H2 Mathematics (9758)
Chapter 6B
3D Vector Geometry (Lines & Planes)
Discussion Questions Solutions

Level 1

1 Find the equation of the following planes in parametric form, scalar product form and Cartesian form.

(a) The plane passing through the points $A(0, 1, 1)$, $B(1, -3, 2)$ and $C(1, 0, 1)$.

(b) The plane containing the lines $x = 3, \frac{y+1}{2} = z - 3$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}$.

(c) The plane that includes the line $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$ and the point with position vector $2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

In each case, find the coordinates of the point of intersection of the plane and the line $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}, \alpha \in \mathbb{R}$.

Q1	Solution
(a)	The plane is parallel to $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Vector equation of plane is $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \lambda, \mu \in \mathbb{R}$. In **parametric form**, $x = \lambda + \mu, y = 1 - 4\lambda - \mu, z = 1 + \lambda$, where $\lambda, \mu \in \mathbb{R}$ A vector normal to the plane is $\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$. $\mathbf{r} \cdot \mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0 + 1 + 3 = 4$.

$\therefore$ In **<u>scalar product form</u>**, the equation of the plane is $\mathbf{r}\bullet\begin{pmatrix}1\\1\\3\end{pmatrix}=4$.

$$\mathbf{r}\bullet\begin{pmatrix}1\\1\\3\end{pmatrix}=4 \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}1\\1\\3\end{pmatrix}=4$$

$\therefore$ The **<u>cartesian equation</u>** of the plane is $x+y+3z=4$

$$\mathbf{r}=\begin{pmatrix}1\\-2\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\-5\\3\end{pmatrix}, \alpha\in\mathbb{R} \text{ - (1)} \qquad \mathbf{r}\bullet\begin{pmatrix}1\\1\\3\end{pmatrix}=4 \text{ -(2)}$$

Substitute (1) into (2)

$$\begin{pmatrix}1+\alpha\\-2-5\alpha\\1+3\alpha\end{pmatrix}\bullet\begin{pmatrix}1\\1\\3\end{pmatrix}=4$$

$$1+\alpha-2-5\alpha+3+9\alpha=4$$

$$\Rightarrow \alpha=\frac{2}{5}$$

Substitute $\alpha=\frac{2}{5}$ into (1)

$$\mathbf{r}=\begin{pmatrix}1+\frac{2}{5}\\-2-5\left(\frac{2}{5}\right)\\1+3\left(\frac{2}{5}\right)\end{pmatrix}=\frac{1}{5}\begin{pmatrix}7\\-20\\11\end{pmatrix}$$

Thus, point of intersection is $\left(\frac{7}{5},-4,\frac{11}{5}\right)$.

Alternative: Using GC to find the point of intersection between line and plane:
Using the Cartesian equation of the line and plane, we have

$$\begin{aligned} &x+y+3z=4 & & & x+y+3z \quad &=4\\ &x=1+\alpha & &\text{Rearranging} & x \quad\quad -\alpha&=1\\ &y=-2-5\alpha & &\Rightarrow & y \quad\quad +5\alpha&=-2\\ &z=1+3\alpha & & & z \quad\quad -3\alpha&=1 \end{aligned}$$

Using GC,

$$x=\frac{7}{5}, y=-4, z=\frac{11}{5}, \alpha=\frac{2}{5}$$

Thus, point of intersection is $\left(\frac{7}{5},-4,\frac{11}{5}\right)$.

(b)	$x=3, \frac{y+1}{2}=z-3 \Rightarrow \mathbf{r}=\begin{pmatrix}3\\-1\\3\end{pmatrix}+\beta\begin{pmatrix}0\\2\\1\end{pmatrix}, \beta\in\mathbb{R}$ Vector equation of plane is $\mathbf{r}=\begin{pmatrix}3\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}0\\2\\1\end{pmatrix}+\mu\begin{pmatrix}2\\4\\3\end{pmatrix}, \lambda,\mu\in\mathbb{R}$. In **<u>parametric form</u>**, $x=3+2\mu, y=-1+2\lambda+4\mu, z=3+\lambda+3\mu$, where $\lambda,\mu\in\mathbb{R}$ $\begin{pmatrix}0\\2\\1\end{pmatrix}\times\begin{pmatrix}2\\4\\3\end{pmatrix}=\begin{pmatrix}2\\2\\-4\end{pmatrix}=2\begin{pmatrix}1\\1\\-2\end{pmatrix}$ is a normal vector to the plane. $\therefore$ vector equation of plane in **<u>scalar product form</u>** is $\mathbf{r}\bullet\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}3\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}1\\1\\-2\end{pmatrix}=-4$, i.e. $\mathbf{r}.\begin{pmatrix}1\\1\\-2\end{pmatrix}=-4$. From $\mathbf{r}.\begin{pmatrix}1\\1\\-2\end{pmatrix}=-4$, let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$. $\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}1\\1\\-2\end{pmatrix}=-4$ $x+y-2z=-4$ **<u>Cartesian</u>** equation of plane is $x+y-2z=-4$. For point of intersection between $\mathbf{r}=\begin{pmatrix}1\\-2\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\-5\\3\end{pmatrix}$ — (1) and $\mathbf{r}\bullet\begin{pmatrix}1\\1\\-2\end{pmatrix}=-4$ — (2) Substitute (1) into (2) $\begin{pmatrix}1+\alpha\\-2-5\alpha\\1+3\alpha\end{pmatrix}.\begin{pmatrix}1\\1\\-2\end{pmatrix}=-4 \Rightarrow \alpha=\frac{1}{10}$. Substitute $\alpha=\frac{1}{10}$ into (1)

$$\mathbf{r}=\begin{pmatrix}1+\frac{1}{10}\\-2-5\left(\frac{1}{10}\right)\\1+3\left(\frac{1}{10}\right)\end{pmatrix}=\begin{pmatrix}\frac{11}{10}\\-\frac{5}{2}\\\frac{13}{10}\end{pmatrix}$$

Thus, point of intersection is $\left(\frac{11}{10},-\frac{5}{2},\frac{13}{10}\right)$.

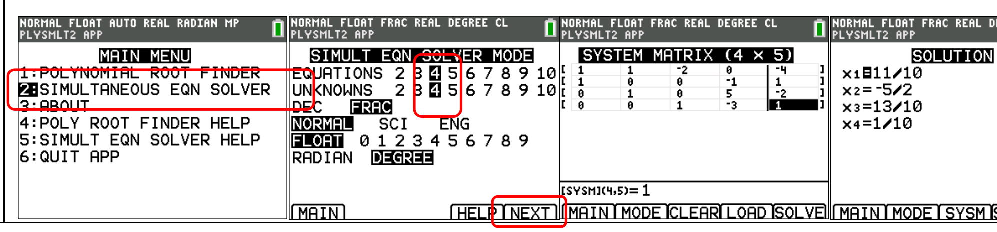
Alternative: Using GC to find the point of intersection between line and plane:
Using the Cartesian equation of the line and plane, we have

$$\begin{array}{lcl} x+y-2z=-4 & & x+y-2z \quad =-4 \\ x=1+\alpha & \text{Rearranging} & x \quad -\alpha=1 \\ y=-2-5\alpha & \Rightarrow & y \quad +5\alpha=-2 \\ z=1+3\alpha & & z \quad -3\alpha=1 \end{array}$$

Using GC,

$x=\frac{11}{10}, y=-\frac{5}{2}, z=\frac{13}{10}, \alpha=\frac{1}{10}$

Thus, point of intersection is $\left(\frac{11}{10},-\frac{5}{2},\frac{13}{10}\right)$.



