1 The roots of the quadratic equation $x^2 - 4x + 6 = 0$ are $2\alpha$ and $2\beta$. Find

(i) the value of $\alpha^3 + \beta^3$. [3]

Sum of roots $= 2\alpha + 2\beta$

$\frac{4}{1} = 2(\alpha+\beta)$

$4 = 2(\alpha+\beta)$

$\alpha+\beta = 2$

product of roots $= 2\alpha \times 2\beta$

$\frac{6}{1} = 4\alpha\beta$

$6 = 4\alpha\beta$

$\alpha\beta = \frac{6}{4}$

$\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$= [(2)][(2)^2 - 2(\frac{6}{4}) - \frac{6}{4}]$

$= -1$

~~Sum of new roots $= \alpha^3+\beta^3$~~

$\alpha^3+\beta^3 = -1$

3

(ii) a quadratic equation with roots $\frac{3}{\alpha^3}$ and $\frac{3}{\beta^3}$. [3]

Sum of new roots $= \frac{3\beta^3}{\alpha^3\beta^3} + \frac{3\alpha^3}{\beta^3\alpha^3}$

$= \frac{3(\alpha^3+\beta^3)}{(\alpha\beta)^3}$

$= \frac{3(-1)}{(\frac{6}{4})^3}$

$= -\frac{3}{\frac{27}{8}}$

$= -\frac{8}{9}$ ✱

Product of new roots $= \frac{3}{\alpha^3} \times \frac{3}{\beta^3}$

$= \frac{9}{(\alpha\beta)^3}$

$= \frac{9}{(\frac{6}{4})^3}$

$= \frac{8}{3}$ ✱

$x^2 + \frac{8}{9}x + \frac{8}{3} = 0$ 2×9

$9x^2 + 8x + 24 = 0$ ✱

3

2 (i) Write down and simplify the first three terms in the binomial expansion of $\left(2+\frac{x}{4}\right)^n$, where $n$ is a positive integer greater than 2. [3]

$(2+\frac{x}{4})^n$

$= 2^n + 2^{n-1}\binom{n}{1}\left(\frac{x}{4}\right) + 2^{n-2}\binom{n}{2}\left(\frac{x}{4}\right)^2$

$= 2^n + 2^{n-1} \times n \times \frac{x}{4} + 2^{n-2}\,\frac{n(n-1)}{1\times2} \times \frac{x^2}{16}$

$= 2^n + 2^{n-1}\frac{\times n \times x}{4} + \frac{2^{n-2}(n^2-n)x^2}{32}$

$= 2^n + 2^{n-1-2} \times n \times x + 2^{n-2-5}(n^2-n)x^2$

$= 2^n + 2^{n-3}nx + 2^{n-7}(n^2-n)x^2$ ✱ + ...

3

The first two terms in the expansion, in ascending powers of $x$, of $(1-x)\left(2+\frac{x}{4}\right)^n$ are $a+bx^2$, where $a$ and $b$ are constants.

(ii) Find the value of $n$. [3]

$(1-x)\left(2^n+2^{n-3}nx+2^{n-7}(n^2-n)x^2\right)$

$= 2^n+2^{n-3}nx+2^{n-7}(n^2-n)x^2-\cancel{2^n x}-2^{n-3}nx^2-2^{n-7}(n^2-n)x^3-$

comparing coefficients of $x$, $2^{n-3}n-2^n=0$

$2^{n-3}n-2^n=0$

$2^{n-3}n=2^n$

$n=\frac{2^n}{2^{n-3}}$

$n=2^{n-n+3}$

$n=2^3$

$n=8$ ※

3

(iii) Hence find the value of $a$ and of $b$. [3]

finding for a

$2^n=a$

$2^8=256$

$a=256$※

finding for b (comparing coefficient of $x^2$)

$2^{n-7}(n^2-n)-2^{n-3}n$

$=2^{8-7}(8^2-8)-2^{8-3}(8)$

$=-144$

$b=-144$ ※

3

3 (i) Prove that $\dfrac{1+\cos 2\theta+\sin 2\theta}{1-\cos 2\theta+\sin 2\theta}=\cot\theta$. **[3]**

