2023 NYJC/TJC/VJC Prelim H3 Math Paper 1

1(i)

For all $t \in \mathbb{R}$, $\sum_{i=1}^{n}(a_i t - b_i)^2 \geq 0 \Rightarrow \left(\sum_{i=1}^{n} a_i^2\right)t^2 - 2\left(\sum_{i=1}^{n} a_i b_i\right)t + \sum_{i=1}^{n} b_i^2 \geq 0$ Note that coefficient of t^2, $\sum_{i=1}^{n} a_i^2 > 0$ since a_i is non-zero. For the quadratic function to be non-negative for all values of *t*, we have discriminant ≤ 0.

$$4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq 0$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

1(ii)

Given $p \leq \frac{b_i}{a_i} \leq q \quad \Rightarrow \quad \left(p - \frac{b_i}{a_i}\right) \leq 0$ and $\left(q - \frac{b_i}{a_i}\right) \geq 0$.

Thus

$$\left(p - \frac{b_i}{a_i}\right)\left(q - \frac{b_i}{a_i}\right) \leq 0$$

$$\frac{1}{a_i^2}(pa_i - b_i)(qa_i - b_i) \leq 0$$

$$pqa_i^2 - (p+q)a_i b_i + b_i^2 \leq 0 \text{ since } a_i^2 > 0$$

Taking summation

$$\Rightarrow \quad \sum_{i=1}^{k}\left[pqa_i^2 - (p+q)a_i b_i + b_i^2\right] \leq 0$$

$$\Rightarrow \quad pq\sum_{i=1}^{k}(a_i^2) - (p+q)\sum_{i=1}^{k}(a_i b_i) + \sum_{i=1}^{k}(b_i^2) \leq 0$$

$$\therefore \quad (p+q)\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} b_i^2 + pq\sum_{i=1}^{n} a_i^2$$

1(iii)

Since $m \leq a_i \leq M$ and $m \leq b_i \leq M$, thus $\frac{m}{M} \leq \frac{b_i}{a_i} \leq \frac{M}{n}$

From **(ii)**, let $p = \frac{m}{M}$ and $q = \frac{M}{m}$ so that

$$\left(\frac{m}{M} + \frac{M}{m}\right)\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} b_i^2 + \underbrace{\left(\frac{m}{M}\right)\left(\frac{M}{m}\right)}_{=1}\sum_{i=1}^{n} a_i^2 .$$

By AM-GM inequality, $\frac{\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2}{2} \geq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$

Thus $\left(\frac{m}{M} + \frac{M}{m}\right)\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n} a_i^2 \geq 2\sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$.

Squaring both sides:

$$\frac{1}{4}\left(\frac{m}{M} + \frac{M}{m}\right)^2 \left\{\sum_{i=1}^{n} a_i b_i\right\}^2 \geq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \text{ (shown)}$$

2(i)	$\frac{dy}{dx} = f(x, y) = f(tx, ty)$ By letting $t = \frac{1}{y}$, we have $\frac{dy}{dx} = f\left(\frac{x}{y}, 1\right) = g\left(\frac{x}{y}\right)$ (shown).	
2(ii)	$2xye^{\left(\frac{x}{y}\right)^2}\frac{dx}{dy} = y^2 + (y^2 + 2x^2)e^{\left(\frac{x}{y}\right)^2} \Leftrightarrow \frac{dy}{dx} = \frac{2xye^{\left(\frac{x}{y}\right)^2}}{y^2 + (y^2 + 2x^2)e^{\left(\frac{x}{y}\right)^2}}$ Let $f(x, y) = \frac{2xye^{\left(\frac{x}{y}\right)^2}}{y^2 + (y^2 + 2x^2)e^{\left(\frac{x}{y}\right)^2}}$. $f(tx, ty) = \frac{2(tx)(ty)e^{\left(\frac{tx}{ty}\right)^2}}{(ty)^2 + \left[(ty)^2 + 2(tx)^2\right]e^{\left(\frac{tx}{ty}\right)^2}} = \frac{2xye^{\left(\frac{x}{y}\right)^2}}{y^2 + (y^2 + 2x^2)e^{\left(\frac{x}{y}\right)^2}} = f(x, y)$ Hence, $2xye^{\left(\frac{x}{y}\right)^2}\frac{dx}{dy} = y^2 + (y^2 + 2x^2)e^{\left(\frac{x}{y}\right)^2}$ is a homogeneous differential equation.	
2(iii)	Let $u = \frac{x}{y} \Rightarrow x = yu$, $\frac{dx}{dy} = u + y\frac{du}{dy}$. $2uy^2e^{u^2}\left(u + y\frac{du}{dy}\right) = y^2 + (y^2 + 2u^2y^2)e^{u^2}$ $2uy^3e^{u^2}\frac{du}{dy} = y^2 + y^2e^{u^2}$ $\int \frac{2ue^{u^2}}{1 + e^{u^2}}\, du = \int \frac{1}{y}\, dy$ $\ln(1 + e^{u^2}) = \ln\lvert y\rvert + \ln\lvert c\rvert = \ln\lvert cy\rvert$ $y = A(1 + e^{u^2})$ where $A = \pm\frac{1}{c}$ General solution: $y = A\left[1 + e^{\left(\frac{x}{y}\right)^2}\right]$.	
	Tangent at $(4, -2)$ is perpendicular to the line $y = mx$. When $x = 4$, $y = -2$, $\frac{dy}{dx} = -\frac{1}{m} \Rightarrow \frac{dx}{dy} = -m$ Substitute into DE: $2(-8)e^4(-m) = 4 + \left[4 + 2(16)\right]e^4$ $\therefore \quad m = \frac{1}{4e^4}(1 + 9e^4)$.	