(c)	A direction vector parallel to plane $=\begin{pmatrix}1\\-2\\-4\end{pmatrix}-\begin{pmatrix}2\\5\\-6\end{pmatrix}=\begin{pmatrix}-1\\-7\\2\end{pmatrix}$. Vector equation of plane is $\mathbf{r}=\begin{pmatrix}1\\-2\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-6\end{pmatrix}+\mu\begin{pmatrix}-1\\-7\\2\end{pmatrix}$, $\lambda,\mu\in\mathbb{R}$. In **<u>parametric form,</u>** $x=1+2\lambda-\mu, y=-2+3\lambda-7\mu, z=-4-6\lambda+2\mu$, where $\lambda,\mu\in\mathbb{R}$ $\begin{pmatrix}-1\\-7\\2\end{pmatrix}\times\begin{pmatrix}2\\3\\-6\end{pmatrix}=\begin{pmatrix}36\\-2\\11\end{pmatrix}$ is a normal vector to the plane. $\therefore$ vector equation of plane in **<u>scalar product form</u>** is $\mathbf{r}\bullet\begin{pmatrix}36\\-2\\11\end{pmatrix}=\begin{pmatrix}1\\-2\\-4\end{pmatrix}\bullet\begin{pmatrix}36\\-2\\11\end{pmatrix}=-4$ i.e $\mathbf{r}\bullet\begin{pmatrix}36\\-2\\11\end{pmatrix}=-4$ From $\mathbf{r}.\begin{pmatrix}36\\-2\\11\end{pmatrix}=-4$, let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$. $\mathbf{r}\bullet\begin{pmatrix}36\\-2\\11\end{pmatrix}=-4$ $36x-2y+11z=-4$ **<u>Cartesian</u>** equation of plane is $36x-2y+11z=-4$. For point of intersection between $\mathbf{r}=\begin{pmatrix}1\\-2\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\-5\\3\end{pmatrix}$ — (1) and $\mathbf{r}\bullet\begin{pmatrix}36\\-2\\11\end{pmatrix}=-4$ — (2) Substitute (1) into (2) $\begin{pmatrix}1+\alpha\\-2-5\alpha\\1+3\alpha\end{pmatrix}.\begin{pmatrix}36\\-2\\11\end{pmatrix}=-4 \Rightarrow \alpha=-\frac{55}{79}$. Substitute $\alpha=-\frac{55}{79}$ into (1)

$$\mathbf{r} = \begin{pmatrix} 1-\frac{55}{79} \\ -2-5\left(-\frac{55}{79}\right) \\ 1+3\left(-\frac{55}{79}\right) \end{pmatrix} = \begin{pmatrix} \frac{24}{79} \\ \frac{117}{79} \\ -\frac{86}{79} \end{pmatrix}$$

Thus, point of intersection is $\left(\frac{24}{79}, \frac{117}{79}, -\frac{86}{79}\right)$.

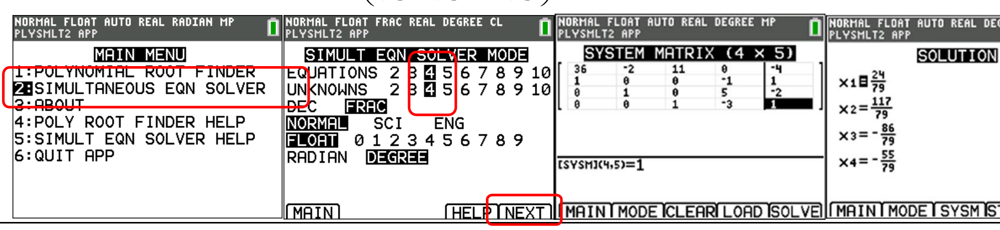
Alternative: Using GC to find the point of intersection between line and plane:
Using the Cartesian equation of the line and plane, we have

$$\begin{array}{lcl} 36x-2y+11z=-4 & & 36x-2y+11z \qquad =-4 \\ x=1+\alpha & \text{Rearranging} & x \qquad\qquad -\alpha=1 \\ y=-2-5\alpha & \Rightarrow & y \qquad\qquad +5\alpha=-2 \\ z=1+3\alpha & & z \qquad\qquad -3\alpha=1 \end{array}$$

Using GC,

$$x=\frac{24}{79}, y=\frac{117}{79}, z=-\frac{86}{79}, \alpha=-\frac{55}{79}$$

Thus, point of intersection is $\left(\frac{24}{79}, \frac{117}{79}, -\frac{86}{79}\right)$.



2 Find the coordinates of the point where the line $\mathbf{r} = \mathbf{i} + \lambda(\mathbf{i} - \mathbf{k})$ intersects the plane with equation $2x - 3y + z = 1$.

Q2	**Solution**
	Line: $\mathbf{r} = \mathbf{i} + \lambda(\mathbf{i} - \mathbf{k}) = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}$, Plane: $2x - 3y + z = 1 \Rightarrow \mathbf{r}\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix} = 1$ Since line intersects the plane, $\left[\begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}\right]\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix} = 1$ $\Rightarrow 2 + 2\lambda - \lambda = 1 \Rightarrow \lambda = -1$ $\therefore$ The position vector of the point of intersection is $\begin{pmatrix}0\\0\\1\end{pmatrix}$ and the coordinates are (0, 0, 1).

3 Find the equation of the line of intersection between the two planes with equations

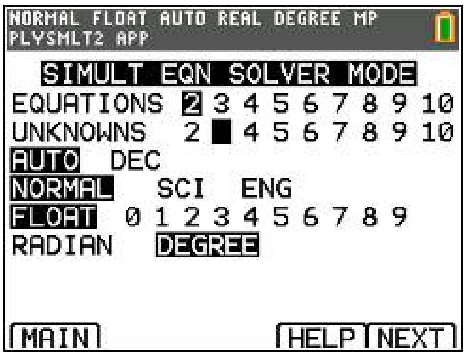
$$\mathbf{r}\bullet\begin{pmatrix}3\\2\\-1\end{pmatrix}=10 \text{ and } \mathbf{r}\bullet\begin{pmatrix}5\\-1\\2\end{pmatrix}=6 \text{ respectively.}$$

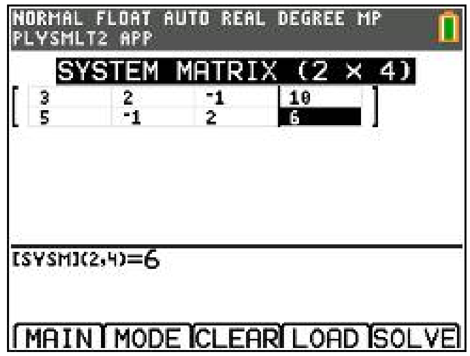
Q3	**Solution**

The cartesian equations of the two planes are

$$3x+2y-z=10 \qquad \text{----------} \ (1)$$

$$5x-y+2z=6 \qquad \text{----------} \ (2)$$





Using the G.C, an equation of the line of intersection is

$$\mathbf{r} = \begin{pmatrix}22/13\\32/13\\0\end{pmatrix}+\lambda\begin{pmatrix}-3\\11\\13\end{pmatrix}, \lambda\in\mathbb{R}.$$

Learning point:

Although the GC gives $\mathbf{r} = \begin{pmatrix}22/13\\32/13\\0\end{pmatrix}+\lambda\begin{pmatrix}-3/13\\11/13\\1\end{pmatrix}, \lambda\in\mathbb{R}$, it is helpful to factorise out $\frac{1}{13}$ from the direction vector so that the remaining numbers are nicer to work with.

4 Find the acute angle between

(a) the line $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$, and the plane $x - 3y + z = 2$;

(b) the planes $\mathbf{r} \bullet \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 6$ and $x + 2y - 5z = 8$.

Q4	**Solution**
(a)	$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$, and $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = 2$ Let θ be acute angle between the line and the plane. $\sin\theta = \dfrac{\left\lvert \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\rvert}{\sqrt{38}\sqrt{11}} = \dfrac{10}{\sqrt{38}\sqrt{11}} \quad \Rightarrow \quad \theta = 29.3°$ Required angle is $29.3°$.
(b)	$\mathbf{r} \bullet \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 6$ and $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = 8$ Let θ be acute angle between the two planes. $\cos\theta = \dfrac{\left\lvert \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \right\rvert}{\sqrt{29}\sqrt{30}} = \dfrac{24}{\sqrt{29}\sqrt{30}} \quad \Rightarrow \quad \theta = 35.5°$ Required angle is $35.5°$.