LHS

$$\frac{1+(2\cos^2\theta-1)+2\sin\theta\cos\theta}{1-(1-2\sin^2\theta)+2\sin\theta\cos\theta}$$

$$=\frac{2\cos^2\theta+2\sin\theta\cos\theta}{2\sin^2\theta+2\sin\theta\cos\theta}$$

$$=\frac{\cos\theta(2\cos\theta+2\sin\theta)}{\sin\theta(2\sin\theta+2\cos\theta)}$$

$$=\frac{\cos\theta}{\sin\theta}$$

$= \cot\theta$ * (RHS proven) *

3

(ii) Hence, solve the equation $\dfrac{1+\cos 2\theta+\sin 2\theta}{1-\cos 2\theta+\sin 2\theta}=2\tan\theta$ for $0\le\theta\le\pi$. **[3]**

$\cot\theta = 2\tan\theta$

$\dfrac{1}{\tan\theta}=\dfrac{2\tan\theta}{1}$

$2\tan^2\theta = 1$

$\tan^2\theta = \frac{1}{2}$

$\tan\theta = \sqrt{\frac{1}{2}}$ or $\tan\theta = -\sqrt{\frac{1}{2}}$

Basic ∡ $= 0.61548$ rad $\quad = -0.61548$ rad

$\pi + 0.61548$
$= 3.7571$ rad

$\pi - (0.61548)$
$= 2.5261$ rad
$2\pi - (0.61548)$
$= 5.6677$ rad

2

$\theta = 0.615$ rad, $2.53$ rad, ~~5.67 rad~~ $3.76$ rad, $5.67$ rad *

4 The table below shows experimental values of $x$ and $y$ which are related by the equation, $y = e^{-A}b^x$, where $A$ and $b$ are constants.

| $x$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $y$ | 0.623 | 4.73 | 35.9 | 273 | 2073 |

(i) By drawing a straight line graph of $\ln y$ against $x$, estimate the value of $A$ and of $b$. [5]

$y = e^{-A} b^x$

$\ln y = \ln(e^{-A} b^x)$

$\ln y = \ln e^{-A} + \ln b^x$

$\ln y = -A \ln e + x \ln b$

$\ln y = -A + x \ln b$

$\ln y = x \ln b - A$

$\ln b = 0.40769$

$b = e^{0.40769}$

$\approx 1.50$ (3sf)

$\ln b = \dfrac{9.8 - (-0.8)}{30 - 4}$

$= 0.40769$ (5sf)

$A = 2.4$

$b = 1.50$ (3sf)

1

2

(ii) Estimate the value of $x$ when $y = 100$. [2]

when $y = 100$

$\ln y = \ln 100$

$\ln y = 4.61$

$x \approx 17.5$

2

(iii) On the same axes, draw the straight line representing $y^5 = e^{10-x}$ and hence find the value of $x$ for which $e^{2-\frac{x}{5}} = e^{-A}b^x$. [3]