3(i)

$$\int_a^x \mathrm{f}'(t)\,\mathrm{d}t = \left[\mathrm{f}(t)\right]_a^x = \mathrm{f}(x) - \mathrm{f}(a)$$

which on rearranging gives

$$\begin{aligned}\mathrm{f}(x) &= \mathrm{f}(a) + \int_a^x \mathrm{f}'(t)\,\mathrm{d}t \\ &= \mathrm{f}(a) + \frac{1}{0!}\int_a^x \mathrm{f}'(t)(x-t)^0\,\mathrm{d}t\end{aligned}$$

which implies Taylor's theorem holds for the case $n = 0$.

3(ii)

Assume Taylor's theorem holds for $n = k \in \mathbb{Z}^+ \cup \{0\}$. That is,

$$\mathrm{f}(x) = \sum_{r=1}^{k} \frac{\mathrm{f}^{(r)}(a)}{r!}(x-a)^r + \frac{1}{k!}\int_a^x \mathrm{f}^{(k+1)}(t)(x-t)^k\,\mathrm{d}t\,.$$

Using integration by parts with $u = \mathrm{f}^{(k+1)}(t)$; $\dfrac{\mathrm{d}v}{\mathrm{d}x} = (x-t)^k$:

$$\begin{aligned}\int_a^x \mathrm{f}^{(k+1)}(t)(x-t)^k\,\mathrm{d}t &= -\frac{1}{k+1}\left[\mathrm{f}^{(k+1)}(t)(x-t)^{k+1}\right]_a^x \\ &\quad + \frac{1}{k+1}\int_a^x \mathrm{f}^{(k+2)}(t)(x-t)^{k+1}\,\mathrm{d}t \\ &= \frac{1}{k+1}\mathrm{f}^{(k+1)}(a)(x-a)^{k+1} \\ &\quad + \frac{1}{k+1}\int_a^x \mathrm{f}^{(k+2)}(t)(x-t)^{k+1}\,\mathrm{d}t\end{aligned}$$

$$\begin{aligned}\mathrm{f}(x) &= \sum_{r=1}^{k} \frac{\mathrm{f}^{(r)}(a)}{r!}(x-a)^r \\ &\quad + \frac{1}{k!}\left\{\frac{1}{k+1}\mathrm{f}^{(k+1)}(a)(x-a)^{k+1} + \frac{1}{k+1}\int_a^x \mathrm{f}^{(k+2)}(t)(x-t)^{k+1}\,\mathrm{d}t\right\} \\ &= \sum_{r=1}^{k} \frac{\mathrm{f}^{(r)}(a)}{r!}(x-a)^r \\ &\quad + \frac{\mathrm{f}^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + \frac{1}{(k+1)!}\int_a^x \mathrm{f}^{(k+2)}(t)(x-t)^{k+1}\,\mathrm{d}t \\ &= \sum_{r=1}^{k+1} \frac{\mathrm{f}^{(r)}(a)}{r!}(x-a)^r + \frac{1}{(k+1)!}\int_a^x \mathrm{f}^{(k+2)}(t)(x-t)^{k+1}\,\mathrm{d}t\end{aligned}$$

which establishes the theorem for the case $n = k+1$ and hence proves Taylor's theorem.

