Anglo-Chinese School

(Independent)

PRELIMINARY EXAMINATION 2019

YEAR 6 IB DIPLOMA PROGRAMME

MATHEMATICS

HIGHER LEVEL

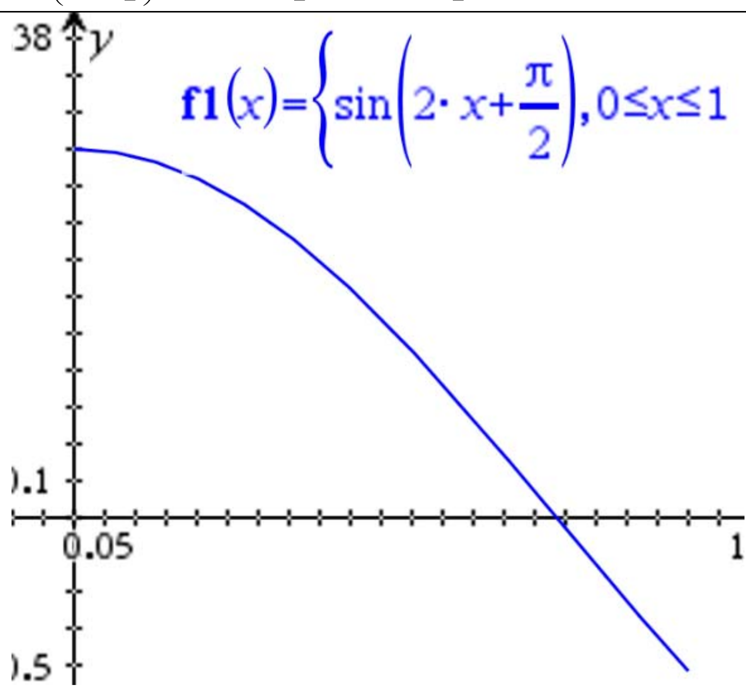
Paper 1 **SOLUTIONS**

SECTION A

Qn	Solution
1.	
(a)	Since the total probability is 1, $k\left(\frac{2}{5}\right)^2 + k\left(\frac{2}{5}\right)^3 + \ldots = 1$
	this takes the form of the sum to infinity of an geometric progression and so $k\left[\frac{\left(\frac{2}{5}\right)^2}{1-\frac{2}{5}}\right] = 1$
	Therefore $k\left(\frac{4}{25}\right) = \frac{3}{5}$ and so $k = \frac{15}{4}$. (note that sum to infinity does exist because $\lvert r\rvert = \left\lvert\frac{2}{5}\right\rvert = \frac{2}{5} < 1$)

Qn	Solution
(b)	$P(X \le x) = \frac{15}{4}\left(\left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \ldots + \left(\frac{2}{5}\right)^x\right)$ $= \frac{15}{4}\left(\frac{\left(\frac{2}{5}\right)^2\left(1-\left(\frac{2}{5}\right)^{x-1}\right)}{1-\frac{2}{5}}\right)$ $= \frac{15}{4}\left(\frac{5}{3}\right)\left(\frac{2}{5}\right)^2\left(1-\left(\frac{2}{5}\right)^{x-1}\right)$ $= 1-\left(\frac{2}{5}\right)^{x-1}$.

	Comments :
	For (b), many students used integration when it is a discrete random variable.
	For (b), some students incorrectly concluded that there are x terms in the sum when there are actually $x-1$ terms.

2.	
(a)	The composite function exists since $R_g = \left[\frac{\pi}{2}, 2+\frac{\pi}{2}\right] \subset D_f$
	$fg(x) = \sin\left(2x+\frac{\pi}{2}\right), x \in [0,1]$ Also accept $fg(x) = \cos 2x, x \in [0,1]$ since $\sin\left(2x+\frac{\pi}{2}\right) = \sin 2x \cos\frac{\pi}{2} + \cos 2x \cos\frac{\pi}{2} = \sin 2x(0) + \cos 2x(1) = \cos 2x$
(b)	
	Comment : Generally well done.