1.64

$y^5 = e^{10-x}$

$y = e^{-A} b^x$

$5 \ln y = \ln(e^{10-x})$

$5 \ln y = 10 - x \ln e$

$\ln y = \dfrac{10 - x \ln e}{5}$

$\ln y = 2 - \frac{x}{5}$

$\ln y = 2 - \frac{1}{5}x$

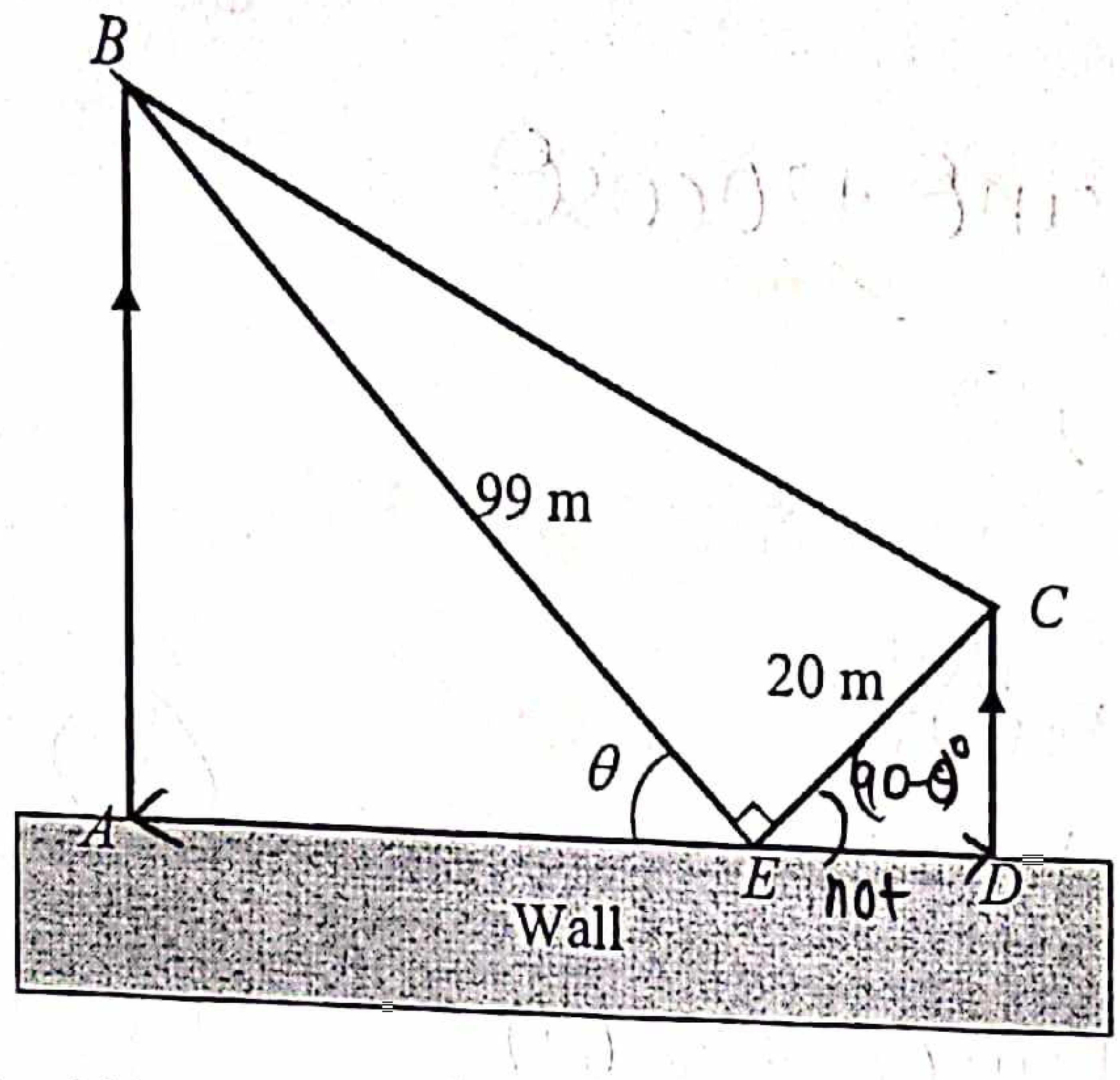
~~x ≈ 5.3~~

$x \approx 7.5$

1

1

5




A farmer fences part of his land. He puts fences around the perimeter of the trapezium field *ABCD* and also from *B* to *E* and from *E* to *C*. The side *AD* which is against a wall is not fenced.

*AB* is parallel to *DC*. *BE* = 99 m, *CE* = 20 m, $\angle BAE = \angle EDC = \angle BEC = 90°$ and $\angle AEB = \theta$, where $\theta$ is an acute angle in degrees.

