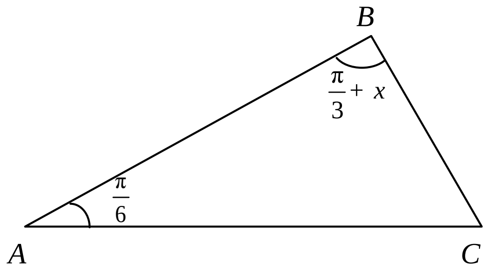
1	Using sine rule, $\dfrac{BC}{\sin\frac{\pi}{6}}=\dfrac{AC}{\sin\left(\frac{\pi}{3}+x\right)}$  $\dfrac{BC}{AC}=\dfrac{\sin\frac{\pi}{6}}{\sin\left(\frac{\pi}{3}+x\right)}$ $=\dfrac{\sin\frac{\pi}{6}}{\sin\frac{\pi}{3}\cos x+\cos\frac{\pi}{3}\sin x}$ $=\dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x}$ $\approx\dfrac{1}{\sqrt{3}\left(1-\frac{x^2}{2}\right)+x}$ $=\dfrac{1}{\sqrt{3}}\left[1+\dfrac{x}{\sqrt{3}}-\dfrac{x^2}{2}\right]^{-1}$ $=\dfrac{1}{\sqrt{3}}\left[1-\left(\dfrac{x}{\sqrt{3}}-\dfrac{x^2}{2}\right)+\left(\dfrac{x}{\sqrt{3}}-\dfrac{x^2}{2}\right)^2+\ldots\right]$ $\approx\dfrac{1}{\sqrt{3}}\left[1-\dfrac{x}{\sqrt{3}}+\dfrac{5}{6}x^2\right]$ $=\dfrac{1}{\sqrt{3}}-\dfrac{1}{3}x+\dfrac{5}{6\sqrt{3}}x^2$ or $\dfrac{\sqrt{3}}{3}-\dfrac{1}{3}x+\dfrac{5\sqrt{3}}{18}x^2$ (shown) Hence, $a=\frac{1}{\sqrt{3}}$, b = $-\frac{1}{3}$, $c=\frac{5}{6\sqrt{3}}$ (Ans).
2	(i) $u_k=\dfrac{k+1}{k-1}u_{k-1}$ $=\dfrac{k+1}{k-1}\dfrac{k}{k-2}u_{k-2}$ $=\dfrac{k+1}{k-1}\dfrac{k}{k-2}\dfrac{k-1}{k-3}u_{k-3}$ $=\dfrac{k+1}{k-1}\dfrac{k}{k-2}\dfrac{k-1}{k-3}\cdots\dfrac{5}{3}\dfrac{4}{2}\dfrac{3}{1}u_1$ $=\dfrac{(k+1)k}{2}u_1$

(ii) Let P(n) be the statement "$S_n = \frac{n}{3}(n+1)(n+2)$ for all positive integer $n \geq 1$"

When $n = 1$,
L.H.S = $S_1 = u_1 = 2$

R.H.S = $\frac{1}{3}(1+1)(1+2) = 2$

$\therefore$ P(1) is true

Assume P(k) is true for some $k \geq 1$, $S_k = \frac{k}{3}(k+1)(k+2)$

To prove P($k+1$) is true:

$$S_{k+1} = \frac{k+1}{3}(k+2)(k+3)$$

$$\begin{aligned}
L.H.S &= S_{k+1} \\
&= S_k + u_{k+1} \\
&= \frac{k}{3}(k+1)(k+2) + \frac{k+2}{k}u_k \\
&= \frac{k}{3}(k+1)(k+2) + \frac{k+2}{k} \times \frac{(k+1)k}{2}u_1 \\
&= \frac{k}{3}(k+1)(k+2) + \frac{(k+2)(k+1)}{2}(2) \\
&= \left(\frac{k}{3}+1\right)(k+1)(k+2) \\
&= \left(\frac{k+3}{3}\right)(k+1)(k+2) \\
&= \left(\frac{k+1}{3}\right)(k+2)(k+3) = R.H.S
\end{aligned}$$