Qn	Solution
3.	
(a)	$\overrightarrow{PQ}=\begin{pmatrix}0\\2\\-1\end{pmatrix}-\begin{pmatrix}2\\5\\1\end{pmatrix}=\begin{pmatrix}-2\\-3\\-2\end{pmatrix}$ $\overrightarrow{PR}=\begin{pmatrix}3\\7\\-2\end{pmatrix}-\begin{pmatrix}2\\5\\1\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}$ $\overrightarrow{QR}=\begin{pmatrix}3\\7\\-2\end{pmatrix}-\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}3\\5\\-1\end{pmatrix}$
	A vector perpendicular to the plane PQR is $-\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{pmatrix}2\\3\\2\end{pmatrix}\times\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}-9-4\\-(-6-2)\\4-3\end{pmatrix}=\begin{pmatrix}-13\\8\\1\end{pmatrix}$
	Therefore the vector equation of the plane is $\boldsymbol{r}\cdot\begin{pmatrix}-13\\8\\1\end{pmatrix}=\begin{pmatrix}0\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}-13\\8\\1\end{pmatrix}=15$
	The cartesian equation is $-13x+8y+z=15$
	(Accept $13x-8y-z=-15$)
	NB : can also take $\overrightarrow{PR}\times\overrightarrow{QR}$ or $\overrightarrow{PQ}\times\overrightarrow{QR}$
(b)	The distance of the plane from the origin is $\dfrac{\lvert 15\rvert}{\sqrt{13^2+8^2+1^2}}=\dfrac{15}{\sqrt{169+64+1}}=\dfrac{15}{\sqrt{234}}$
	The area of the plane PQR is $\frac{1}{2}\left\lvert\overrightarrow{PQ}\times\overrightarrow{PR}\right\rvert=\dfrac{1}{2}\left\lvert\begin{pmatrix}-13\\8\\1\end{pmatrix}\right\rvert=\dfrac{\sqrt{234}}{2}$
	So the volume of the pyramid formed is $\dfrac{1}{3}\dfrac{\sqrt{234}}{2}\left(\dfrac{15}{\sqrt{234}}\right)=\dfrac{5}{2}\left(=\dfrac{15}{6}\right)$
	Comment : Some students did not give the proper Cartesian equation answer to (a)
4.	
	$e^{x+2y}=2$
	Differentiate with respect to x,
	$\left(1+2\dfrac{dy}{dx}\right)e^{x+2y}=0$
	When $x=0, e^{2y}=2, \therefore y=\dfrac{1}{2}\ln 2$
	So $\left(1+2\dfrac{dy}{dx}\right)(1)(2)=0 \quad \Rightarrow \quad \dfrac{dy}{dx}=-\dfrac{1}{2}$
	The equation of the tangent at $\left(0,\dfrac{1}{2}\ln 2\right)$ is

Qn	Solution
	$y-\frac{1}{2}\ln 2=-\frac{1}{2}(x-0)$ $2y-\ln 2=-x$ $2y+x-\ln 2=0$
	(accept $y=\frac{1}{2}\ln 2-\frac{1}{2}x$)
	Comment : Students are not confident with implicit differentiation. Some students found the equation of the normal instead of the equation of the tangent.
5.	
(a)	Put the system of linear equations into a matrix and use row operations to solve
	$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 3 & 2 \\ 3 & -10 & k & -1 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & -13 & k-3 & -1 \end{array}\right) \xrightarrow{R_3+13R_2} \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & k+10 & 25 \end{array}\right)$
	For no unique solution, $k+10=0$, so $k=-10$
(b)	If $k=-9$, the last equation is $z=25$
	Then $y=2-z=2-(25)=-23$
	and $x=-y-z=-(-23)-(25)=-2$
	The coordinates of the point of intersection is $(-2,-23,25)$
	Comment : For (a), a common mistake is to equate $k+10=25$. This is conceptually incorrect as it assumes there is a unique solution.
6.	
	Let $u=\sqrt{x}$, then $\frac{du}{dx}=\frac{1}{2\sqrt{x}}$ and so $\frac{dx}{du}=2u$
	Replacing dx with $2u\ du$ and $\sqrt{x}$ with u,
	$\int \sqrt{x}\cos\sqrt{x}\ dx=\int u\cos u(2u)\,du$ $=\int 2u^2\cos u\ du$ $=2u^2\sin u-\int 4u\sin u\ du$ $=2u^2\sin u-\left[-4u\cos u+\int 4\cos u\ du\right]$ $=2u^2\sin u+4u\cos u-4\sin u+c$ $=(2x-4)\sin\sqrt{x}+4\sqrt{x}\cos\sqrt{x}+c$
	Comment : Students made careless mistakes doing the integration by parts.
7.	
(a)	Let P_n be the proposition that $\sum_{r=1}^{n}(-1)^r\frac{r^2+r+1}{r!}=(-1)^n\frac{n+1}{n!}-1$ for any $n\in\mathbb{Z}^+$.
	To prove P_1 :