(i) Show that *L* m, the length of the fences, can be expressed in the form $p + q\sin\theta + r\cos\theta$, where *p*, *q* and *r* are constants to be found. [3]

$L = AB + BE + BC + CD + CE$

$AB \Rightarrow \sin\theta = \frac{AB}{99}$

$AB = 99\sin\theta$

$BE \Rightarrow 99$

$BC \Rightarrow$ By pythagoras Thm,

$99^2 + 20^2 = BC^2$

$BC = 101\,m$

$180° - 90° - \theta = 90 - \theta$

$\angle CED = (90-\theta)°$

$CD \rightarrow \sin(90-\theta) = \frac{CD}{20}$

$20\sin(90-\theta) = CD$

$20\cos\theta = CD$

$CE \rightarrow 20m$

$L = 99\sin\theta + 99 + 101 + 20\cos\theta + 20$

$= 99\sin\theta + 20\cos\theta + 220$

$L = 220 + 99\sin\theta + 20\cos\theta$

$q = 99$

$r = 20$

$p = 220$

3

(ii) Express $L$ in the form $p+R\sin(\theta+\alpha)$, where $R>0$ and $\alpha$ is an acute angle. [3]

$\angle 90^\circ$

$L = 220 + 99\sin\theta + 20\cos\theta$

$R = \sqrt{99^2+20^2}$

$= 101$

$\alpha = \tan^{-1}\left(\frac{99}{20}\right)$

$\tan^{-1}\left(\frac{20}{99}\right)$

$\alpha = 78.579^\circ$ (5sf) ✗

$= 11.421^\circ$

1

$\alpha = 78.6^\circ$ (3sf)

$L = 220 + 101\sin(\theta + 78.6^\circ)$ ✗

$101\sin(\theta + 11.421^\circ) + 220$

$101\sin(\theta + 11.4^\circ) + 220$

(iii) Explain, with proper justification, if it is possible for the total length of the fences [3] to be 232 m.