Level 2

5 Find the position vector of F, the foot of the perpendicular from $B(2, 3, -4)$ to the plane p, whose equation is $2x+y-2z+9=0$.

Q5	**Solution**
	$2x+y-2z+9=0 \Rightarrow \mathbf{r}\bullet\begin{pmatrix}2\\1\\-2\end{pmatrix}=-9$ ---------- (1) Let l be the line through point B perpendicular to the plane p. $l:\ \mathbf{r}=\begin{pmatrix}2\\3\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix},\ \lambda\in\mathbb{R}$ ---------- (2) Solving (1) and (2), we have $\left[\begin{pmatrix}2\\3\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}\right]\bullet\begin{pmatrix}2\\1\\-2\end{pmatrix}=-9$ $\Rightarrow (4+3+8)+\lambda(4+1+4)=-9 \Rightarrow \lambda=-\frac{8}{3}$ $\quad\therefore \overrightarrow{OF}=\begin{pmatrix}2\\3\\-4\end{pmatrix}-\frac{8}{3}\begin{pmatrix}2\\1\\-2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-10\\1\\4\end{pmatrix}$

6 Find the perpendicular distance from point $A(4,5,6)$ to the plane $\mathbf{r}\bullet\mathbf{k}=2$.

Q6	**Solution**
	Observe that point $B(0,0,2)$ lies on the plane because $\begin{pmatrix}0\\0\\2\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=2$ $\overrightarrow{AB}=\begin{pmatrix}-4\\-5\\-4\end{pmatrix}$ Hence, perpendicular distance from A to plane is $\left\lvert\overrightarrow{AB}\bullet\hat{\mathbf{n}}\right\rvert=\frac{\left\lvert\begin{pmatrix}-4\\-5\\-4\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{1}=4$

7 The plane π contains the point $P(2,2,4)$ and the line $l_1:\ \mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix},\ \lambda\in\mathbb{R}$.

(i) Find the vector equation of the plane π in scalar product form.

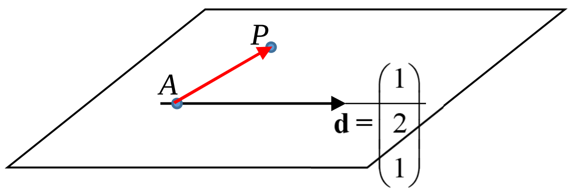
(ii) Find the position vector of the foot of the perpendicular from the point $S(1,0,-4)$ to π.

(iii) Verify that $S(1,0,-4)$ lies on line $l_2:\mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}0\\1\\-5\end{pmatrix},\ \mu\in\mathbb{R}$.

(iv) Find the equation of the image obtained by reflecting the line l_2 in the plane π.

Q7	Solution

(i)



$A(1,-1,1)$ lies on π $\Rightarrow$ $\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\begin{pmatrix}2\\2\\4\end{pmatrix}-\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\3\\3\end{pmatrix}$ is a vector on π.

To find the normal vector to π, we use

$$\mathbf{d}\times\overrightarrow{AP}$$
$$=\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\3\\3\end{pmatrix}=\begin{pmatrix}3\\-2\\1\end{pmatrix}$$

Learning point:
To obtain the equation of a plane, we need the normal vector as well as the position vector of a fixed point that lies on the plane.

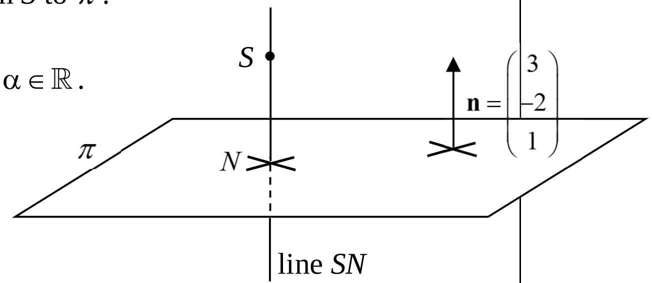
$$\pi:\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$$

$\therefore$ vector equation of π in scalar product form is

$$\mathbf{r}\bullet\begin{pmatrix}3\\-2\\1\end{pmatrix}=\begin{pmatrix}2\\2\\4\end{pmatrix}\bullet\begin{pmatrix}3\\-2\\1\end{pmatrix}=6,\quad \text{i.e. } \mathbf{r}\bullet\begin{pmatrix}3\\-2\\1\end{pmatrix}=6$$

(ii) Let N be the foot of the perpendicular from S to π.

Equation for line SN is $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}$.



Since N lies on this line, $\overrightarrow{ON} = \begin{pmatrix} 1+3\alpha \\ -2\alpha \\ -4+\alpha \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1+3\alpha \\ -2\alpha \\ -4+\alpha \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 6$$

$$\Rightarrow 14\alpha = 7$$

$$\Rightarrow \alpha = \frac{1}{2}$$

Learning Point:

Step 1:
Form the line SN as we have a point on the line and a vector parallel to it.

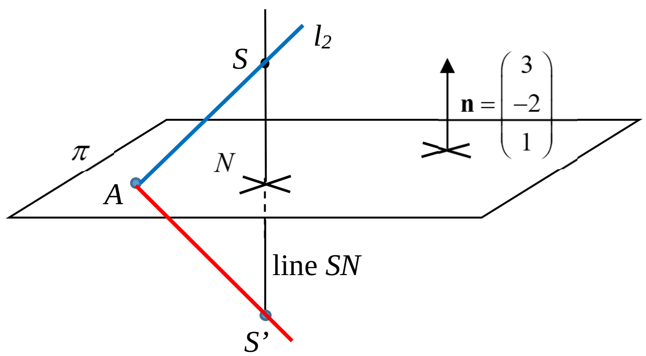
Step 2:
Find the intersection point between the line and plane by substituting the line into the plane.

$\therefore$ position vector of the foot of the perpendicular is $\overrightarrow{ON} = \begin{pmatrix} 1+3\left(\frac{1}{2}\right) \\ -2\left(\frac{1}{2}\right) \\ -4+\left(\frac{1}{2}\right) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 \\ -2 \\ -7 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} \Rightarrow \mu \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} \Rightarrow \mu = 1$

$\therefore S(1, 0, -4)$ lies on l_2.

(iv)



Let S' be the reflection of S in π.

$$\overrightarrow{ON} = \frac{\overrightarrow{OS} + \overrightarrow{OS'}}{2}$$

$$\Rightarrow \quad \overrightarrow{OS'} = 2\overrightarrow{ON} - \overrightarrow{OS} = \begin{pmatrix} 5 \\ -2 \\ -7 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}$$

Note also that $A(1,-1,1)$ lies on both l_2 and π.

$\therefore$ direction vector for reflected line is $\overrightarrow{AS'} = \overrightarrow{OS'} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$

$\therefore$ reflected line has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}, \beta \in \mathbb{R}$.

8 **2011(9740)/I/11**

The plane p passes through the points with coordinates $(4,-1,-3)$, $(-2,-5,2)$ and $(4,-3,-2)$.

(i) Find the Cartesian equation of p. [4]

The line l_1 has equation $\dfrac{x-1}{2}=\dfrac{y-2}{-4}=\dfrac{z+3}{1}$ and the line l_2 has equation $\dfrac{x+2}{1}=\dfrac{y-1}{5}=\dfrac{z-3}{k}$, where k is a constant. It is given that l_1 and l_2 intersect.

