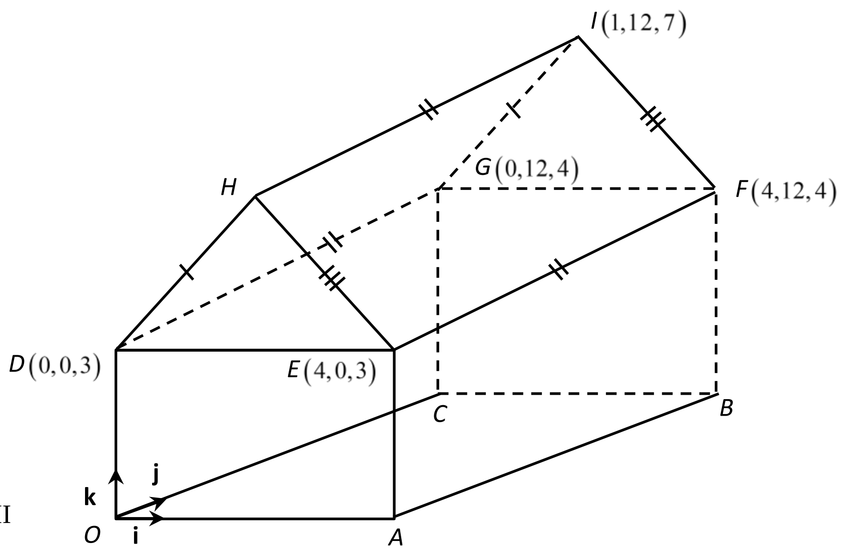
**(iv)** Find $k$ for the material of this prism. [1]

**(v)** What can be said about the value of $k$ for a material for which $\beta > \theta$? [1]

[Ans: (i) $Q\left(\frac{8}{11},\frac{1}{11},\frac{2}{11}\right)$, $R\left(-\frac{37}{11},-\frac{39}{11},-\frac{23}{11}\right)$ (ii) $\cos\theta = \frac{11\sqrt{3}}{21}$, $\cos\beta = \frac{11\sqrt{510}}{255}$ (iii) $\frac{10\sqrt{3}}{3}$
(iv) $k = 1.86$ or $k = \frac{\sqrt{170}}{7}$ ]

11. CJC Prelim 9758/2020/01/Q11

A temporary isolation centre is built to manage the increasing number of COVID-19 cases. The roof takes the shape of a triangular prism. Points $(x, y, z)$ are defined relative to an origin, $O$, with unit vectors $\mathbf{i}$ along $\overrightarrow{OA}$, j along $\overrightarrow{OC}$, and $\mathbf{k}$ along $\overrightarrow{OD}$ (see diagram). The coordinates of $D$, $E$, $F$, $G$ and $I$ are (0,0,3), (4,0,3), (4,12,4), (0,12,4) and (1,12,7) respectively. The units are measured in metres.



**(i)** Find a cartesian equation of the plane that contains the roof section *EFIH*. [3]

It is given that the roof section *DGIH* is part of the plane with equation $36x+y-12z=-36$.

**(ii)** Find a cartesian equation of the line that contains the roof ridge *HI*. [3]

Steel cables are used to hold the isolation centre in place. Cables are laid in straight lines and the widths of cables can be neglected. It is given that cable 1 passes through $F$ and $D$ and cable 2 passes through $G$ and $M$, where $M$ is the mid point of *EF*.

A builder needs to locate the point $J$ where both cables meet.

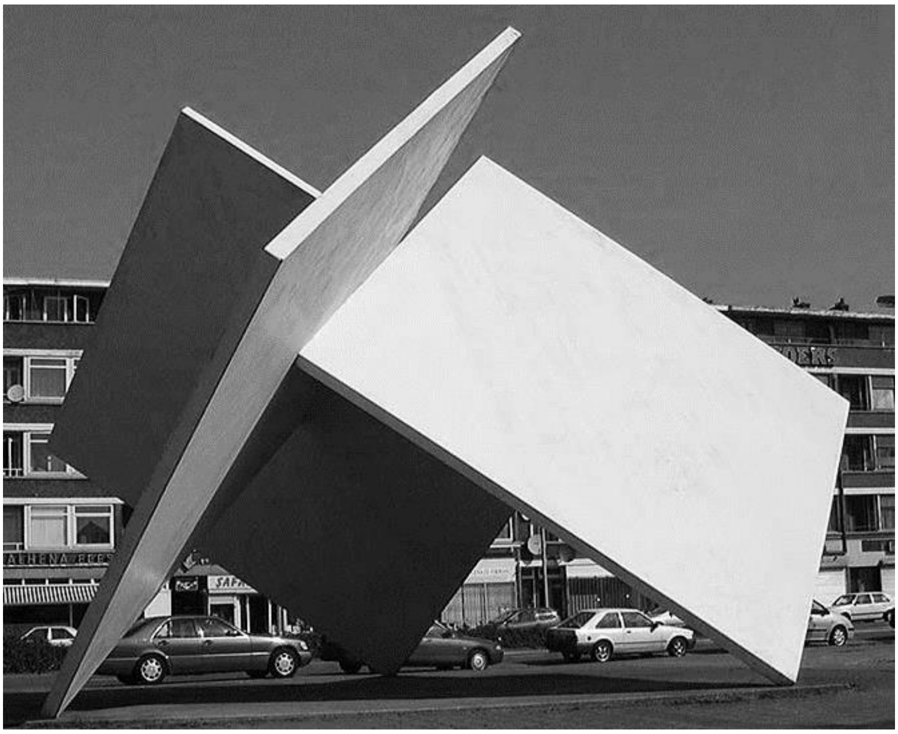
**(iii)** Find the coordinates of $J$. [4]