3(iii)

For *x* close to *a*, $x - a \approx 0$. Then

$$\mathrm{f}(x) = \mathrm{f}(a) + \frac{\mathrm{f}'(a)}{1!}(x-a) + \frac{\mathrm{f}''(a)}{2!}(x-a)^2 \text{ approximately.}$$

Substitute $\mathrm{f}(x) = \sin x$, $x = 1.6$, $a = \dfrac{\pi}{2}$ into the above equation:

$$\sin 1.6 \approx \sin\frac{\pi}{2} + \cos\frac{\pi}{2}\left(1.6 - \frac{\pi}{2}\right) - \frac{1}{2}\sin\frac{\pi}{2}\left(1.6 - \frac{\pi}{2}\right)^2 = 1 - \frac{1}{2}\left(1.6 - \frac{\pi}{2}\right)^2.$$

<table>
<tr><td>3(iv)</td><td>Since $0 < x < \hat{x}$ where $\hat{x}$ is some finite positive integer, for $n > 2\hat{x}$,
$\frac{x^n}{n!} < \frac{\hat{x}^n}{n!}$
$= \underbrace{\frac{\hat{x}}{1}\cdot\frac{\hat{x}}{2}\cdot\frac{\hat{x}}{3}\cdots\frac{\hat{x}}{2\hat{x}}}_{2\hat{x}\text{ terms}}\cdot\underbrace{\frac{\hat{x}}{2\hat{x}+1}\cdot\frac{\hat{x}}{2\hat{x}+2}\cdots\frac{\hat{x}}{n}}_{n-2\hat{x}\text{ terms}}$
$< k\left(\frac{1}{2}\right)^{n-2\hat{x}}$ where $k = \frac{\hat{x}}{1}\cdot\frac{\hat{x}}{2}\cdot\frac{\hat{x}}{3}\cdots\frac{\hat{x}}{2\hat{x}}$
The last inequality holds since $0 < \frac{\hat{x}}{2\hat{x}+1}, \frac{\hat{x}}{2\hat{x}+2}, \cdots, \frac{\hat{x}}{n} < \frac{1}{2}$.</td><td></td></tr>
<tr><td>3(v)</td><td>For $x = 0$, $\lim_{n\to\infty}\frac{x^n}{n!} = \lim_{n\to\infty} 0 = 0$.
For each fixed $x \in \mathbb{R}^+$, k is finite since $\hat{x}$ is a finite number.
As $n \to \infty$, $k\left(\frac{1}{2}\right)^{n-2\hat{x}} \to 0$ and $0 \le \lim_{n\to\infty}\frac{x^n}{n!} \le 0 \Rightarrow \lim_{n\to\infty}\frac{x^n}{n!} = 0$.
For each fixed $x \in \mathbb{R}^-$, let $x = -y, y \in \mathbb{R}^+$.
$\lim_{n\to\infty}\frac{x^n}{n!} = \lim_{n\to\infty}\frac{(-1)^n y^n}{n!} = \pm\lim_{n\to\infty}\frac{y^n}{n!} = 0$ by the above result.
We therefore conclude that
$\forall x \in \mathbb{R},\ \lim_{n\to\infty}\frac{x^n}{n!} = 0$.</td><td></td></tr>
<tr><td>3(vi)</td><td>By (v), For all $x \in \mathbb{R}$, $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty}\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = f^{(n+1)}(c)\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!} = 0$.
By Taylor's theorem, $e^x = T_n(x) + R_n(x)$.
For each $x \in \mathbb{R}$, letting $n \to \infty$ and putting $a = 0$ in particular, gives
$e^x = \lim_{n\to\infty} T_n(x) + \underbrace{\lim_{n\to\infty} R_n(x)}_{0} = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{x^r}{r!} = \sum_{r=0}^{\infty}\frac{x^r}{r!}$.

