National Junior College
2016 – 2017 H2 Further Mathematics
Topic F7: Matrices and Linear Spaces (Tutorial Set 1)

This tutorial set is for the following sections from the notes:

§1 System of Linear Equations
§2 Matrices and Matrix Operations
§3 Inverse Matrix and Its Applications
§4 Determinants

Basic Mastery Questions

1 Without using a graphic calculator, find a row-echelon form and the reduced row-echelon form of the matrix

$$\begin{pmatrix} 2 & 3 & 3 & 25 \\ 3 & 2 & 3 & 24 \\ 4 & 1 & 2 & 21 \end{pmatrix}.$$

2 It is given that $\mathbf{A} = \begin{pmatrix} a & 3 & 2 \\ 0 & -1 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ 0 & -3 & -2 \end{pmatrix}$. Find $\mathbf{AB}^T$ and $\mathbf{BA}^T$ in terms of a.

3 Which of the following are elementary matrices? State the inverses of these elementary matrices.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

4 Find the determinant and the inverse of $\begin{pmatrix} 2016 & 2017 \\ 2018 & 2019 \end{pmatrix}$.

5 Show that $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{pmatrix}$ is singular.

6 Let $\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 2 & 7 & 0 \end{pmatrix}$.

(i) Find $\det(\mathbf{A})$ using

(a) cofactor expansion;
(b) the special rule that only works for 3×3 matrix;
(c) row reductions (to a upper-triangular or diagonal matrix);
(c) a graphic calculator.

(ii) Find $\mathbf{A}^{-1}$ using

(a) elementary row operations;
(b) its adjoint;
(c) a graphic calculator.

7 Solve the following system of linear equations

$$\begin{aligned} 2x &+3y &+3z &= 25 \\ 3x &+2y &+3z &= 24 \\ 4x &+y &+2z &= 21 \end{aligned}$$

using

(a) Gaussian elimination method;
(b) Gauss-Jordan elimination method;
(c) the inverse of the coefficient matrix;
(d) Cramer's Rule,
(e) a graphic calculator.

What can you say about the relationship among the three planes with these equations?

8 Solve the following linear system

$$\begin{aligned} x &+2y &+3z &+4w &= 5 \\ x &+3y &+5z &+7w &= 11 \\ x & &-3z &-2w &= -7 \end{aligned}$$

using

(a) Gauss-Jordan elimination method;
(b) a graphic calculator.

9 Express the matrix $\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$ as a product of elementary matrices.

Practice Questions

10 Determine the value(s) of a such that $\begin{pmatrix} 2 & -5 & -1 \\ 1 & a & 1 \\ -3 & 10 & 2a \end{pmatrix}$ is singular.

Hence, discuss how the value of b affects the number of the solutions of the linear system, and give a geometrical interpretation of the solutions for each case.

$$\begin{aligned} 2x \quad -5y \quad -z \quad &= 0 \\ x \quad +by \quad +z \quad &= -2 \\ -3x \quad +10y + 2bz &= 10 \end{aligned}$$

11 Use row reduction to show that $\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix} = (a_2 - a_1)(a_3 - a_2)(a_3 - a_1)$.

12 **(a)** Given $b \neq 0$, prove that $\begin{vmatrix} a & a+b & a+2b \\ a+b & a+2b & a \\ a+2b & a & a+b \end{vmatrix} = 0$ if and only if $a = -b$.

(b) Prove that $\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} = 2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

(1973 A Level / FM / Nov / P2)

13 Show that if $\mathbf{A}^n = \mathbf{O}$ for some integer $n \geq 1$, then $\mathbf{A}$ is not invertible.

14 Let $\mathbf{A}$ be the matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

Show that if $a+b+c = d+e+f = g+h+i = 0$, then $\mathbf{A}$ is not invertible.

15 Show that there does not exist an $n \times n$ matrix $\mathbf{A}$, where n is odd, such that $\mathbf{A}^2 + \mathbf{I} = \mathbf{O}$.