$232 = 220 + 101\sin(\theta + 78.6^\circ)$

$12 = 101\sin(\theta + 78.6^\circ)$

$\theta + 11.4^\circ = \sin^{-1}\left(\frac{12}{101}\right)$

$\frac{12}{101} = \sin(\theta + 78.6^\circ)$

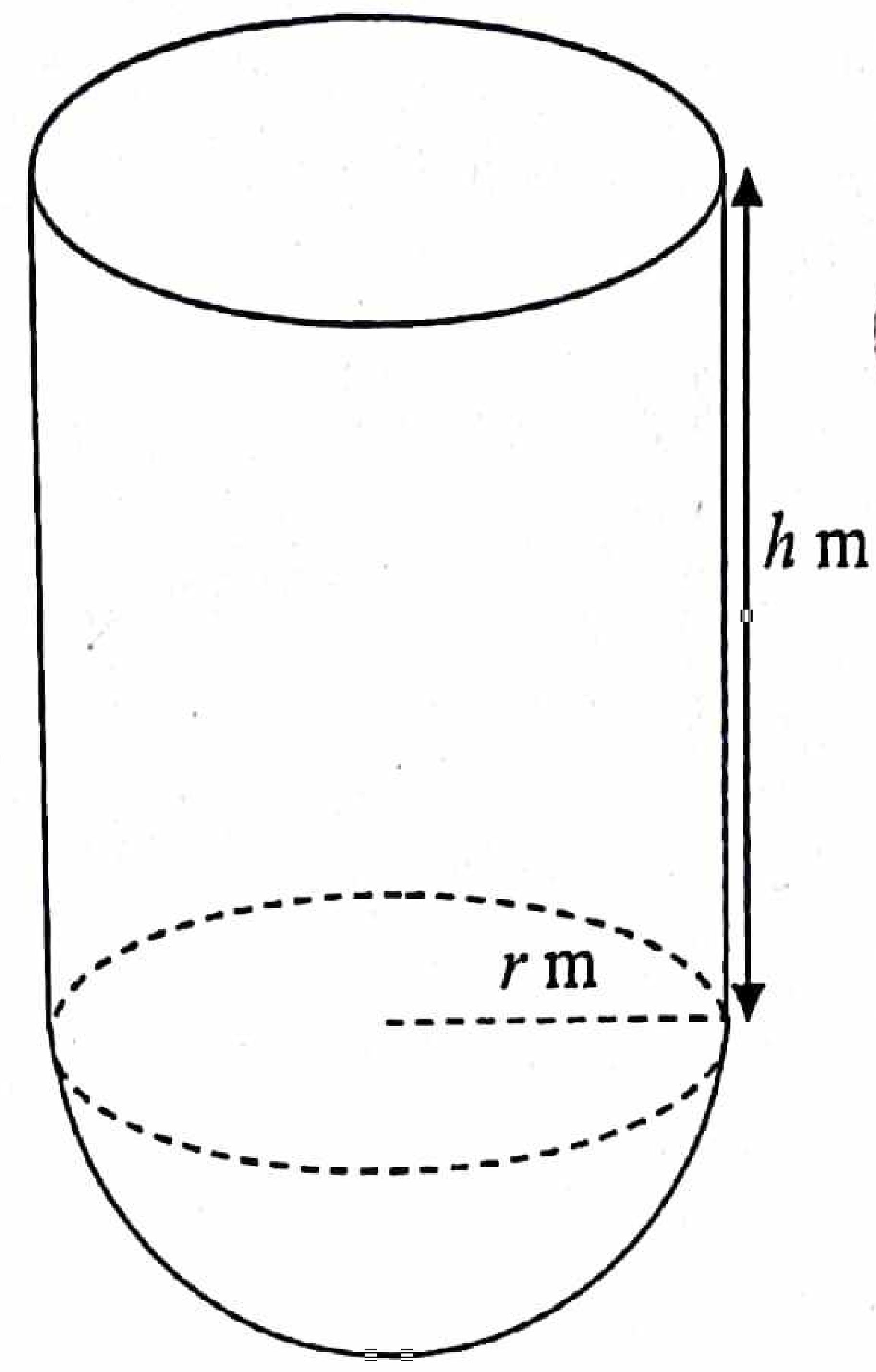
$\theta + 78.6 = 6.8235$ or $173.2$

$\theta = -71.776^\circ$

No. ~~TA~~ $\theta$ would then be $-71.8^\circ$ 1

which is negative and incorrect.

6 A container with an open top is made up of two parts: a right circular cylinder of radius $r$ m and height $h$ m and a hemisphere at the bottom of the cylinder of radius $r$ m, as shown in the diagram.




(i) Given that volume of the container is $\frac{3}{4}\pi$ m$^3$, express $h$ in terms of $r$. [2]

cylinder → $\pi r^2 \times h$
sphere → $\frac{4}{3}\pi r^3$
hemisphere → $\frac{2}{3}\pi r^3$

$$\pi r^2 h + \frac{2}{3}\pi r^3 = \frac{3}{4}\pi m^3$$

$$\pi r^2 (h + \frac{2}{3} r) = \frac{3}{4}\pi$$

$$r^2 h + \frac{2}{3} r^3 = \frac{3}{4}$$

$$\cancel{r^2(h+\frac{2}{3}r} \quad h + \frac{2}{3} r = \frac{3}{4r^2}$$

$$h = \frac{3 \times 3}{4r^2 \times 3} - \frac{2r \times 4r^2}{3 \times 4r^2}$$

$$\cancel{h = \frac{9-8r^3}{12r^2}}$$

$$h = \left(\frac{3}{4r^2} - \frac{2r}{3}\right) m$$

2

Area = $2\pi r^2 + 2\pi rh$

(ii) The container is made up of some thin metal sheets with negligible thickness. The cost of the cylindrical surface is \$3 per m$^2$ and that of the hemisphere surface is \$6 per m$^2$. Let \$$C$ be the total cost of making the container. [3]

$\frac{4}{3}\pi r^3$

(a) Show that $C = 8\pi r^2 + \dfrac{9\pi}{2r}$.

Cost = $2\pi r^2 \times 6 + 2\pi rh \times 3$

SA of cylinder → $2\pi r \times \left(\frac{3}{4r^2} - \frac{2r}{3}\right)$ ?