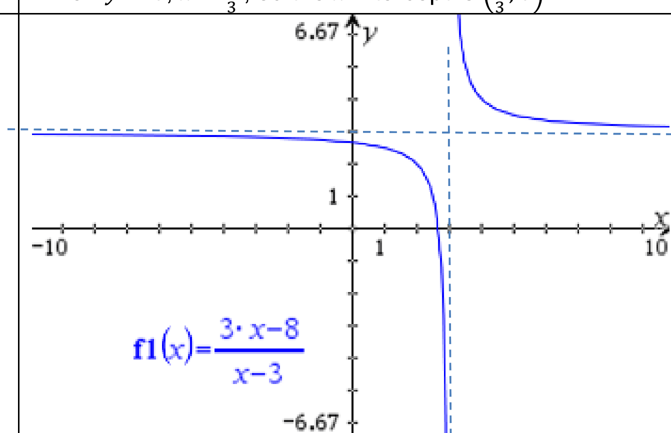
Qn	Solution
	LHS of P_1 : $(-1)^1 \frac{1^2+1+1}{1!} = -3$
	RHS of $P_1 : (-1)^1 \frac{1+1}{1!} - 1 = -2-1 = -3$. Thus P_1 is true.
	Assume that P_k is true for **<u>some</u>** $k \in \mathbb{Z}^+$. That is, $\sum_{r=1}^{k} (-1)^r \frac{r^2+r+1}{r!} = (-1)^k \frac{k+1}{k!} - 1$
	LHS of P_{k+1} :
	$\sum_{r=1}^{k+1} (-1)^r \frac{r^2+r+1}{r!}$ $= \sum_{r=1}^{k} (-1)^r \frac{r^2+r+1}{r!} + (-1)^{k+1} \frac{(k+1)^2+(k+1)+1}{(k+1)!}$ $= (-1)^k \frac{k+1}{k!} - 1 + (-1)^{k+1} \frac{(k+1)^2+(k+1)+1}{(k+1)!}$ $= \frac{(-1)^k}{(k+1)!}\left[(k+1)^2 - \left((k+1)^2+(k+1)+1\right)\right] - 1$ $= \frac{(-1)^{k+1}}{(k+1)!}[k+2] - 1 = (-1)^{k+1}\frac{(k+1)+1}{(k+1)!} - 1 = \text{RHS of } P_{k+1}$
	Since P_1 is true and P_k true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
(b)	$\sum_{r=3}^{20} (-1)^r \frac{r^2+r+1}{r!} = \sum_{r=1}^{20} (-1)^r \frac{r^2+r+1}{r!} - \sum_{r=1}^{2} (-1)^r \frac{r^2+r+1}{r!}$ $= \left((-1)^{20}\frac{20+1}{(20)!} - 1\right) - \left((-1)^2 \frac{2+1}{(2)!} - 1\right)$ $= \frac{21}{20!} - 1 - \frac{3}{2!} + 1$ $= \frac{21}{20!} - \frac{3}{2}.$
	Comment : Generally alright. For (b), many students made the fatal mistake of computing $\sum_{r=1}^{20} (-1)^r \frac{r^2+r+1}{r!} - \sum_{r=1}^{3} (-1)^r \frac{r^2+r+1}{r!}$ instead.
8.	
(a)	$P(0 \le X \le 2) = \int_0^1 e^{-x}\, dx + \int_1^2 \frac{1}{2e} dx$ $= \left[\frac{e^{-x}}{-1}\right]_0^1 + \left[\frac{x}{2e}\right]_1^2$ $= -e^{-1} + 1 + \frac{2}{2e} - \frac{1}{2e}$ $= 1 - \frac{1}{2e}$