16 What can you say about a and b if the matrix $\mathbf{A} = \begin{pmatrix} 1 & 4 & 3 \\ -2 & 8 & a \\ 1 & b & 3 \end{pmatrix}$ is singular?

17 Let $\mathbf{D} = \left(d_{ij}\right)$ be an $n \times n$ diagonal matrix.

(i) Show that $\mathbf{D}^m$ is also an $n \times n$ diagonal matrix with entries on the main diagonal, $d_{11}{}^n$, $d_{22}{}^n$, …, $d_{nn}{}^m$, for all positive integer m.

(ii) It is further given $d_{ii} \neq 0$ for all $1 \leq i \leq n$. Explain whether **(i)** holds if m is a negative integer instead.

18 Given matrices **A** and **B** where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \\ 3 & 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 12 \\ 3 & 1 & 0 \end{pmatrix},$$

find non-singular matrices **P** and **Q** such that **PA** and **QB** are echelon matrices in which the first non-zero entry in any row is unity.

For each of **A** and **B** find its inverse, if it exists.

For each of **(a)** and **(b)** below, solve, if possible, the equation:

(a) $\mathbf{Ax} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$, **(b)** $\mathbf{Bx} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$.

(1985 A Level / FM / Nov / P2)

19 Given that **A** is an invertible 3×3 matrix, show that the first column of $\mathbf{A}^{-1}$ is the solution of the equation

$$\mathbf{Ax} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Give equations whose solutions are the second and third columns of $\mathbf{A}^{-1}$ respectively. Hence or otherwise, find the inverse of **P**, where

$$\mathbf{P} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the matrix **B** given that $\mathbf{BP} = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & 5 \end{pmatrix}$.

(1973 A Level / FM / Jun / P1)

20 **(i)** By considering $\begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix}$ and $\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix}$ (from **PQ 11**), form a conjecture for

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \end{vmatrix}.$$

(ii) Using factor theorem or otherwise, prove your conjecture.

(iii) Generalise and prove the result for the $n \times n$ matrix

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}.$$

21 **(a)** The matrices **A** and **B** are such that $\mathbf{AB} = \mathbf{BA}$. Show that

$$(\mathbf{A}+\mathbf{B})^3 = \mathbf{A}^3 + 3\mathbf{A}^2\mathbf{B} + 3\mathbf{A}\mathbf{B}^2 + \mathbf{B}^3.$$

(b) Let **C** be a given matrix where $\mathbf{C} \neq \mathbf{O}$. Find two different matrices **P** satisfying both the equations $\mathbf{CP} = \mathbf{PC}$ and $\mathbf{P}^2 - \mathbf{PC} - 6\mathbf{C}^2 = \mathbf{O}$.

(c) Given that $\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, show that all the solutions of the equation $\mathbf{QD} = \mathbf{DQ}$ are of the form $\mathbf{Q} = \alpha\mathbf{I} + \beta\mathbf{D}$ where α and β are scalars.

(1982 A Level / FM / Jun / P2)

22 A square matrix **A** is said to be ***symmetric*** if $\mathbf{A}^T = \mathbf{A}$ and ***skew-symmetric*** **if** $\mathbf{A}^T = -\mathbf{A}$. For each of the following statements, either prove it, or give a counterexample to show that it is false.

(i) **A** is a 2×2 non-singular symmetric matrix $\Rightarrow$ $\mathbf{A}^{-1}$ is symmetric.

(ii) **A** is a 2×2 skew-symmetric matrix $\Rightarrow \det(\mathbf{A}) = 0$.

(iii) **A** is a 3×3 skew-symmetric matrix $\Rightarrow \det(\mathbf{A}) = 0$.

(iv) $\mathbf{A} - \mathbf{A}^T$ is a skew-symmetric matrix, where **A** is any 2×2 matrix.