(ii) Find the value of k. [4]

(iii) Show that l_1 lies in p and find the coordinates of the point at which l_2 intersects p. [4]

(iv) Find the acute angle between l_2 and p. [3]

Q8	**Solution**
(i)	$\begin{pmatrix}4\\-1\\-3\end{pmatrix}-\begin{pmatrix}-2\\-5\\2\end{pmatrix}=\begin{pmatrix}6\\4\\-5\end{pmatrix}$ $\begin{pmatrix}4\\-1\\-3\end{pmatrix}-\begin{pmatrix}4\\-3\\-2\end{pmatrix}=\begin{pmatrix}0\\2\\-1\end{pmatrix}$ Find 2 direction vectors parallel to the plane. $\begin{pmatrix}6\\4\\-5\end{pmatrix}\times\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}6\\6\\12\end{pmatrix}=6\begin{pmatrix}1\\1\\2\end{pmatrix}$ Use the cross product of the 2 vectors parallel to the plane, to find the normal to the plane. $\mathbf{r}\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}4\\-1\\-3\end{pmatrix}\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}=-3.$ Equation of plane passing through point A, with position vector **a**: $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$ Let r $=\begin{pmatrix}x\\y\\z\end{pmatrix}$: $\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}=-3$ Leave your answer in Cartesian form. Hence the equation of the plane, p is $x+y+2z=-3$

(ii)

Convert equations of both lines to vector form:

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R} \quad \text{and} \quad l_2 : \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ k \end{pmatrix}, \quad \mu \in \mathbb{R}$$

Since the two lines intersect,

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ k \end{pmatrix}$$

If l_1 and l_2 intersects say, at point *F*, then there exist λ and $\mu \in \mathbb{R}$, such that

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \overrightarrow{OF} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ k \end{pmatrix}.$$

$1 + 2\lambda = -2 + \mu$ ------(1)

$2 - 4\lambda = 1 + 5\mu$ ------(2)

$-3 + \lambda = 3 + k\mu$ ------(3)

Solving (1) and (2) gives $\lambda = -1$ and $\mu = 1$.

Substitute these values into (3) to get $k = -7$.

(iii)

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \qquad p : \mathbf{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = -3$$

Method 1:

$$\mathbf{n} \bullet \mathbf{d} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 2 - 4 + 2 = 0$$

For l_1 to lie in *p*,
Condition 1: l_1 must be parallel to *p*
AND
Condition 2: Any point on *l* must lie in *p*.

Substitute $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ into equation of *p*:

$$\text{LHS} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1 + 2 - 6 = -3 = \text{RHS}$$

Since l_1 is parallel to *p* and $(1, 2, -3)$ lies in *p*, l_1 lies in *p*. (shown)

Method 2:

Let $\mathbf{r}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\1\end{pmatrix}$ which is a vector representing any point on l_1.

Denote $p:\mathbf{r}\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}=-3$ ------(1)

To show that l_1 lies in *p*, we need to show that any point that lies in l_1 also lies in *p* i.e., position vector of any point in l_1 also satisfies the equation of *p*.

Substitute this vector into LHS of (1):

$$\left[\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\1\end{pmatrix}\right]\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}=-3 \text{ (RHS of (1))}$$

Thus l_1 lies on *p*.

Finding Coordinates of intersection point of l_1 and l_2

Method 1

Substitute $\lambda=-1$ into l_1,

$$\begin{pmatrix}1\\2\\-3\end{pmatrix}-(1)\begin{pmatrix}2\\-4\\1\end{pmatrix}=\begin{pmatrix}-1\\6\\-4\end{pmatrix}$$

Since l_1 lies on *p*, and l_2 intersects l_1, hence to find the intersection point of l_2 and *p*, we can just find the intersection point of l_2 and l_1.

The coordinates of the intersectio

Remember to leave your answer in coordinate form.

Method 2

To find the point of intersection of l_2 and *p*,

$$\left[\begin{pmatrix}-2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\5\\-7\end{pmatrix}\right]\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}=-3$$

$$\begin{pmatrix}-2\\1\\3\end{pmatrix}\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}+\mu\begin{pmatrix}1\\5\\-7\end{pmatrix}\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}=-3$$

$5-8\mu=-3$

$\mu=1$

If l_2 and *p* intersects at say, point *Q*, then $\overrightarrow{OQ}$ must satisfy both the equations of l_2 and *p*. i.e., $\overrightarrow{OQ}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\5\\-7\end{pmatrix}$ for some $\mu\in\mathbb{R}$ and $\overrightarrow{OQ}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=-3$.

Use this value on $\mathbf{r}=\begin{pmatrix}-2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\5\\-7\end{pmatrix}$ to obtain $\mathbf{r}=\begin{pmatrix}-1\\6\\-4\end{pmatrix}$.

The coordinates of the intersection point is (−1, 6, −4).

Remember to leave your answer in coordinate form.

(iv)	Let θ be acute angle between l_2 and p. $\sin\theta = \left\|\dfrac{\mathbf{n}\bullet\mathbf{d}}{\|\mathbf{n}\|\|\mathbf{d}\|}\right\| = \dfrac{\left\|\begin{pmatrix}1\\5\\-7\end{pmatrix}\bullet\begin{pmatrix}1\\1\\2\end{pmatrix}\right\|}{\sqrt{75}\sqrt{6}} = \dfrac{8}{\sqrt{450}}$ $\therefore \theta = 22.2^\circ$ Question asks for acute angle, so remember to place a modulus. For angle, answer must be corrected to 1 decimal place.

9 2013(9740)/II/4 (modified)

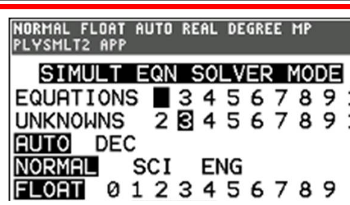
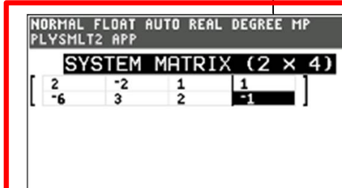
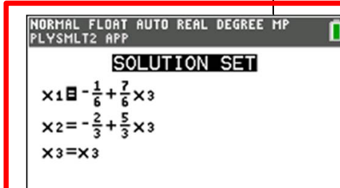
The planes p_1 and p_2 have equations $\mathbf{r} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1$ and $\mathbf{r} \bullet \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} = -1$ respectively, and meet in the line *l*.

(i) Find the shortest distance from the origin to plane p_2. [2]

(ii) Find the acute angle between p_1 and p_2. [3]

(iii) Find a vector equation for *l*. [4]

(iv) The point $A(4, 3, c)$ is equidistant from the planes p_1 and p_2. Calculate the two possible values of *c*. [6]

Q9	Solution
(i)	Shortest distance from the origin to plane p_2 $= \frac{\lvert -1 \rvert}{\sqrt{(-6)^2 + 3^2 + 2^2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$. Shortest distance from O to plane $p: \mathbf{r}\cdot\mathbf{n} = \mathrm{d}$ is $\frac{\lvert \mathrm{d} \rvert}{\lvert \mathbf{n} \rvert}$
(ii)	$\cos\theta = \left\lvert \frac{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}}{\left\lVert \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\rVert \left\lVert \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} \right\rVert} \right\rvert = \left\lvert \frac{-12-6+2}{\sqrt{4+4+1}\sqrt{36+9+4}} \right\rvert = \frac{16}{21}$ Question asks for acute angle, so remember to place a modulus. $\theta = 40.4^\circ$ (1d.p.) For angle, answer must be corrected to 1 decimal place.
(iii)	$p_1: 2x - 2y + z = 1$ $p_2: -6x + 3y + 2z = -1$   Using GC, the line of intersection is $\mathbf{r} = \begin{pmatrix} -\frac{1}{6} \\ -\frac{2}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{7}{6} \\ \frac{5}{3} \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ 

(iv)	Let B denote the point $(0,0,1)$ on p_1 and C denote the point $\left(0,0,-\frac{1}{2}\right)$ on p_2.