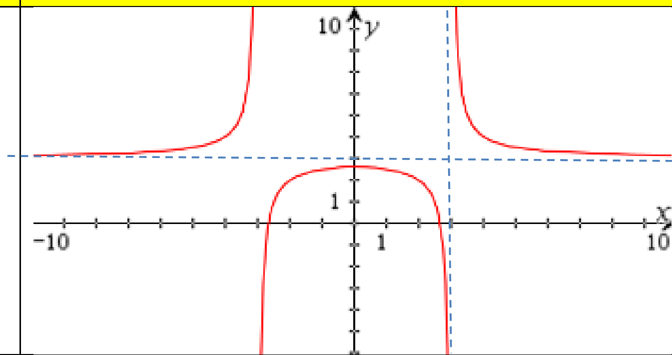
Qn	Solution
(b)	$P(0 \le X \le 2) = 1 - \frac{1}{2}e^{-1} \approx 1 - \frac{1}{2}\left(\frac{1}{2.71}\right) > 1 - \frac{1}{2}\left(\frac{1}{3}\right) = \frac{5}{6} > \frac{1}{2}$
	So $P(0 \le X \le 2) > 0.5$
	Also, $P(0 \le X \le 1) = 1 - \frac{1}{e} > 1 - \frac{1}{2} = \frac{1}{2}$
	So the median lies in $0 \le x \le 1$
	(NB : Do not accept "otherwise" method)
(c)	Let *m* be the median of X.
	Then $\int_0^m e^{-x}\,dx = \frac{1}{2}$
	$-e^{-m} + 1 = \frac{1}{2}$ $e^{-m} = \frac{1}{2}$ $m = \ln 2$
	Comment : For (b), many students did not refer to the answer in (a) when justifying why the median lies in $0 \le x \le 1$, as required by the question.

SECTION B

Qn	Solution
9.	
(a)	$\frac{dy}{dx} = \frac{e^{x-3}(1) - (x-1)e^{x-3}}{\left(e^{x-3}\right)^2}$ $= \frac{e^{x-3}\left(1-(x-1)\right)}{\left(e^{x-3}\right)^2}$ $= \frac{2-x}{e^{x-3}}$ $= \frac{1}{e^{x-3}} - \frac{x-1}{e^{x-3}}$ $= e^{3-x} - y$
	METHOD 2 : $y\left(e^{x-3}\right) = x - 1$ Differentiate w.r.t. *x*, $\frac{dy}{dx}\left(e^{x-3}\right) + y\left(e^{x-3}\right) = 1$ $\frac{dy}{dx} + y = \frac{1}{e^{x-3}}$ $\frac{dy}{dx} = e^{3-x} - y$

Qn	Solution
(b)	From earlier, $\frac{dy}{dx}=\frac{2-x}{e^{x-3}}$,
	ALTERNATIVELY : $\frac{dy}{dx}=e^{3-x}-y$ $=e^{3-x}-\frac{x-1}{e^{x-3}}$ $=\frac{1}{e^{x-3}}-\frac{x-1}{e^{x-3}}=\frac{2-x}{e^{x-3}}$
	THEN : when $\frac{dy}{dx}=0, x=2$.
	When $x=2, y=\frac{1}{e^{2-3}}=e$. So a turning point is at $y=e$.
	Since we are finding the maximum and minimum points in a bounded interval $[1,4]$, there is a chance that the endpoints might take a y value that is higher or lower than $y=e$. Checking the endpoints, when $x=1, y=0$ when $x=4, y=\frac{3}{e}$
	Since $0<\frac{3}{e}<2<e$
	The maximum value is e and the minimum value is 0
	Comments : Students knew that they had to compute $\frac{dy}{dx}=0$, but incorrectly deduced that $e^{x-3}=0$, and further concluding that $x=3$. For those who did correctly find that $x=2$, they forgot to check the endpoints of the bounded interval $[1,4]$ to check the values of y at the endpoints. Some students found the maximum and minimum values of $\frac{dy}{dx}$ instead of y
(c)	From $\frac{dy}{dx}=\frac{2-x}{e^{x-3}}$,
	$\frac{d^2y}{dx^2}=\frac{e^{x-3}(-1)-(2-x)e^{x-3}}{\left(e^{x-3}\right)^2}$ $=\frac{-1-2+x}{e^{x-3}}$ $=\frac{x-3}{e^{x-3}}$
	When $\frac{d^2y}{dx^2}=0, x=3$
	So $y=\frac{3-1}{e^0}=2$