Remark:

In the above computation, we put in $a = 0$. For any arbitrary $a \in \mathbb{R}$, Taylor's theorem gives
$e^x = \lim_{n\to\infty} T_n(x) + \underbrace{\lim_{n\to\infty} R_n(x)}_{0} = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{e^a (x-a)^r}{r!}$
$\Rightarrow e^{x-a} = \sum_{r=0}^{\infty}\frac{(x-a)^r}{r!}$ on dividing by e^a
$\overset{\text{replace } x-a \text{ with } x}{\Rightarrow} e^x = \sum_{r=0}^{\infty}\frac{x^r}{r!}$ as before
So the choice of a is arbitrary.</td><td></td></tr>
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4(i)	Since p is prime and $1 \le k-1 < p$, $\gcd(k-1, p) = 1$. There exist $a, b \in \mathbb{Z}$ such that $pa + (k-1)b = 1$. For any $m \in \mathbb{Z}$, $p(a - m(k-1)) + (k-1)(b + mp) = 1$. Choose m large enough such that $y_k := b + mp > 0$ and $x_k := a - m(k-1)$. Hence, there exist $x_k, y_k \in \mathbb{Z}$ with $y_k > 0$ such that $px_k + (k-1)y_k = 1$.	
4(ii)	Since $(k-1) \mid k(k-1)y_k$, $u_k = k(k-1)y_k \equiv 0$ (mod $k-1$). From part **(i)**, $px_k + (k-1)y_k = 1$ implies $(k-1)y_k \equiv 1$ (mod p), and so, $u_k = k(k-1)y_k \equiv k$ (mod p).	
4(iii)	Suppose $\frac{u_i}{i-1} \equiv \frac{u_j}{j-1}$ (mod p) for some $2 \le i \le j \le p$. Hence, in modulo p, we have the following. $(j-1)u_i \equiv (i-1)u_j$ $(j-1)i \equiv (i-1)j$ by (ii), $u_k \equiv k \pmod p$ $ij - i \equiv ij - j$ $i \equiv j$ Since $2 \le i \le j \le p$, $i = j$ and so for each $2 \le k \le p$, each $\frac{u_k}{k-1}$ is distinct in modulo p.	
4(iv)	Let $v_1 = 1$ and for each $2 \le k \le p-1$, let v_k be the remainder when $\frac{u_k}{k-1}$ is divided by p. Note that $\frac{u_p}{p-1} = py_p \equiv p$ (mod p), and so, we define $v_p = p$. Only $v_1 \equiv 1$ (mod p). From part **(i)**, for all $2 \le k \le p$, $(k-1)y_k \equiv 1$ (mod p) and it is clear that $y_k \ne 1$. Hence, $v_k = ky_k \not\equiv 1$ (mod p) for all $2 \le k \le p$. Together with the result in part **(ii)**, we see that all v_k's, $1 \le k \le p$, are unique and form the set $\{1, 2, \ldots, p\}$. $v_1 v_2 \ldots v_k = \frac{u_2 u_3 \ldots u_k}{(k-1)!}$ $\equiv \frac{2 \times 3 \times \ldots \times k}{(k-1)!} \pmod p$ (by part (i)) $= k \pmod p$ This shows that all v_1, $v_1 v_2$, $v_1 v_2 v_3$, …, and $v_1 v_2 \ldots v_p$ leave different remainders when divided by p. **Alternative to working in grey:** $v_k = \frac{u_k}{k-1} = ky_k = 1 + y_k - px_k \equiv 1 + y_k \pmod p$ Hence $v_k \not\equiv 1 \pmod p$ since $y_k := b + mp$ in (i) and b is not a multiple of p **Or**	