(v) Any 2×2 matrix can be expressed as the sum of a symmetric an a skew-symmetric matrix.

(1977 A Level / FM / Jun / P2)

23 Show that the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

satisfies the equation

$$\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I} = \mathbf{O},$$

where **I** and **O** denote the 2×2 identity and zero matrices respectively.

Prove by induction that, for each positive integer n, there are real numbers b_n and c_n such that

$$\mathbf{A}^n = b_n \mathbf{A} + c_n \mathbf{I}.$$

Hence or otherwise, find the matrix **B** without using a calculator, where

$$\mathbf{B} = \mathbf{A}^4 - 3\mathbf{A}^3 - 7\mathbf{A}^2 - 10\mathbf{A} - 6\mathbf{I}.$$

(1976 A Level / FM / Jun / P2)

24 **(i)** Let **A** be the matrix $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}$.

Write down matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ such that

$$\mathbf{P}_1\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 + a_1 & b_2 + a_2 & b_3 + a_3 & b_4 + a_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix};$$

$$\mathbf{P}_2\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 + a_1 & b_2 + a_2 & b_3 + a_3 & b_4 + a_4 \\ c_1 + ka_1 & c_2 + ka_2 & c_3 + ka_3 & c_4 + ka_4 \end{pmatrix}.$$

(ii) Given that $\mathbf{B} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & -3 & -6 \\ 4 & -9 & -13 & -25 \end{pmatrix}$, find a matrix **P** such that

$$\mathbf{PB} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & x & y \\ 0 & 0 & s & t \end{pmatrix}.$$

Solve the equation $\mathbf{Bx} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$.

(1982 A Level / FM / Nov / P1)

25 The matrix $\mathbf{A}$ is given by $\mathbf{A}=\begin{pmatrix}6&3&2\\3&2&1\\8&4&3\end{pmatrix}$.

By performing elementary row operations on the matrix $(\mathbf{A}\,|\,\mathbf{I})$, find $\mathbf{A}^{-1}$.

Hence, or otherwise, obtain $\mathbf{B}^{-1}$, where $\mathbf{B}=\begin{pmatrix}1&\frac{1}{2}&\frac{1}{3}\\\frac{1}{2}&\frac{1}{3}&\frac{1}{6}\\\frac{1}{3}&\frac{1}{6}&\frac{1}{8}\end{pmatrix}$.

Given that the real numbers x_1, x_2, x_3 satisfy the equation $\mathbf{B}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$, show that the solution of the equation $\mathbf{Bx}=\begin{pmatrix}c_1+\delta\\c_2-\delta\\c_3-\delta\end{pmatrix}$ is $\mathbf{x}=\begin{pmatrix}x_1+42\delta\\x_2-18\delta\\x_3-96\delta\end{pmatrix}$.

(1983 A Level / FM / Nov / P2)

26 The matrix $\mathbf{A}$ is given by $\mathbf{A}=\begin{pmatrix}-5&4&3\\10&-7&-6\\-8&6&5\end{pmatrix}$.

By performing elementary row operations on the matrix $(\mathbf{A}\,|\,\mathbf{I})$, find $\mathbf{A}^{-1}$.

(a) Solve the equation $\mathbf{XA}=\mathbf{K}$, where $\mathbf{K}=\begin{pmatrix}-1&2&3\end{pmatrix}$.

(b) Solve the equation

$$\begin{pmatrix}x&y&z&t\end{pmatrix}\begin{pmatrix}-1&2&3\\-5&4&3\\10&-7&-6\\-8&6&5\end{pmatrix}=\begin{pmatrix}2&-2&1\end{pmatrix}$$

by multiplying both sides the equation by $\mathbf{A}^{-1}$, or otherwise.