$$\frac{\left|\overrightarrow{BA}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|}{\sqrt{4+4+1}}=\frac{\left|\overrightarrow{CA}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|}{\sqrt{36+9+4}}$$

The point A is equidistant from the planes p_1 and p_2.
$\Rightarrow$ the shortest distance from A to p_1 = the shortest distance from A to p_2.
Thus, we can use the concept of finding shortest distance from a point to a plane to solve this question.

$$\frac{\left|\overrightarrow{BA}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|}{\sqrt{4+4+1}}=\frac{\left|\overrightarrow{CA}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|}{\sqrt{36+9+4}}$$

$$\frac{\left|\begin{pmatrix}4\\3\\c-1\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|}{3}=\frac{\left|\begin{pmatrix}4\\3\\c+\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|}{7}$$

$$\frac{|8-6+c-1|}{3}=\frac{|-24+9+2c+1|}{7}$$

$$\frac{|1+c|}{3}=\frac{|2c-14|}{7}$$

$$|7+7c|=|6c-42|$$

$$7+7c=\pm(6c-42)$$

Recall:
$|x|=|y|\Rightarrow x=y$ or $x=-y$

$7+7c=6c-42$ or $7+7c=-(6c-42)$

$c=-49$ $\qquad$ $13c=35$

$$c=\frac{35}{13}$$

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The plane p has equation $\mathbf{r}=\begin{pmatrix}1\\-3\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}a\\4\\-2\end{pmatrix}$, and the line l has equation $\mathbf{r}=\begin{pmatrix}a-1\\a\\a+1\end{pmatrix}+t\begin{pmatrix}-2\\1\\2\end{pmatrix}$, where a is a constant and λ, μ and t are parameters.

(i) In the case where $a=0$,

(a) show that l is perpendicular to p and find the values of λ, μ and t which give the coordinates of the point at which l and p intersect, [5]

(b) find the cartesian equations of the planes such that the perpendicular distance from each plane to p is 12. [5]

(ii) Find the value of a such that l and p do not meet in a unique point. [3]

Q10	Solution
(i)(a)	<u>**Method 1**</u> $\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\4\\-2\end{pmatrix}=\begin{pmatrix}-4\\2\\4\end{pmatrix}=2\begin{pmatrix}-2\\1\\2\end{pmatrix}$ $\mathbf{n}=2\begin{pmatrix}-2\\1\\2\end{pmatrix}=2\mathbf{d}$ Since normal vector of plane is parallel to the direction vector of *l*, *l* is perpendicular to *p*. <u>**Method 2**</u> When $a=0$, $\begin{pmatrix}-2\\1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\2\\0\end{pmatrix}=-2+2=0$ $\begin{pmatrix}-2\\1\\2\end{pmatrix}\bullet\begin{pmatrix}0\\4\\-2\end{pmatrix}=4-4=0$ Therefore, *l* is perpendicular to *p*. If *l* is perpendicular to *p*, the direction vector of *l* must be perpendicular to **both** direction vectors parallel to *p*.

Solving equations of l and p,

$$\begin{pmatrix}1\\-3\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}0\\4\\-2\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}+t\begin{pmatrix}-2\\1\\2\end{pmatrix}$$

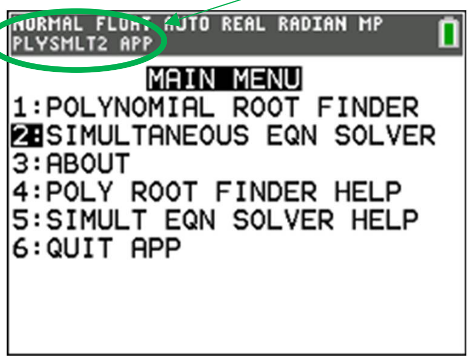
Since the question wants us to find the values of λ, μ and t, we can equate l and p to obtain three equations to solve for the three unknowns.

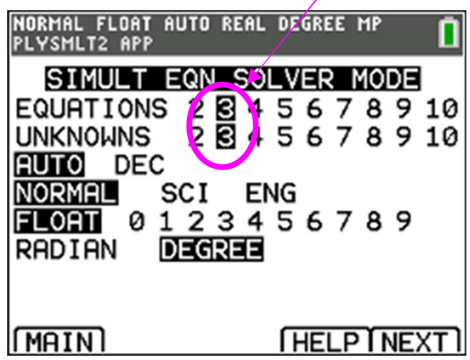
$$\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}0\\4\\-2\end{pmatrix}+t\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}-2\\3\\-1\end{pmatrix}$$

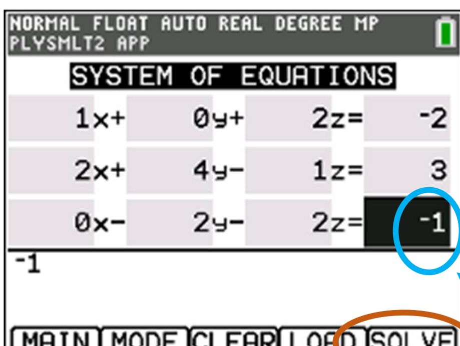
$$\begin{aligned}\lambda \quad\quad +2t &= -2\\ 2\lambda+4\mu-t &= 3\\ -2\mu-2t &= -1\end{aligned}$$

Under apps, select PLYSMLT2, followed by "Simultaneous Eqn Solver"

There are 3 equations and 3 unknowns







Press graph to solve

Remember to press Enter after last input

Using GC,

$\lambda=-\dfrac{8}{9}$, $\mu=\dfrac{19}{18}$ and $t=-\dfrac{5}{9}$.

(i)(b)

$$\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\4\\-2\end{pmatrix}=\begin{pmatrix}-4\\2\\4\end{pmatrix}=2\begin{pmatrix}-2\\1\\2\end{pmatrix}$$

$$\begin{aligned}\mathbf{r}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}&=\begin{pmatrix}1\\-3\\2\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}\\&=1\end{aligned}$$

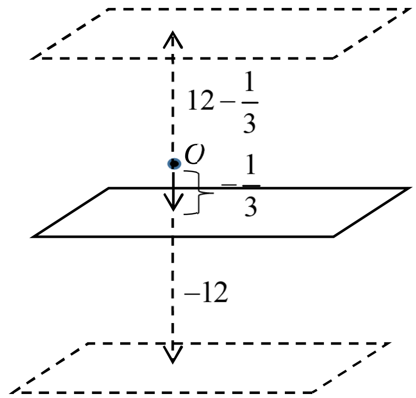
Since we need to find cartesian equation of the planes, it is easier for us to work in scalar product form rather than the given parametric from.

<u>Method 1</u>

$$\mathbf{r}\bullet\frac{\begin{pmatrix}-2\\1\\2\end{pmatrix}}{\left|\begin{pmatrix}-2\\1\\2\end{pmatrix}\right|}=\frac{-1}{\left|\begin{pmatrix}-2\\1\\2\end{pmatrix}\right|}$$

$$\Rightarrow\mathbf{r}\bullet\frac{\begin{pmatrix}-2\\1\\2\end{pmatrix}}{3}=\frac{-1}{3}$$

Note: This tells us that the distance from the origin to the plane is $\frac{1}{3}$.



$\therefore$ Equation of planes are

$$\Rightarrow\mathbf{r}\bullet\left[\frac{1}{3}\begin{pmatrix}-2\\1\\2\end{pmatrix}\right]=\frac{-1}{3}+12\quad\text{or}\quad\mathbf{r}\bullet\left[\frac{1}{3}\begin{pmatrix}-2\\1\\2\end{pmatrix}\right]=\frac{-1}{3}-12$$

$$\Rightarrow\mathbf{r}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}=35\quad\text{or}\quad\mathbf{r}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}=-37$$

$$\Rightarrow-2x+y+2z=35\quad\text{or}\quad-2x+y+2z=-37$$

<u>Method 2</u>

$$\begin{aligned}\mathbf{r}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}&=\begin{pmatrix}1\\-3\\2\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}\\&=-2-3+4=-1\end{aligned}$$

$$p:\mathbf{r}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}=-1$$

Let (x,y,z) be a point on a plane 12 units from p.