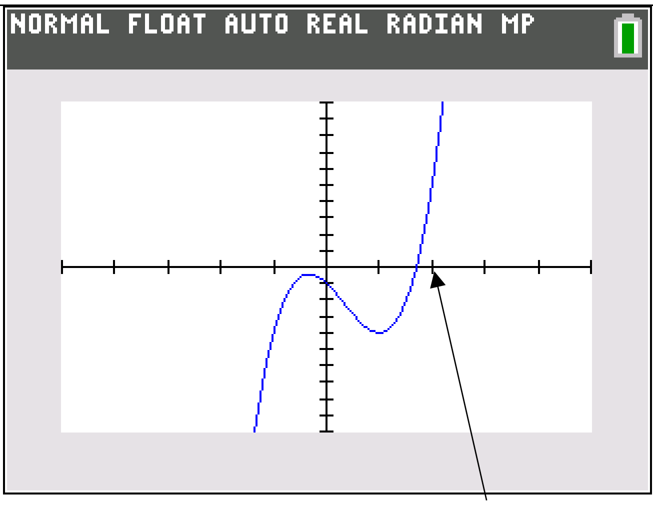
	From (i), there exist $x_k, y_k \in \mathbb{Z}$, $y_k := b + mp > 0$ such that $px_k + (k-1)y_k = 1$. Hence $(k-1)y_k \equiv 1$ (mod p). $ky_k \equiv 1 + y_k$ (mod p) $\not\equiv 1 \pmod{p}$ since $y_k := b + mp$ in (i) and b is not a multiple of p	
4(v)	A permutation is 1, 2, 7, 5, 4, 10, 3, 9, 8, 6, 11. *Check that* v_1, v_1v_2, $v_1v_2v_3$, *…, and* $v_1v_2...v_{11}$ *in modulo 11 are unique:* *1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.* $v_1 = 1$ $v_1v_2 = \frac{u_2}{(2-1)!}$ $\equiv \frac{2}{(2-1)!} \pmod{11}$ $= 2 \pmod{11}$ $\Rightarrow v_2 = 2$ $2 \cdot 7 \equiv 3 \pmod{11}$ $\quad\Rightarrow v_3 = 7$ $2 \cdot 7 \cdot 5 \equiv 3 \cdot 5 \pmod{11}$ $\equiv 4 \pmod{11}$ $\quad\Rightarrow v_4 = 5$ $4 \cdot 4 \equiv 5 \pmod{11}$ $\quad\Rightarrow v_5 = 4$ $5 \cdot 10 \equiv 6 \pmod{11}$ $\quad\Rightarrow v_6 = 10$ $6 \cdot 3 \equiv 7 \pmod{11}$ $\quad\Rightarrow v_7 = 3$ $7 \cdot 9 \equiv 8 \pmod{11}$ $\quad\Rightarrow v_8 = 9$ $8 \cdot 8 \equiv 9 \pmod{11}$ $\quad\Rightarrow v_9 = 8$ $9 \cdot 6 \equiv 10 \pmod{11}$ $\quad\Rightarrow v_{10} = 6$ $v_{11} = 11$	