(1984 A Level / FM / Nov / P2)

27 Given that $\mathbf{M}$ is a 2×2 matrix such that $\mathbf{M}^2=\mathbf{I}$, show that $\mathbf{M}=\mathbf{M}^{-1}$.

Show also that if $\mathbf{M}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ then

<u>either</u> **(i)** $\mathbf{M}=\mathbf{I}$ or $\mathbf{M}=-\mathbf{I}$,

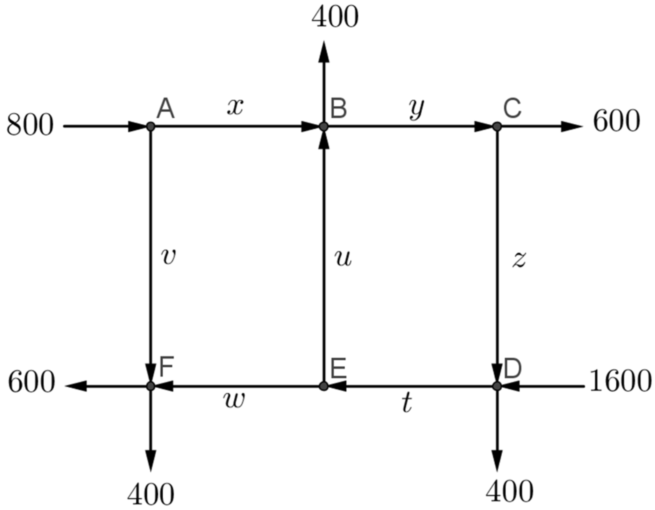
or **(ii)** $\begin{vmatrix}a&b\\c&d\end{vmatrix}=-1$ and $a+d=0$.

(1977 A Level / FM / Nov / P2)

Application Problems

28 (Traffic Network Problem)

The diagram below represents the traffic flow through a certain block of streets. The numbers represent the average flows into and out of the network at peak traffic hours.



Kirchhoff's first law states that the sum of the currents flowing into a node is equal to the sum of the current flowing out.

(i) By considering the traffic flows at each of the nodes A, B, C, D, E and F, set up a system of 6 linear equations for unknowns (x, y, z, u, v, w, t) in the diagram.

(ii) Solve the linear solution.

(iii) Suppose the streets between A and B, and between B and C must be closed. State the value of (x, y, z, u, v, w, t) and explain the significance of the *polarity* of z.

29 (Equation of Plane)

Prove the follow result:

An equation of the plane pass through the distinct points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is ,

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x & y & z & 1 \end{vmatrix} = 0.$$

Hence state, in a similar form, the condition for four distinct points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) to be coplanar.

30 (Encryption Problem)

Because of the heavy use of the Internet to conduct business, Internet security is of the utmost importance. If a malicious party should receive confidential information such as passwords, personal identification numbers, credit card numbers, social security numbers, bank account details, or corporate secrets, the effects can be damaging. To protect the confidentiality and integrity of such information, the most popular forms of Internet security use data *encryption*, the process of encoding information so that the only way to decode it, apart from a brute force "exhaustion attack," is to use a *key*. Data encryption technology uses algorithms based on the material presented here, but on a much more sophisticated level, to prevent malicious parties from discovering the key.

A *cryptogram* is a message written according to a secret code. The following describes a simplified method of using matrix multiplication to *encode* and *decode* messages.

To begin, assign a number to a blank space, '_', and each letter in the alphabet as follows:

Letter	A	B	C	D	E	F	G	H	I
Number Assigned	-13	-12	-11	-10	-9	-8	-7	-6	-5

Letter	J	K	L	M	N	O	P	Q	R
Number Assigned	-4	-3	-2	-1	1	2	3	4	5

Letter	S	T	U	V	W	X	Y	Z	_
Number Assigned	6	7	8	9	10	11	12	13	0

Table of Conversion

A message such as "YES SIR" is first converted and partitioned into *uncoded row matrices of size* 1×3:

$$\begin{pmatrix}12 & -9 & 6\end{pmatrix}, \begin{pmatrix}0 & 6 & -5\end{pmatrix}, \begin{pmatrix}5 & 0 & 0\end{pmatrix},$$

in which 0's are used to fill up the last uncoded row matrix if necessary.