	$\dfrac{\left\lvert\left[\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}1\\-3\\2\end{pmatrix}\right]\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}\right\rvert}{\sqrt{(-2)^2+1^2+2^2}}=12$ We can use the formula to find distance of point to plane. $\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}-(-1)=\pm 36$ Remember to add a "$\pm$" on the RHS after you remove modulus on the LHS. $-2x+y+2z=-1\pm 36$ Remember to answer the question and leave in the equations in Cartesian form. The Cartesian equations of the planes are $-2x+y+2z=35$ and $-2x+y+2z=-37$.
(ii)	$\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}a\\4\\-2\end{pmatrix}=\begin{pmatrix}-4\\2\\4-2a\end{pmatrix}=2\begin{pmatrix}-2\\1\\2-a\end{pmatrix}$ Must take note that for part **(ii)**, a is an unknown to be found again. For l and p to not meet in a unique point, l must be parallel to p. $\begin{pmatrix}-2\\1\\2-a\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}=0$ Recall that if l is parallel to p, the direction vector of l must perpendicular to the normal vector for p. $4+1+4-2a=0$ $a=\dfrac{9}{2}$

Level 3

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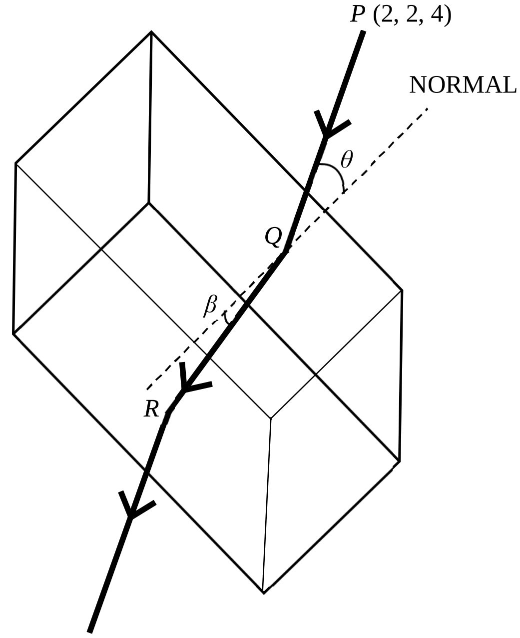


A ray of light passes from air into a material made into a rectangular prism. The ray of light is sent in direction $\begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix}$ from a light source at the point P with coordinates (2, 2, 4). The prism is placed so that the ray of light passes through the prism, entering at the point Q and emerging at the point R and is picked up by a sensor at point S with coordinates (−5, −6, −7). The acute angle between PQ and the normal to the top of the prism at Q is θ and the acute angle between QR and the same normal is β (see diagram).

It is given that the top of the prism is a part of the plane $x+y+z=1$, and that the base of the prism is a part of the plane $x+y+z=-9$. It is also given that the ray of light along PQ is parallel to the ray of light along RS so that P, Q, R and S lie in the same plane.

(i) Find the exact coordinates of Q and R. [5]

(ii) Find the values of $\cos\theta$ and $\cos\beta$. [3]

(iii) Find the thickness of the prism measured in the direction of the normal at Q. [3]

Snell's law states that $\sin\theta = k\sin\beta$, where k is a constant called the refractive index.

(iv) Find k for the material of this prism. [1]

(v) What can be said about the value of k for a material for which $\beta > \theta$? [1]

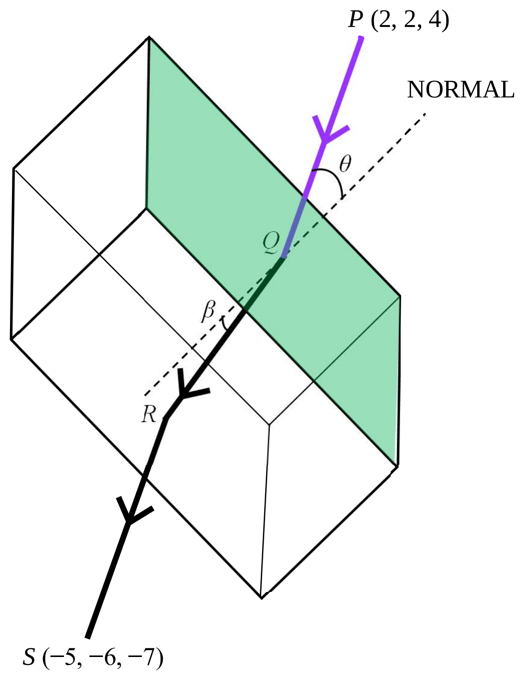
Q11	**Solution**

(i)

Objective: To find the exact coordinates of Q where Q is the point of intersection between the line l_{PQ} with the top of prism.

Top of prism: $\mathbf{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1$

$l_{PQ}:\mathbf{r}=\begin{pmatrix}2\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-2\\-3\\-6\end{pmatrix},\lambda\in\mathbb{R}$



Since point Q lies on l_{PQ},

$\overrightarrow{OQ}=\begin{pmatrix}2-2\lambda\\2-3\lambda\\4-6\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

Learning Point:
If a point lies on a line, then the position vector of that point satisfies the equation of the line

Since point Q lies on the top of the prism,

$$\begin{pmatrix}2-2\lambda\\2-3\lambda\\4-6\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1$$

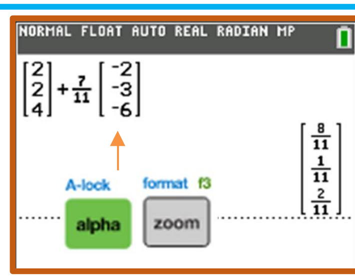
$$2-2\lambda+2-3\lambda+4-6\lambda=1$$

$$\lambda=\frac{7}{11}$$

$$\overrightarrow{OQ}=\begin{pmatrix}2\\2\\4\end{pmatrix}+\frac{7}{11}\begin{pmatrix}-2\\-3\\-6\end{pmatrix}=\begin{pmatrix}\frac{8}{11}\\\frac{1}{11}\\\frac{2}{11}\end{pmatrix}$$

Note:
Leave answer in exact coordinate form

Coordinates of Q is $\left(\frac{8}{11},\frac{1}{11},\frac{2}{11}\right)$



Useful GC Tip:
Use F3 to create 3 rows and 1 column to represent column vector to compute tedious operations.

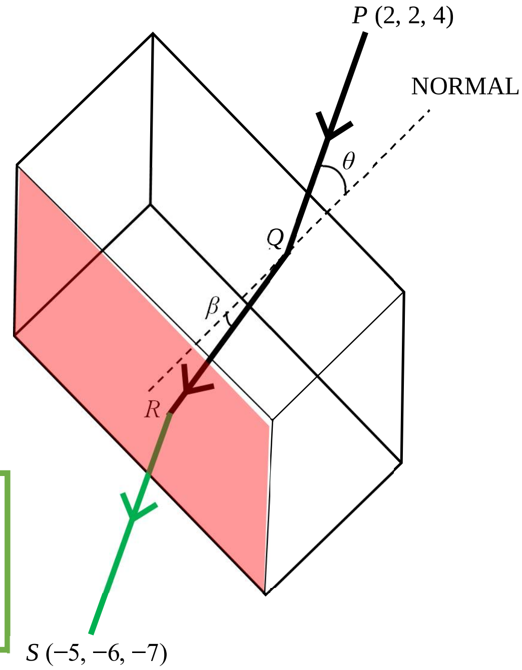
Objective: To find the exact coordinates of R where R is the point of intersection between the line l_{RS} with the bottom of prism.