To strengthen the structure, it is recommended that another steel cable should be extended from $J$ to the closest point on the roof ridge *HI*. The builder is left with 3.2 metres of steel cable.

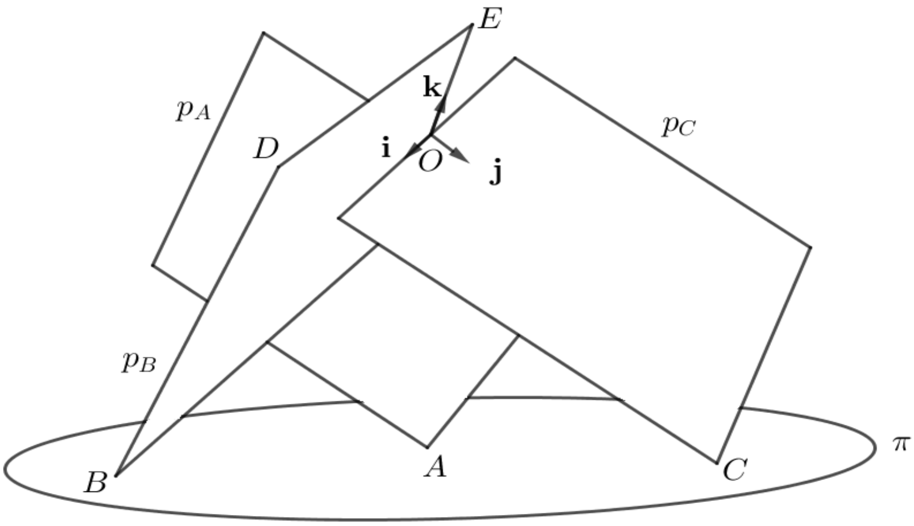
**(iv)** Determine whether the remaining steel cable is long enough to connect $J$ to the roof ridge *HI*. [4]

[Ans: (i) $12x-y+12z=84$ (ii) $x=1, \frac{y+72}{12}=z$ (iii) $J\left(\frac{8}{3},8,\frac{11}{3}\right)$]

12. NJC/2020/Prelims/P1/Q12



The picture shows a concrete sculpture in Rotterdam, Netherlands, made of three identical square slabs of side 8 m. The three slabs are assembled perpendicular to each other such that they meet at one common point *O*. Each slab has one vertex fixed to the flat horizontal ground, represented by the plane $\pi$, and one vertex fixed to the centre of another slab.




- The slab $p_A$ is part of the plane $x = 0$. It has one vertex, point *A*, on $\pi$ and another vertex on the centre of $p_C$.
- The slab $p_B$ is part of the plane $y = 0$. It has one vertex, point *B*, on $\pi$ and another vertex on the centre of $p_A$. The remaining two vertices are denoted as *D* and *E*.

- The slab $p_C$ is part of the plane $z=0$. It has one vertex, point *C*, on $\pi$ and another vertex on the centre of $p_B$.

The point *O* is taken as the origin and perpendicular unit vectors **i**, **j** and **k** are defined, as shown in the diagram, with **i**, **j**, **k** along one edge of $p_C$, $p_A$, $p_B$ respectively. For example, point *D* has position vector $8\mathbf{i}+4\mathbf{k}$. It is assumed that the slabs are made of materials with negligible thickness.

**(i)** Show that a Cartesian equation of $\pi$ is $x+y-z=12$. [4]

**(ii)** To make the sculpture more stable, plans are made to construct a vertical support from *O* to a point *F* on $\pi$, such that *OF* is perpendicular to $\pi$. Find the position vector of *F*. [3]

A pigeon is at a point *P* $(2,11,1)$. It flies in a straight line and rests on the sculpture along the edge *DE*.