of hemisphere →

$\frac{dC}{dr}$

$C = 8\pi r^2 + \frac{9\pi}{2r} = 8\pi r^2 + \frac{9\pi}{2} r^{-1}$

$\frac{dC}{dr} = 16\pi r - \frac{9\pi}{2r^2}$

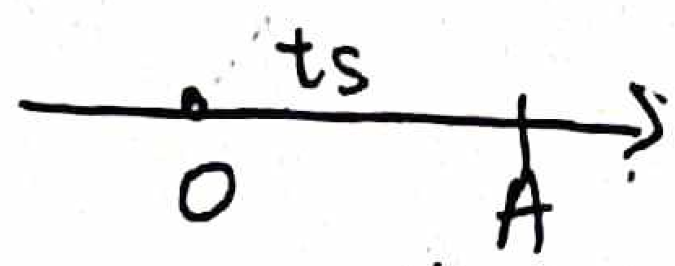
(b) Given that $r$ and $h$ can vary, find the value of $r$ for which $C$ has a stationary value. Show that this value of $r$ gives the minimum cost of making the container. [4]

0

7 A particle moves in a straight line, so that, $t$ seconds after passing a fixed point $O$, its velocity, $v$ m/s, is given by $v = 2e^{0.1t} - 6e^{0.1-0.4t}$. The particle comes to an instantaneous rest at the point $A$.

(i) Show that the particle reaches $A$ when $t = 2\ln 3 + \frac{1}{5}$. [3]

$2e^{0.1t} - 6e^{0.1-0.4t} = 0$




~~$\ln(2e^{0.1t} - 6e^{0.1-0.4t}) = \ln 1$~~

$2e^{0.1t} = 6e^{0.1-0.4t}$

$\frac{2}{6} = e^{0.1-0.4t-0.1t}$

$\frac{2}{6} = e^{0.1-0.5t}$

$\ln\frac{2}{6} = 0.1 - 0.5t \ln e$

$\ln\frac{2}{6} - 0.1 = -0.5t$

~~$t = 2.3972$~~ $2 \times (-2)$

$-2\ln(\frac{1}{3}) + 0.2 = t$

$t = 2\ln 3 + 0.2$ ~~(as shown)~~

$= 2\ln 3 + \frac{1}{5}$ (as shown)

3

(ii) Find the acceleration of the particle at $A$. [3]

$\frac{dv}{dt} = a$

$\frac{dv}{dt} = 0.1 \times 2e^{0.1t} - 6(-0.4)e^{0.1-0.4t}$

$a = 0.2e^{0.1t} + 2.4e^{0.1-0.4t}$

when $t = 2\ln 3 + \frac{1}{5}$,

$a = 0.2e^{0.1(2\ln 3 + \frac{1}{5})} + 2.4e^{0.1-0.4(2\ln 3 + \frac{1}{5})}$

$= 1.2709$

$= 1.27$ m/s$^2$

3

(iii) Find the distance $OA$. [4]

$s = \int v\,dt$

$\int 2e^{0.1t} - 6e^{0.1-0.4t}\,dt$

$= \left\{ \frac{2e^{0.1t}}{0.1} - \frac{6e^{0.1-0.4t}}{-0.4} \right\} + c$, where $c$ is a constant

$s = 20e^{0.1t} + 15e^{0.1-0.4t} + c$

When $t=0$, $s=0$

$0 = 20e^{0.1(0)} + 15e^{0.1-0.4(0)} + c$

$c = -36.578$

$s = 20e^{0.1(2\ln 3 + \frac{1}{5})} + 15e^{0.1-0.4(2\ln 3+\frac{1}{5})} - 36.578$

$= 4.81\text{m}$

4

(iv) Explain whether the particle is again at $O$ at some instant during the sixth second after first passing through $O$. [2]

$s = 20e^{0.1(}$

?

NO.

At $t=5$

5.0 − 6.0s

$s = 20e^{0.1(5)} + 15e^{0.1-0.4(5)} - (20+15e^{0.1})$

$= -1.36\text{m}$

At $t=6$,

$s = 20e^{0.1(6)} + 15e^{0.1-0.4(6)} - (20+15e^{0.1})$

$= 1.37\text{m}$

Since displacement changes from negative to positive, the particle ~~then~~ passes through O during the sixth second.