Bottom of prism: $\mathbf{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9$

$l_{RS}:\mathbf{r}=\begin{pmatrix}-5\\-6\\-7\end{pmatrix}+\mu\begin{pmatrix}-2\\-3\\-6\end{pmatrix},\mu\in\mathbb{R}$

Since point R lies on l_{RS},

$\overrightarrow{OR}=\begin{pmatrix}-5-2\mu\\-6-3\mu\\-7-6\mu\end{pmatrix}$ for some $\mu\in\mathbb{R}$.



Learning Point:
If a point lies on a line, then the position vector of that point satisfies the equation of the line

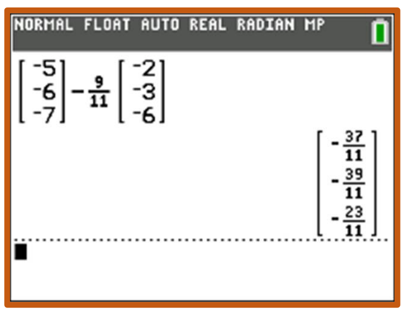
Since point R lies on the bottom of the prism,

$$\begin{pmatrix}-5-2\mu\\-6-3\mu\\-7-6\mu\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9$$

$$-5-2\lambda-6-3\lambda-7-6\lambda=-9$$

$$\lambda=-\frac{9}{11}$$

$$\overrightarrow{OR}=\begin{pmatrix}-5\\-6\\-7\end{pmatrix}-\frac{9}{11}\begin{pmatrix}-2\\-3\\-6\end{pmatrix}=\begin{pmatrix}-\frac{37}{11}\\-\frac{39}{11}\\-\frac{23}{11}\end{pmatrix}$$



Coordinates of R is $\left(-\frac{37}{11},-\frac{39}{11},-\frac{23}{11}\right)$

Note:
Leave answer in exact coordinate form

(ii)

Recall:

$$\cos\theta = \frac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}$$ where θ is the actue angle between the two lines l_1 and l_2

Direction vector of l_{PQ}

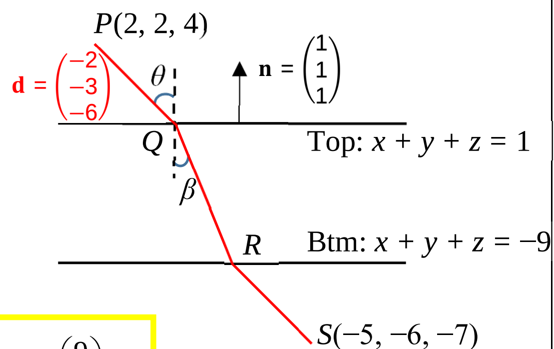
Normal vector of top prism

$$\cos\theta = \frac{\left|\begin{pmatrix}-2\\-3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{1^2+1^2+1^2}\sqrt{(-2)^2+(-3)^2+(-6)^2}}$$

$$= \frac{|-2-3-6|}{\sqrt{3}\sqrt{49}}$$

$$= \frac{11}{7\sqrt{3}} = \frac{11}{21}\sqrt{3}$$

θ is the angle between the line l_{PQ} and the normal.



$$\overrightarrow{QR} = \frac{-1}{11}\begin{pmatrix}37\\39\\23\end{pmatrix} - \frac{1}{11}\begin{pmatrix}8\\1\\2\end{pmatrix} = -\frac{5}{11}\begin{pmatrix}9\\8\\5\end{pmatrix}$$

$-\frac{5}{11}\begin{pmatrix}9\\8\\5\end{pmatrix}$ is parallel to $\begin{pmatrix}9\\8\\5\end{pmatrix}$, so we can use $\begin{pmatrix}9\\8\\5\end{pmatrix}$

Direction vector of l_{QR}

Normal vector of top prism

$$\cos\beta = \frac{\left|\begin{pmatrix}9\\8\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{1^2+1^2+1^2}\sqrt{9^2+8^2+5^2}}$$

$$= \frac{|9+8+5|}{\sqrt{3}\sqrt{170}}$$

$$= \frac{22}{\sqrt{510}} = \frac{11}{255}\sqrt{510}$$

β is the angle between the line l_{QR} and the normal.

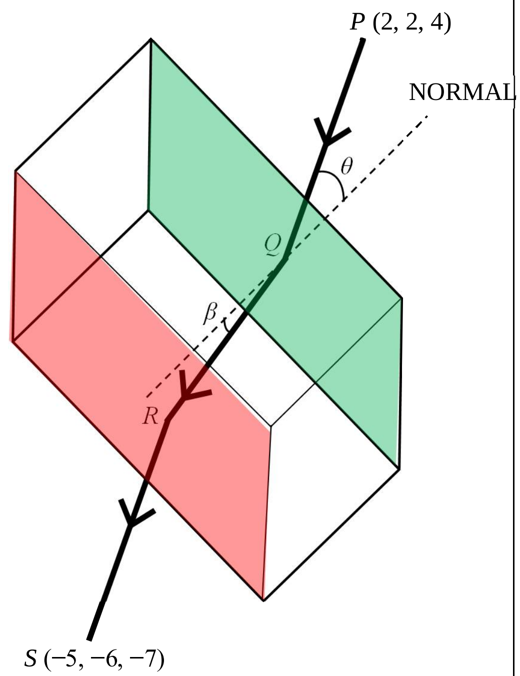
(iii)

Objective: To find the thickness of the prism measured in the direction of normal at Q means to find the distance between 2 planes (Top and bottom of prism)

<u>Method 1 (Using concept of distance between 2 planes)</u>

Top of prism: $\mathbf{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1$

Bottom of prism: $\mathbf{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9$



Distance between 2 planes $=\left|\frac{d_1-d_2}{|\mathbf{n}|}\right|$

$$\text{Thickness of prism}=\left|\frac{1-(-9)}{\sqrt{1^2+1^2+1^2}}\right|$$
$$=\frac{10}{\sqrt{3}}=\frac{10}{3}\sqrt{3}$$

<u>Method 2 (Using concept of length of projection)</u>

The point $A(1,0,0)$ lies on the top of the prism because $\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1$.

The point $B(-9,0,0)$ lies on the bottom of the prism because $\begin{pmatrix}-9\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9$.

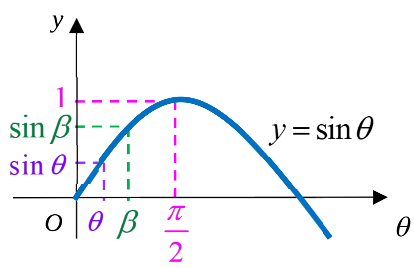
$$\overrightarrow{AB}=\begin{pmatrix}-9\\0\\0\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}-10\\0\\0\end{pmatrix}$$

Thickness of prism = Distance between top of prism and bottom of prism

= Length of projection of $\overrightarrow{AB}$ onto normal

$$=\frac{\left|\begin{pmatrix}-10\\0\\0\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{1^2+1^2+1^2}}$$
$$=\frac{10}{\sqrt{3}}=\frac{10}{3}\sqrt{3}$$

We could consider the length of projection of $\overrightarrow{QR}$ onto normal as well, but $\overrightarrow{AB}$ would yield a less tedious working.

(iv)	$k=\dfrac{\sin\theta}{\sin\beta}$ $=\dfrac{\sin\left(\cos^{-1}\dfrac{11}{7\sqrt{3}}\right)}{\sin\left(\cos^{-1}\dfrac{22}{\sqrt{510}}\right)}$ $=1.862629$ $=1.86$ (3 s.f.)
(v)	Since $0<\theta<\beta<\dfrac{\pi}{2}$,  $0<\sin\theta<\sin\beta<1$ $0<\dfrac{\sin\theta}{\sin\beta}<1$ $\therefore 0<k<1$

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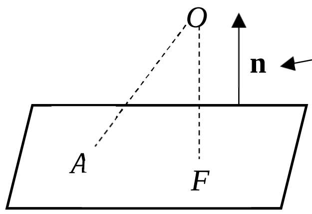
(i) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter. [2]

(ii) Interpret geometrically the vector equation $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{n}$ is a constant unit vector and d is a constant scalar, stating what d represents. [3]

(iii) Given that $\mathbf{b} \cdot \mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} \cdot \mathbf{n} = d$ to find $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ and d. Interpret the solution geometrically. [3]

Q12	Solution
(i)	This is a line parallel to **b** containing **the point *A* with** position vector **a**.
(ii)	This is a plane perpendicular to **n** and |d| units away from the origin. If d is positive, the plane is |d| units from origin in the direction of **n**. If d is negative, the plane is |d| units from origin in the direction of −**n**. If d is zero, the plane contains the origin.