5(i)	Simple sketch of the two graphs to show that $\mathrm{R_f} = \mathrm{R_g} = \mathbb{R} \setminus \{0, 1\} = \mathrm{D_f} = \mathrm{D_g}$. Hence, fg and gf exist.	
5(ii)	$\mathrm{f}^2(x) = \dfrac{1}{1-\dfrac{1}{1-x}} = \dfrac{1-x}{1-x-1} = \dfrac{x-1}{x} = 1-\dfrac{1}{x} = \mathrm{g}(x)$ $\mathrm{g}^2(x) = 1-\dfrac{1}{1-\dfrac{1}{x}} = 1-\dfrac{x}{x-1} = \dfrac{x-1-x}{x-1} = \dfrac{1}{1-x} = \mathrm{f}(x)$.	
5(iii)	Since $\mathrm{R_f} = \mathrm{R_g} = \mathbb{R} \setminus \{0, 1\} = \mathrm{D_h}$, hf and hg exist. $\mathrm{h}(x)+\mathrm{h}\left(\dfrac{1}{1-x}\right) = x \Rightarrow \mathrm{h}(x)+\mathrm{hf}(x) = x$ -------- (1) Replace x with f(x) in (1): $\mathrm{hf}(x)+\mathrm{hg}(x) = \mathrm{f}(x)$ $\left(\because \mathrm{f}^2 = \mathrm{g}\right)$ -------- (2) (2) – (1): $\mathrm{hg}(x)-\mathrm{h}(x) = \mathrm{f}(x)-x$ -------- (3) Replace x with g(x) in (3): $\mathrm{hf}(x)-\mathrm{hg}(x) = \mathrm{fg}(x)-\mathrm{g}(x)$ $\left(\because \mathrm{g}^2 = \mathrm{f}\right)$ -------- (4) (3) + (4): $\mathrm{hf}(x)-\mathrm{h}(x) = \mathrm{fg}(x)+\mathrm{f}(x)-\mathrm{g}(x)-x$ -------- (5) (1) – (5): $2\mathrm{h}(x) = 2x-\mathrm{fg}(x)-\mathrm{f}(x)+\mathrm{g}(x)$ $\mathrm{h}(x) = \dfrac{1}{2}\left[2x-\dfrac{1}{1-\left(1-\dfrac{1}{x}\right)}-\dfrac{1}{1-x}+1-\dfrac{1}{x}\right]$ $= \dfrac{1}{2}\left(x+1+\dfrac{1}{x-1}-\dfrac{1}{x}\right)$ $= \dfrac{(x+1)(x)(x-1)+x-(x-1)}{2x(x-1)}$ $= \dfrac{x^3-x+1}{2x(x-1)}$	
5(iii)	Alternatively, $\mathrm{fg}(x) = \dfrac{1}{1-\left(1-\dfrac{1}{x}\right)} = x$ Since $\mathrm{R_f} = \mathrm{R_g} = \mathbb{R} \setminus \{0, 1\} = \mathrm{D_h}$, hf and hg exist. $\mathrm{h}(x)+\mathrm{h}\left(\dfrac{1}{1-x}\right) = x \Rightarrow \mathrm{h}(x)+\mathrm{hf}(x) = x$ $\Rightarrow \mathrm{hf}(x)+\mathrm{hf}^2(x) = \mathrm{f}(x)$ --- (1) and $\mathrm{hg}(x)+\mathrm{hfg}(x) = \mathrm{g}(x) \Rightarrow \mathrm{hg}(x)+\mathrm{h}(x) = \mathrm{g}(x)$ --- (2) $\mathrm{h}(x) = x-\mathrm{hf}(x)$ $= x-\left[\mathrm{f}(x)-\mathrm{hf}^2(x)\right]$ by (1) $= x-\mathrm{f}(x)+\mathrm{hg}(x)$ $= x-\mathrm{f}(x)+\left[\mathrm{g}(x)-\mathrm{h}(x)\right]$ by (2) $2\mathrm{h}(x) = x-\mathrm{f}(x)+\mathrm{g}(x)$ $\mathrm{h}(x) = \dfrac{1}{2}\left[x-\dfrac{1}{1-x}+1-\dfrac{1}{x}\right]$	

6(a)(i)	$xy \equiv 1 \pmod z$ There exists $m \in \mathbb{Z}$ such that $xy = mz + 1$. Since $1 = xy - mz$ where $y, m \in \mathbb{Z}$, $\gcd(x, z) = 1$.	
6(a)(ii)	$x \equiv 1 \pmod y$ and $x \equiv 1 \pmod z$ $y \mid (x-1)$ and $z \mid (x-1)$ Since there are no common factors greater than 1 between *y* and *z*, all the prime factors of *y* and *z* are contained in $x-1$. Hence, $yz \mid (x-1)$, i.e. $x \equiv 1 \pmod{yz}$.	
6(b)(i)	Since $ab \equiv 1 \pmod c$, $ac \equiv 1 \pmod b$, and $bc \equiv 1 \pmod a$, by part **(a)(i)**, we have $\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1$. **Claim:** $ab + ac + bc \equiv 1$ modulo *a*, *b* and *c*. **Proof:** $ab \equiv 1 \pmod c \Rightarrow ab + ac + bc \equiv 1 + ac + bc \pmod c$ $\equiv 1 + (a+b)c \pmod c$ $\equiv 1 \pmod c$ since $(a+b)c \equiv 0 \pmod c$ Similarly, $ab + ac + bc \equiv 1$ modulo a, b. By part (a)(ii), since *a*, *b* and *c* are pairwise coprime, $ab + ac + bc \equiv 1 \pmod{abc}$. Hence, $ab + ac + bc = kabc + 1$ for some $k \in \mathbb{Z}^+$ since $1 < a < b < c$. $\frac{1}{c} + \frac{1}{b} + \frac{1}{a} = k + \frac{1}{abc}$ $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = k + \frac{1}{abc} > k \geq 1$.	
6bii	By the result in part **(b)(i)**, $1 < \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{3}{a}$ since $1 < a < b < c$. Note that $a = 2$ is the only integer solution greater than 1. Now, $1 < \frac{1}{2} + \frac{1}{b} + \frac{1}{c}$ and so, $\frac{1}{2} < \frac{1}{b} + \frac{1}{c} < \frac{2}{b}$. Also note that $b = 3$ is the only integer solution greater than $a = 2$. Now, $\frac{1}{6} < \frac{1}{c}$ implies that $c = 4$ or $c = 5$. But since *a*, *b* and *c* are pairwise coprime, $c = 5$. Hence, $(a,b,c) = (2,3,5)$ is the only solution.	