Qn	Solution
	The coordinates of the point of inflexion are $(3,2)$
(d)(i)	The curve is concave down when $\frac{d^2y}{dx^2}<0$
	From the expression of $\frac{d^2y}{dx^2}=\frac{x-3}{e^{x-3}}$, the denominator is always positive. So we only need to consider the numerator.
	Therefore, the curve is concave down when $x-3<0$ i.e. $0<x<3$.
	Comment : Some students incorrectly stated that the curve is concave down when $\frac{dy}{dx}<0$
(ii)	From (b), the maximum point is $(2,e)$. So the graph is decreasing for $x>2$.
	OR : $\frac{dy}{dx}<0$
	OR : $x-1$ is increasing slower than e^{x-3} is increasing
(iii)	As $x\to\infty$, $\frac{1}{e^{x-3}}\to 0$ at a faster rate than $x-1\to\infty$
	So the horizontal asymptote is $y=0$.
	Comment : Some students lost marks because they did not show working why the horizontal asymptote is $y=0$.
10.	
(a)(i)	$y=\frac{3x-8}{x-3}=\frac{3(x-3)+1}{x-3}=3+\frac{1}{x-3}$. As $x\to\infty, y\to 3+0=3$
	OR : $y=\frac{3x-8}{x-3}=\frac{3-\frac{8}{x}}{1-\frac{3}{x}}$. As $x\to\infty, y\to\frac{3-0}{1-0}=3$.
	So the vertical asymptote is $x=3$ and the horizontal asymptote is $y=3$
	When $x=0, y=\frac{8}{3}$, so the y intercept is $\left(0,\frac{8}{3}\right)$
	When $y=0$, $x=\frac{8}{3}$, so the x intercept is $\left(\frac{8}{3},0\right)$
	
(ii)	The graph of $y=\frac{3\lvert x\rvert-8}{\lvert x\rvert-3}$ is the graph of $y=f(\lvert x\rvert)$

Qn	Solution
	
	• y-intercept is $\left(0, \frac{8}{3}\right)$ • x-intercept is $\left(\frac{8}{3}, 0\right)$ AND $\left(-\frac{8}{3}, 0\right)$ (from the reflection in the y axis) • Vertical asymptote is $x = 3$ AND $x = -3$ • Horizontal asymptote is $y = 3$
(iii)	$f(x) \xrightarrow{A} f(x-1)+2 \xrightarrow{B} 2f(x-1)+4$
	So the new function is
	$2f(x-1)+4 = 2\left[\frac{3(x-1)-8}{(x-1)-3}\right]+4$ $= \frac{6x-6-16}{x-4}+4$ $= \frac{6x-22}{x-4}+4$
	Comments: • A lot of students did not know how to sketch functions of the form $y = \frac{ax+b}{cx+d}$ • Weak in sketching graphs of $y = f(\lvert x \rvert)$ • Students were unable to apply transformation of graphs correctly, especially when it came to stretching parallel to y-axis.
(b)	$f(x) = x^2$ $\frac{3x-8}{x-3} = x^2$ $3x-8 = x^3-3x^2$ $x^3-3x^2-3x+8=0$
(i)	$\alpha+\beta+\gamma = -\frac{(-3)}{1} = 3$
	$\alpha\beta\gamma = -\frac{8}{1} = -8$
(ii)	$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $= 3^2 - 2(-3)$ $= 9+6$ $= 15$
(iii)	**METHOD 1**
	Let $y = \frac{1}{x}$, so $x = \frac{1}{y}$ and so the new equation is