8 Given that $x^2+x-2$ is a factor of $f(x)=2x^3+ax^2+bx-2$.

(i) Find the value of $a$ and of $b$. [4]

$x^2+x-2=(x-1)(x+2)$

$x-1=0$ or $x+2=0$

$x=1$ $x=-2$

$a=-(-3)$

$a=3$

$b=-3$

when $f(x)=2x^3+ax^2+bx-2$

$f(1)=2(1)^3+a(1)^2+b(1)-2=0$

$=2+a+b-2=0$

$a+b=0$

$f(-2)=2(-2)^3+a(-2)^2+b(-2)-2$

$0=-16+4a-2b-2$

$0=-18+4a-2b$

$18=4a-2b$

$a=-b$

$18=4(-b)-2b$

$18=-6b$

$b=-3$

4

(ii) Using the values of $a$ and $b$ found in (i), find the remainder when $f(x)$ is divided by $(2x-3)$. [2]

$2x-3=0$, $x=\frac{3}{2}$

$2x=3$

when $f(x)=2x^3+3x^2-3x-2$

$$\begin{array}{r} x^2+3x+3 \\ 2x-3\,\overline{)\,2x^3+3x^2-3x-2} \\ -(2x^3-3x^2) \\ \hline 6x^2-3x \\ -(6x^2-9x) \\ \hline 6x-2 \\ -(6x-9) \\ \hline 7 \end{array}$$

when $f(x)$ is divided by $2x-3$,

remainder $=7$

$f(x)=2x^3+3x^2-3x-2$

$f(\frac{3}{2})=2(\frac{3}{2})^3+3(\frac{3}{2})^2-3(\frac{3}{2})-2$

$=7$

$f(x)=2x^3+3x^2-3x-2$

$f(\frac{3}{2})=$

2

Use remainder theorem

more than −4.5

(iii) Show that $f'(x) \geq -4.5$ for all real values of $x$. [4]

Compleete the square

$f(x): 2x^3+3x^2-3x-2$

$\cancel{f'(x)}, \frac{df(x)}{dx} = 6x^2+6x-3$

1

~~when~~ $6x^2+6x-3 \geq -4.5$ ?

$6x^2+6x+1.5 \geq 0$

1

when $6x^2+6x+1.5 \geq 0$,

~~real~~ ~~D<0~~ $D=0$ ~~for all~~

$6^2-4(6)(1.5)=0$ ∗(as shown)

since $D=0$, $f'(x) \geq -4.5$ is true.