7(a)(i)	Consider the complementary case when the Primary 1 classroom holds at least 4 tables. First, place 4 tables in the Primary 1 classroom. Distribute the remaining 16 tables into all 6 rooms, and this can be done in $\binom{16+5}{5}=\binom{21}{5}$ ways. Hence the required answer $=$ Total $-\binom{21}{5}=\binom{20+5}{5}-\binom{21}{5}=32781$ _Alternatively_ The solution to the problem is similar to finding the number of integer solutions to $x_1+x_2+x_3+x_4+x_5+x_6=20$ where $0\le x_1\le 3$ and $x_i\ge 0$ for each $i=2,3,4,5,6$. Now $0\le x_1\le 3$ implies that $17\le x_2+x_3+x_4+x_5+x_6\le 20$. Since the number of non-negative integer solutions to $x_2+x_3+x_4+x_5+x_6=n$ is $\binom{n+4}{4}$, the required answer is $\binom{17+4}{4}+\binom{18+4}{4}+\binom{19+4}{4}+\binom{20+4}{4}=32781$.	
7(a)(ii)	The solution to the problem is similar to finding the number of integer solutions to $x_1+x_2+x_3+x_4+x_5+x_6=20$ where $x_5\ge 3$, $x_6\ge 3$ and $x_i\ge 0$ for each $i=1,2,3,4$. Set $y_5=x_5-3$, $y_6=x_6-3$, $y_i=x_i$ for each $i=1,2,3,4$. Then the required number of integer solutions to $x_1+x_2+x_3+x_4+x_5+x_6=20$ is equal to the number of integer solutions to $x_1+x_2+x_3+x_4+y_5+y_6=14$. Hence the answer is $\binom{14+5}{5}=\binom{19}{5}=11628$.	
7(a)(iii)	Let A_1 be the event that the first, second, and third students are in ordered level, A_2 be the event that the second, third, and fourth student are in ordered level, A_3 be the event that the third, fourth, and fifth students are in ordered level, and A_4 be the event that the fourth, fifth and sixth students are in ordered level. By a combination of complementary counting and PIE, we have that our answer will be $\lvert A_1'\cap A_2'\cap A_3'\cap A_4'\rvert$ $=\lvert S\rvert-\lvert A_1\cup A_2\cup A_3\cup A_n\rvert$ $=\lvert S\rvert-\sum_{i=1}^{4}\lvert A_i\rvert+\sum_{i<j}\lvert A_i\cap A_j\rvert-\sum_{i<j<k}\lvert A_i\cap A_j\cap A_k\rvert+\lvert A_1\cap A_2\cap A_3\cap A_4\rvert$ $\lvert S\rvert=6!=720$ $\sum_{i=1}^{4}\lvert A_i\rvert=\binom{6}{3}\times 3!\times 4=480$ $\sum_{i<j}\lvert A_i\cap A_j\rvert=\lvert A_1\cap A_2\rvert+\lvert A_1\cap A_3\rvert+\lvert A_1\cap A_4\rvert+\lvert A_2\cap A_3\rvert+\lvert A_2\cap A_4\rvert+\lvert A_3\cap A_4\rvert$ $=\binom{6}{4}\times 2!+\binom{6}{5}+\binom{6}{3}+\binom{6}{4}\times 2!+\binom{6}{5}+\binom{6}{4}\times 2!$ $=122$	