To encode this message, a 3×3 matrix $\mathbf{A}$ is used to multiply each of the uncoded row matrices on the right to obtain *coded row matrices*. The matrix notation is then removed to obtain the cryptogram.

(a) Given $\mathbf{A}=\begin{pmatrix}6 & 3 & 2\\ 3 & 2 & 1\\ 8 & 4 & 3\end{pmatrix}$, we multiple each of the uncoded row matrices as follows.

$$\begin{pmatrix}12 & -9 & 6\end{pmatrix}\mathbf{A}=\begin{pmatrix}93 & 42 & 33\end{pmatrix}$$
$$\begin{pmatrix}0 & 6 & -5\end{pmatrix}\mathbf{A}=\begin{pmatrix}-22 & -8 & -9\end{pmatrix}.$$
$$\begin{pmatrix}5 & 0 & 0\end{pmatrix}\mathbf{A}=\begin{pmatrix}30 & 15 & 10\end{pmatrix}$$

In this case, the cryptogram is

$$93 \quad 42 \quad 33 \quad -22 \quad -8 \quad -9 \quad 30 \quad 15 \quad 10\,.$$

Using the same matrix,

(i) find the cryptogram for the message "HELLO WORLD",

(ii) recover the message by decoding the cryptogram

$$-46 \quad -23 \quad -16 \quad -33 \quad -12 \quad -14 \quad -32 \quad -20 \quad -11\,.$$

What condition must a 3×3 matrix satisfy so that it can be used to encode a message? Explain your answer.

(b) A hacker manages to get a message of n letters (inclusive of blanks) and its corresponding cryptogram. The 3×3 matrix $\mathbf{B}$ used to encode this message is unknown to him.

Assuming that the table of conversion remains the same,

(i) What is the minimum value of n such that the hacker can *possibly* find the matrix $\mathbf{B}$.

(ii) Explain why the hacker may not be able to find the matrix $\mathbf{B}$ when the length of the message is exactly n.

Suggest two ways in which we can improve the security of this encryption method.

Numerical Answers

Basic Mastery Questions

1 $\begin{pmatrix} 1 & 0.25 & 0.5 & 5.25 \\ 0 & 1 & 0.8 & 5.8 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 3.2 \\ 0 & 1 & 0 & 4.2 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$

2 $\mathbf{AB}^T = \begin{pmatrix} a+4 & -a+9 & -13 \\ 12 & -3 & -9 \end{pmatrix}$, $\mathbf{BA}^T = \begin{pmatrix} a+4 & 12 \\ -a+9 & -3 \\ -13 & -9 \end{pmatrix}$.

3 **A**, **C** and **D**. $\mathbf{A}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{C}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{D}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

4 $-2, \begin{pmatrix} -\frac{2019}{2} & \frac{2017}{2} \\ 1009 & -1008 \end{pmatrix}.$

6 **(i)** $\det(\mathbf{A}) = 1$, **(ii)** $\mathbf{A}^{-1} = \begin{pmatrix} 7 & 0 & -3 \\ -2 & 0 & 1 \\ -2 & -1 & 1 \end{pmatrix}$.

7 $x = 3.2$, $y = 4.2$, $z = 2$. The three planes intersect at exactly one point with coordinates $(3.2, 4.2, 2)$.

8 $x = -7 + 2t, y = 6 - 3t, z = 0, w = t, t \in \mathbb{R}.$

9 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$

Practice Questions

10 $a = -3$ or $a = \dfrac{5}{4}$.

When $b = -3$, there are infinitely many solutions, the three planes intersect in a line.

When $b = \dfrac{5}{4}$, there is no solution, the three planes have no point in common but form a triangular prismatic surface.

When $b \neq -3$ and $b \neq \dfrac{5}{4}$, there is exactly one solution, the three planes intersect at exactly one point.