**(iii)** Given that the pigeon rests at the point nearest to *P*, find the coordinates of this point. Hence find the distance the pigeon flew. [4]

Another pigeon has flight path with equation $\dfrac{x-1}{3}=\dfrac{2-y}{4}=z+3$.

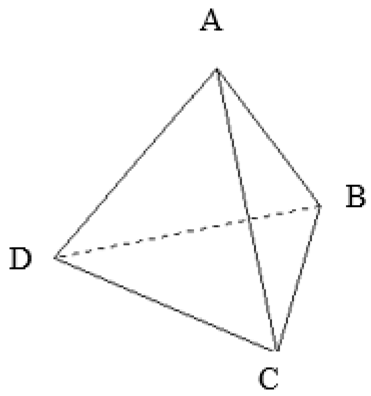
**(iv)** Without using a calculator, determine if the flight paths of the two pigeons cross. [3]

[Ans: (i) $x+y-z=12$ (ii) $\overrightarrow{OF}=\begin{pmatrix}4\\4\\-4\end{pmatrix}$ (iii) $\sqrt{130}$ ]

**<u>Mastery Questions</u>**

1. A tetrahedron is a polyhedron composed of four triangular faces, six straight edges, and four vertex corners.

   A regular tetrahedron is one in which all four faces are equilateral triangles.




The following Cartesian coordinates define the four vertices of a regular tetrahedron $A(1,1,1)$, $B(1,-1,-1)$, $C(-1,1,-1)$, $D(-1,-1,1)$.

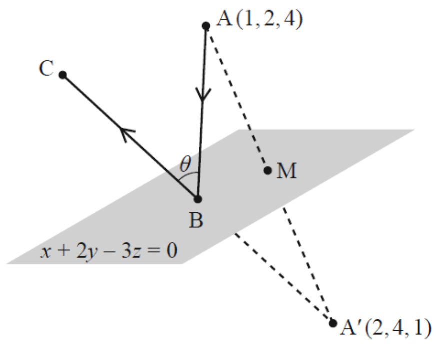
(i) Find the equation of the plane(in cartesian form) which model the surface *ABC* of the tetrahedron.

(ii) Find the length of perpendicular from *D* to the surface *ABC*.

[Ans : (i) $x+y-z=1$ (ii) $\frac{4}{\sqrt{3}}$]

2. [OCR/GCE Math (MEI) PA/JUN 2012/Q8]

A laser beam ABC is fired from the point A(1, 2, 4) and is reflected at point B off the plane with equation $x+2y-3z=0$, as shown in the diagram below. $A'$ is the point (2, 4,1), and M is the midpoint of $AA'$.




(i) Show that $AA'$ is perpendicular to the plane $x+2y-3z=0$, and that M lies on the plane.

The vector equation of the line *AB* is $\mathbf{r}=\begin{pmatrix}1\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$.

(ii) Find the coordinates of B, and a vector equation of the line $A'B$.

(iii) Given that $A'BC$ is a straight line, find the angle $\theta$.

(iv) Find the coordinates of the point where BC crosses the *x*-*z* plane.

[Ans: (ii) (0,3,2), $\mathbf{r}=\begin{pmatrix}2\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}-2\\-1\\1\end{pmatrix}, \lambda\in\mathbb{R}$; (iii) $\theta=80.4^0$; (iv) (-6,0,5)]

3. [VCE/Specialist Math/Nov 2007/Written Exam 2/Section2/Q4]

An aircraft approaching an airport with velocity $\mathbf{v} = 30\mathbf{i} - 40\mathbf{j} - 4\mathbf{k}$ is observed on the control tower radar screen at time $t = 0$ second. Ten seconds later, it passes over a navigation beacon with position vector $-500\mathbf{i} + 2500\mathbf{j}$ relative to the base of the control tower, $O$, at an altitude of 200 metres.

Let $\mathbf{i}$ and $\mathbf{j}$ be horizontal perpendicular unit vectors and let $\mathbf{k}$ be a unit vector in the vertical direction. Displace components are measured in metres.

(a) Show that the position vector of the aircraft relative to $O$ at time $t$ is given by $\mathbf{r} = (30t - 800)\mathbf{i} + (2900 - 40t)\mathbf{j} + (240 - 4t)\mathbf{k}$ .

(b) When does the aircraft land and how far (correct to the nearest metre) from the base of the control tower is the point of landing?

(c) At what angle from the runway, correct to 1 decimal place, does the aircraft land?

(d) At what time, correct to nearest second, is the aircraft closest to the base of the control tower?

(e) What distance does the aircraft travel from the line it is observed on the radar screen to the time it lands? Give your answer to the nearest metre.

[Ans: (b) 60 sec, 1118m; (c) $\theta = 4.6^0$ ; (d) 56.0; (e) 3010]

**Hint for Qn 14** : The following diagram models the situation where the dotted line represents the observed flight path of the aircraft.