	$\sum_{i<j<k} \lvert A_i \cap A_j \cap A_k \rvert$ $= \lvert A_1 \cap A_2 \cap A_3 \rvert + \lvert A_1 \cap A_2 \cap A_4 \rvert + \lvert A_1 \cap A_3 \cap A_4 \rvert + \lvert A_2 \cap A_3 \cap A_4 \rvert$ $= \binom{6}{5} + 1 + 1 + \binom{6}{5}$ $= 14$ $\lvert A_1 \cap A_2 \cap A_3 \cap A_4 \rvert = 1$ $\lvert A_1' \cap A_2' \cap A_3' \cap A_4' \rvert = 720 - 480 + 122 - 14 + 1 = 349$	
7(b)	Consider 4 boxes labelled 0, 1, 2 and 3. For each of the 9 integers in the solution set, place the integer *m* in box *i* if $m \pmod 4) = i$. By PP, there is at least one box with at least $\left\lceil \frac{9}{4} \right\rceil = 3$ integers in it. Let the box with 3 integers be box *j*. If we label these 3 integers m_p , m_q , m_r as $m_p = 4n_p + j$, $m_q = 4n_q + j$ and $m_r = 4n_r + j$ for some integers n_p , n_q , n_r , then the difference between any of these pairs is divisible by 4, i.e. $m_p - m_q = 4(n_p - n_q)$, and similarly for the other two pairs.	
7c	First, start from Y and recognising each letter has 2 presiding letters considering a quadrant first). Next remove the four over-counting VICTORY. Number of ways is $4 \times 2^6 - 4 = 2^8 - 4 = 252$.	

8(i)	For $n = 2$, the grid contains the numbers $2,3,4,5,6,7$ which has no cube.	
8(ii)	 $x = 2$ Clearly, $3x^3 - 3x^2 - 3x - 1 > 0$ for all $x \geq 2$. That is, $k = 2$.	
8(iii)	By way of contradiction, since $n = 1 = 1^3$ is a cube, we will exclude it and consider $n \geq 2$. Suppose on the contrary, there exists a positive integer $n \geq 2$ for which there is no cube in Peter's grid. Let m^3 be the largest cube strictly less than *n*. By hypothesis, the next cube $(m+1)^3$ must be strictly more than $4n$. That is, we have $m^3 < n$ and $(m+1)^3 > 4n$. Firstly, note that if $m = 1$, then $(m+1)^3 > 4n \Rightarrow 8 > 4n \Rightarrow n < 2$ which contradicts the assumption that $n \geq 2$. Thus $m \geq 2$. Putting the two inequalities above together, $(m+1)^3 > 4n > 4m^3$ $\Rightarrow m^3 + 3m^2 + 3m + 1 > 4m^3$ $\Rightarrow 3m^3 - 3m^2 - 3m - 1 < 0$ (*) But from **(ii)**, $3m^3 - 3m^2 - 3m - 1 > 0$ for all $m \geq 2$ which contradicts the above conclusion. Hence Peter's claim is correct.	
8(iv)	**(a)** $n \leq 2^3 \leq 4n \Rightarrow 2 \leq n \leq 8$ **(b)** $n \leq 3^3 \leq 4n \Rightarrow \frac{27}{4} \leq n \leq 27 \Rightarrow 7 \leq n \leq 27$	
8(v)	We want the range of values of *n* for which $n \leq m^3 \leq 4n$. $n \leq m^3 \leq 4n \Rightarrow n \leq m^3$ and $4n \geq m^3 \Rightarrow \frac{m^3}{4} \leq n \leq m^3$. Since *n* is a positive integer, the above inequality becomes $\left\lceil \frac{m^3}{4} \right\rceil \leq n \leq m^3$ where $\left\lceil \frac{m^3}{4} \right\rceil$ is the least integer greater than or equal to $\frac{m^3}{4}$.	

8(vi)	By **(ii)**, for all $m \geq 2$, $$m^3 - \frac{(m+1)^3}{4} = \frac{1}{4}\left(3m^3 - 3m^2 - 3m - 1\right) > 0 \Rightarrow m^3 > \frac{(m+1)^3}{4}.$$ Thus $m^3 \geq \left\lceil \frac{(m+1)^3}{4} \right\rceil$. The range of values of *n* for which Peter's grid contains $(m+1)^3$ is $\left\lceil \frac{(m+1)^3}{4} \right\rceil \leq n \leq (m+1)^3$. We want the intersection of $\left\lceil \frac{m^3}{4} \right\rceil \leq n \leq m^3$ and $\left\lceil \frac{(m+1)^3}{4} \right\rceil \leq n \leq (m+1)^3$. Since $m^3 \geq \left\lceil \frac{(m+1)^3}{4} \right\rceil$, the required range of *n* is $$\left\lceil \frac{(m+1)^3}{4} \right\rceil \leq n \leq m^3.$$	

Mathematics is clothed in truth because it is fabricated with sound axioms; it is adorned with beauty because it is fashioned with

IDEAS