Qn	Solution
	$\left(\frac{1}{y}\right)^3 - 3\left(\frac{1}{y}\right)^2 - 3\left(\frac{1}{y}\right) + 8 = 0$ $\frac{1}{y^3} - \frac{3}{y^2} - \frac{3}{y} + 8 = 0$ $1 - 3y - 3y^2 + 8y^3 = 0$
	(also accept alternative forms like $y^3 - \frac{3}{8}y^2 - \frac{3}{8}y + \frac{1}{8} = 0$)
	METHOD 2
	The sum of the new roots is $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ $= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ $= \frac{-3}{-8} = \frac{3}{8}$
	The sum of roots taken two at a time is
	$\frac{1}{\alpha}\frac{1}{\beta} + \frac{1}{\beta}\frac{1}{\gamma} + \frac{1}{\alpha}\frac{1}{\gamma}$ $= \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma}$ $= \frac{3}{-8} = -\frac{3}{8}$
	The product of roots is $\frac{1}{\alpha}\frac{1}{\beta}\frac{1}{\gamma} = \frac{1}{-8} = -\frac{1}{8}$
	So the equation is $-8y^3 + 3y^2 + 3y - 1 = 0$ $8y^3 - 3y^2 - 3y + 1 = 0$
	Comments: • Many students did not read the question properly and were penalised. Students should learn to read the question within the context of the problem. • For those who attempted, some did not manage to do part (iii).
11.	
(a)(i)	$\left\|\left(\sqrt{3}+i\right)(1-i)\right\| = \left\|\sqrt{3}+i\right\|\left\|1-i\right\|$ $= 2\sqrt{2}$
	$\arg\left(\left(\sqrt{3}+i\right)(1-i)\right) = \arg\left(\sqrt{3}+i\right) + \arg(1-i)$ $= \frac{\pi}{6} - \left(\frac{\pi}{4}\right)$ $= -\frac{\pi}{12}$
(ii)	$\left(\sqrt{3}+i\right)(1-i) = \sqrt{3} + i - \sqrt{3}i - i^2$ $= \sqrt{3} + 1 + i\left(1-\sqrt{3}\right)$
	From (a)(i), we know that $\alpha = 2\sqrt{2}\left(cos\left(-\frac{\pi}{12}\right) + i\, sin\left(-\frac{\pi}{12}\right)\right)$

Qn	Solution
	Comparing imaginary parts, $1-\sqrt{3} = 2\sqrt{2}\,sin\left(-\frac{\pi}{12}\right) = -2\sqrt{2}\,sin\left(\frac{\pi}{12}\right)$
	So $sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$
	Similarly, $2\sqrt{2}\,cos\left(\frac{\pi}{12}\right) = \sqrt{3}+1$
	So $cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$
	$cot\frac{\pi}{12} = \frac{cos\frac{\pi}{12}}{sin\frac{\pi}{12}} = \frac{\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$
	Comments: • In (i), generally ok for modulus. For argument, many students ran into problems because did not know how to approach the problem. • In (ii), students must use 'hence' method in their approach. A lot of students also did not simplify and made mistakes such as $sin\left(\frac{\pi}{12}\right) = \frac{1-\sqrt{3}}{2\sqrt{2}}$
(b)	In Euler form, $\alpha^n = \left(2^{\frac{3}{2}n}\right)e^{-i\frac{n\pi}{12}}$
	for α^n to be positive, $-\frac{n\pi}{12}$ is an even multiple of π
	So the smallest positive integer value of n is 24. (0 is not accepted as it is not positive)
	Comments: Students can apply De Moivre's Theorem properly but many could not get the right answer (they gave the answer as 12).
(c)(i)	$\left(z-re^{i\theta}\right)\left(z-re^{-i\theta}\right) = z^2-\left(re^{i\theta}+re^{-i\theta}\right)z+r^2$ $= z^2-2\left(r\cos\theta\right)z+r^2$
(ii)	$\lvert\alpha\rvert^8 = \left(2\sqrt{2}\right)^8 = 2^{12}$
	So $z^4+2^{12} = 0 \Rightarrow z^4 = -2^{12} = 2^{12}e^{i\pi} = 2^{12}e^{i(\pi+2k\pi)}$
	Therefore $z = 2^3e^{i\left(\frac{\pi+2k\pi}{4}\right)}, k = -2,-1,0,1$
	So $z = 8e^{-i\frac{3\pi}{4}}, 8e^{-i\frac{\pi}{4}}, 8e^{i\frac{\pi}{4}}, 8e^{i\frac{3\pi}{4}}$
(iii)	$z^4+\lvert\alpha\rvert^8 = \left(z-8e^{-i\frac{3\pi}{4}}\right)\left(z-8e^{i\frac{3\pi}{4}}\right)\left(z-8e^{-i\frac{\pi}{4}}\right)\left(z-8e^{i\frac{\pi}{4}}\right)$ $= \left(z^2-16\cos\frac{3\pi}{4}z+64\right)\left(z^2-16\cos\frac{\pi}{4}z+64\right)$ $= \left(z^2+16\frac{\sqrt{2}}{2}z+64\right)\left(z^2-16\frac{\sqrt{2}}{2}z+64\right)$ $= \left(z^2+8\sqrt{2}z+64\right)\left(z^2-8\sqrt{2}z+64\right)$
	Comments: • (i) was generally ok. Remember that you need to show working to demonstrate your understanding. • (ii) and (iii) was poorly done. Many are weak in solving questions involving nth root of a complex number.