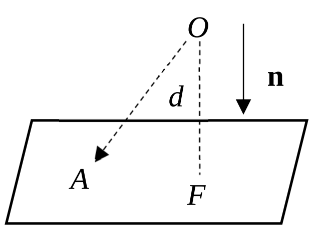
Note that for this question:
n is a unit vector.

Explanation:
In general, the equation of the plane containing the point *A* in scalar product form is $\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$.

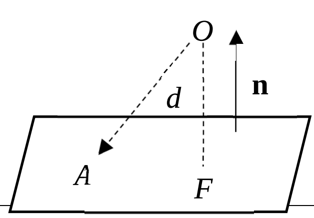
In this case, to find the shortest distance, *OF*, from the origin to the plane, we use the concept of "length of projection".

$$OF = \frac{|\mathbf{a} \bullet \mathbf{n}|}{|\mathbf{n}|} = |\mathbf{a} \bullet \mathbf{n}| \quad (\because \mathbf{n} \text{ is a unit vector})$$

Hence for this question, to interpret geometrically $\mathbf{r} \cdot \mathbf{n} = d$, where *d* is a constant, we can say that |d| is indeed the shortest distance from the origin to the plane with perpendicular vector **n**.



If d is a positive constant, it means $\mathbf{a} \bullet \mathbf{n} > 0$, angle between $\overrightarrow{OA}$ and $\mathbf{n}$ is acute and thus the plane is |d| units from origin in the direction of **n**.



If d is a negative constant, it means $\mathbf{a} \bullet \mathbf{n} < 0$, angle between $\overrightarrow{OA}$ and $\mathbf{n}$ is obtuse and thus the plane is |d| units from origin in the direction of −**n**.

(iii)	To solve the equations $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ (line) and $\mathbf{r}\cdot\mathbf{n}=d$ (plane) is simply using the concept of solving the point of intersection between the line and plane. $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ --------(1) $\mathbf{r}\cdot\mathbf{n}=d$ --------(2) Substitute (1) into (2): $(\mathbf{a}+t\mathbf{b})\bullet\mathbf{n}=d$ $\mathbf{a}\bullet\mathbf{n}+t(\mathbf{b}\bullet\mathbf{n})=d$ Perform expansion here, make sure the "dot product" is included during the expansion. $t(\mathbf{b}\bullet\mathbf{n})=d-\mathbf{a}\bullet\mathbf{n}$ $t=\dfrac{d-\mathbf{a}\bullet\mathbf{n}}{\mathbf{b}\bullet\mathbf{n}}$ $(\text{since } \mathbf{b}\bullet\mathbf{n}\neq 0)$ Our aim here is to solve for the parameter *t*. Substitute $t=\dfrac{d-\mathbf{a}\bullet\mathbf{n}}{\mathbf{b}\bullet\mathbf{n}}$ into (1) to find the point of intersection. $\mathbf{r}=\mathbf{a}+\left(\dfrac{d-\mathbf{a}\bullet\mathbf{n}}{\mathbf{b}\bullet\mathbf{n}}\right)\mathbf{b}$ The solution is the position vector of the point of intersection between the line and the plane.

13 2008/HCI/I/12(b) (modified)

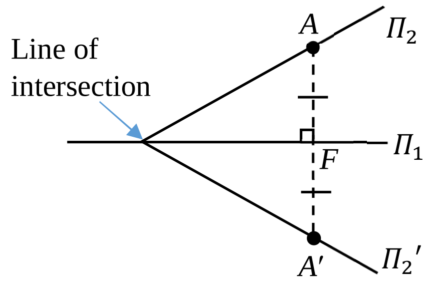
Referring to the origin *O*, two planes Π_1 and Π_2 are given by

$$\Pi_1:\ \mathbf{r}\bullet\begin{pmatrix}1\\2\\-4\end{pmatrix}=13 \text{ and } \Pi_2:\ \mathbf{r}\bullet\begin{pmatrix}1\\3\\3\end{pmatrix}=-8.$$

Find an equation of the plane which is the image of Π_2 when Π_2 is reflected in Π_1. [9]

Q13	**Solution**
	$\Pi_1: x+2y-4z=13$ ……. (1) $\Pi_2: x+3y+3z=-8$ ……. (2) By G.C. solve equations (1) & (2) The vector equation of the line of intersection is $l:\mathbf{r}=\begin{pmatrix}55\\-21\\0\end{pmatrix}+\lambda\begin{pmatrix}18\\-7\\1\end{pmatrix}$ where $\lambda\in\mathbb{R}$. For equation of the plane, you will need 1) Normal vector to plane $\Rightarrow$ 2 non-parallel direction vectors that are parallel to the plane 2) Position vector of a known point of the plane Line of intersection between Π_1 and Π_2 also lies on the reflected plane Π_2'

Let $A(-8,0,0)$ be a point on Π_2 and F be the foot of perpendicular from A to Π_2.



$$l_{AF}: \mathbf{r} = \begin{pmatrix} -8 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \mu \in \mathbb{R}.$$

$$\left(\begin{pmatrix} -8 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \right) \bullet \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 13$$

$$-8 + \mu + 4\mu + 16\mu = 13$$

$$\mu = 1$$

$$\overrightarrow{OF} = \begin{pmatrix} -7 \\ 2 \\ -4 \end{pmatrix}$$

Let A' be the reflection of point A in the plane Π_1

Using ratio theorem,

$$\overrightarrow{OF} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$$

$$\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA}$$

$$\overrightarrow{OA'} = \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix}$$

Let $B(55,-21,0)$ be a point on line l.

$$\overrightarrow{A'B} = \begin{pmatrix} 55 \\ -21 \\ 0 \end{pmatrix} - \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix} = \begin{pmatrix} 61 \\ -25 \\ 8 \end{pmatrix}$$

Explanation of what is done

With the line of intersection, you have 1 direction vector parallel to the reflected plane Π_2' and a position vector of a known point on Π_2'

In order to obtain another direction vector parallel to the reflected plane Π_2', we will reflect a point, A in Π_2 in the plane Π_1.

Then using A' and the position vector of a known point from the line of intersection we can obtain the second direction vector parallel to the reflected plane Π_2'

Equation of reflected plane:

$$\mathbf{r} = \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix} + \alpha \begin{pmatrix} 18 \\ -7 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 61 \\ -25 \\ 8 \end{pmatrix}, \alpha, \beta \in \mathbb{R}. \quad \text{where } \lambda, \mu \in \mathbb{R}$$

Note: There are many possible answers for the above depending on the position vector of the fixed point and direction vectors used. Check whether your scalar product form is same as the one below.

$$\begin{pmatrix} 18 \\ -7 \\ 1 \end{pmatrix} \times \begin{pmatrix} 61 \\ -25 \\ 8 \end{pmatrix} = \begin{pmatrix} -31 \\ -83 \\ -23 \end{pmatrix}$$

Scalar product form: $r \bullet \begin{pmatrix} 31 \\ 83 \\ 23 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix} \bullet \begin{pmatrix} 31 \\ 83 \\ 23 \end{pmatrix} = -38 \Rightarrow r \bullet \begin{pmatrix} 31 \\ 83 \\ 23 \end{pmatrix} = -38.$