Show that determinant =0

$6[x^2+x-\frac{1}{2}]$

$6((x+\frac{1}{2})^2-(\frac{1}{2})^2)-3$

$6(x+\frac{1}{2})^2-4\frac{1}{2}$

$6x^2+6x-3$

$=6(x^2+x)-3$

$=6[x^2+x+\frac{1}{4}-\frac{1}{4}]-3$

$=6[x^2+x+\frac{1}{4}]-\frac{3}{2}-3$

$=6[x+\frac{1}{2}]^2-4\frac{1}{2}$

Least value $=-4\frac{1}{2}$

$\therefore f'(x) \geq -4\frac{1}{2}$

$ax^2+bx+c=0$

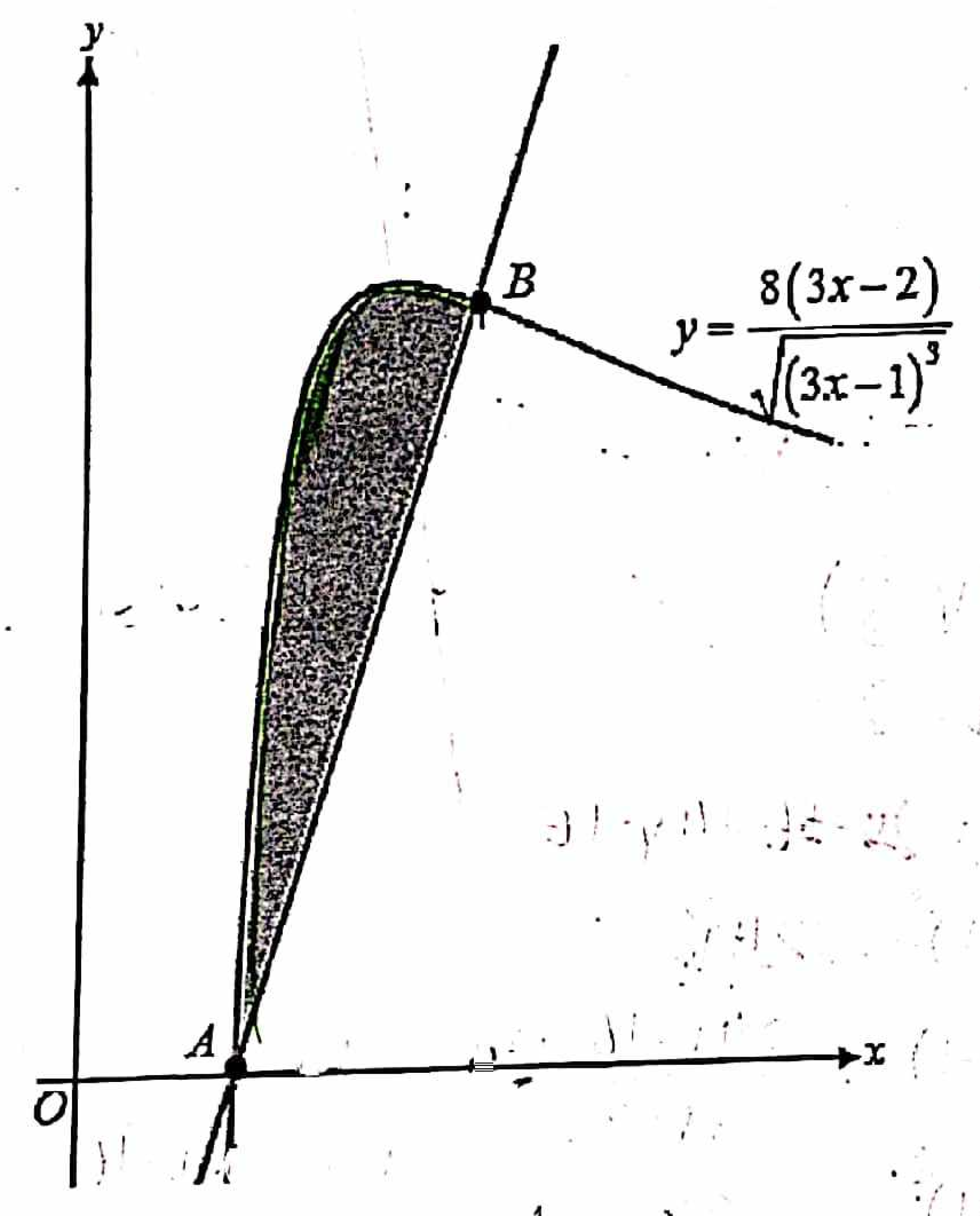
9 The equation of a curve is $y = ax^2 + bx - 3$, where $a$ and $b$ are constants and the curve has a minimum turning point.

(i) Explain why the curve cuts the $x$-axis at two distinct points. [3]

(ii) In the case where $a = 1$, find the range of values of $b$ for which the curve is above the line $y = x - 4$. [3]

(iii) Hence, find the values of $b$ for which the line is a tangent to the curve. [1]

10 (a) Show that $\frac{d}{dx}\left(\frac{x}{\sqrt{3x-1}}\right)=\frac{3x-2}{2\sqrt{(3x-1)^3}}$. [3]




(b) The diagram shows part of the curve $y=\frac{8(3x-2)}{\sqrt{(3x-1)^3}}$. The curve intersects the $x$-axis at point $A$. A line with gradient 3 intersects the curve at points $A$ and $B$.

(i) Verify that the $x$-coordinate of $B$ is $\frac{5}{3}$. [4]

(ii) Find the area of the shaded region. [4]

11 $P$ is the centre of a circle, $C_1$, whose equation is $x^2 + y^2 + 8x + 4y - 5 = 0$.

(i) Find the coordinates of $P$ and the length of the radius of circle $C_1$. [3]

(ii) Find the perpendicular distance of point $P$ to the line $4x + 3y = 28$. [5]

The circle $C_1$ is reflected into circle $C_2$ with the line $4x + 3y = 28$ as the line of reflection.

(iii) Find the equation of the circle $C_2$ in the form $x^2 + y^2 + px + qy + r = 0$. [3]