16 $a=-6$ or $b=4$.

18 $\mathbf{P}=\begin{pmatrix}1&1&0\\1&2&0\\-1&1&1\end{pmatrix}$ (not unique), $\mathbf{Q}=\begin{pmatrix}1&1&0\\\frac{1}{3}&\frac{2}{3}&0\\-\frac{1}{3}&\frac{1}{12}&\frac{1}{4}\end{pmatrix}$ (not unique); $\mathbf{A}^{-1}=\begin{pmatrix}-4&3&4\\12&-9&-11\\-1&1&1\end{pmatrix}$, $\mathbf{B}$ has no inverse.

(i) $\mathbf{x}=\begin{pmatrix}7\\-16\\2\end{pmatrix}$ **(ii)** $\mathbf{x}=\begin{pmatrix}3t+1\\-9t+2\\t\end{pmatrix}$.

19 $\mathbf{P}^{-1}=\begin{pmatrix}1&-a&ac-b\\0&1&-c\\0&0&1\end{pmatrix}$; $\mathbf{B}=\begin{pmatrix}3&-3a-1&3ac-3b+c+2\\1&-a&ac-b+5\end{pmatrix}$.

20 **(i)** $(a_4-a_3)(a_4-a_2)(a_4-a_1)(a_3-a_2)(a_3-a_1)(a_2-a_1)$.

21 **(b)** $3\mathbf{C}$, $-2\mathbf{C}$.

22 **(i)** True, **(ii)** False, **(iii)** True, **(iv)** True, **(v)** True.

23 $\begin{pmatrix}7&6\\12&13\end{pmatrix}$.

24 **(i)** $\mathbf{P}_1=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}$, $\mathbf{P}_2=\begin{pmatrix}1&0&0\\1&1&0\\k&0&1\end{pmatrix}$.

(ii) $\mathbf{P}=\begin{pmatrix}1&0&0\\\frac{2}{7}&-\frac{1}{7}&0\\2&-3&1\end{pmatrix}$; $x=1, y=2, s=0, t=1$; $\mathbf{x}=\begin{pmatrix}2+\lambda\\1-\lambda\\\lambda\\0\end{pmatrix}$.

25 **(i)** $\mathbf{A}^{-1}=\begin{pmatrix}2&-1&-1\\-1&2&0\\-4&0&3\end{pmatrix}$; $\mathbf{B}^{-1}=\begin{pmatrix}12&-6&-24\\-6&12&0\\-24&0&72\end{pmatrix}$.

26 $\mathbf{A}^{-1}=\begin{pmatrix}2&-1&-1\\-1&2&0\\-4&0&3\end{pmatrix}$

(a) $\mathbf{X}=\begin{pmatrix}-7&6&12\end{pmatrix}$, **(b)** $x=\lambda, y=7\lambda-10, z=4-6\lambda, t=11-12\lambda$.

Application Problems

28 **(i)**

$$\begin{array}{ccccccccc} x & +y & & & & & & = & 800 \\ x & -y & & +u & & & & = & 400 \\ & y & -z & & & & & = & 600 \\ & & z & & & & -t & = & -1200 \\ & & & u & & +w & -t & = & 0 \\ & & & & v & +w & & = & 1000 \end{array}.$$

(ii) $(x,y,z,u,v,w,t)=(\lambda-200,\mu-600,\mu-1200,\mu-\lambda,1000-\lambda,\lambda,\mu),\ \lambda,\mu\in\mathbb{R}$.

(iii) $z=-600$. The direction of the traffic flow should be from D to C instead.

29

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0.$$

30 **(a)** **(i)** $-79 \quad -44 \quad -27 \quad -6 \quad -2 \quad -2 \quad 106 \quad 54 \quad 37 \quad -42 \quad -26 \quad -14$.

(ii) I LOVE FM.

(b) **(i)** 9.