$\therefore$ P($k+1$) is true

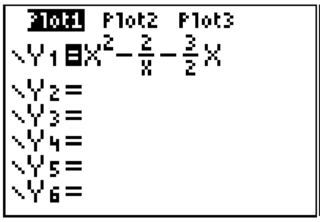
Since P(1) is true and P(k) is true implies P($k+1$) is true, by Mathematical Induction, P(n) is true for all $n \in \mathbb{Z}^+$.

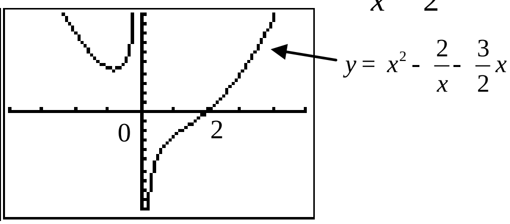
3(i)

$$x^2 - \frac{2}{x} \geq \frac{3}{2}x, \quad x \in \mathbb{R}, \quad x \neq 0$$

$$\Rightarrow x^2 - \frac{2}{x} - \frac{3}{2}x \geq 0$$

Method 1:Using GC to sketch the graphs of $y = x^2 - \frac{2}{x} - \frac{3}{2}x$.





From the graph: $x < 0$ or $x \geq 2$

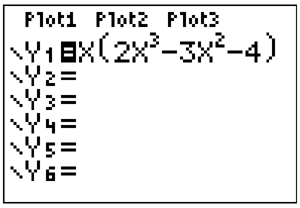
Method 2:

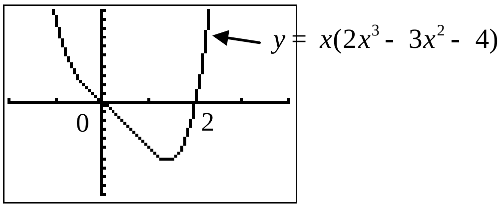
$x^2 - \frac{2}{x} - \frac{3}{2}x \geq 0, \quad x \neq 0$

$\frac{2x^3 - 3x^2 - 4}{2x} \geq 0$

Multiplying by $2x^2$, $x\left(2x^3 - 3x^2 - 4\right) \geq 0$

Sketch the graph of $y = x\left(2x^3 - 3x^2 - 4\right)$ using GC:





From the graph: $x < 0$ or $x \geq 2$

<u>Method 3: Analytical method</u>

$x^2 - \frac{2}{x} - \frac{3}{2}x \geq 0, \quad x \neq 0$

$\frac{2x^3 - 3x^2 - 4}{2x} \geq 0$

Multiplying by $2x^2$,

$x\left(2x^3 - 3x^2 - 4\right) \geq 0$

$x(x-2)(2x^2 + x + 2) \geq 0$

$x(x-2) \geq 0 \quad \text{since} \quad 2x^2 + x + 2 = 2(x + \tfrac{1}{4})^2 + \tfrac{15}{16} > 0$

$x < 0 \quad \text{or} \quad x \geq 2$

(ii)

$$\int_1^a \left| x^2 - \frac{2}{x} - \frac{3}{2}x \right| \mathrm{d}x$$

$$= -\int_1^2 \left(x^2 - \frac{2}{x} - \frac{3}{2}x \right) \mathrm{d}x + \int_2^a \left(x^2 - \frac{2}{x} - \frac{3}{2}x \right) \mathrm{d}x$$

$$= -\left[\frac{x^3}{3} - 2\ln x - \frac{3x^2}{4} \right]_1^2 + \left[\frac{x^3}{3} - 2\ln x - \frac{3x^2}{4} \right]_2^a$$

$$= -\left[\frac{8}{3} - 2\ln 2 - 3 - \left(\frac{1}{3} - 2\ln 1 - \frac{3}{4} \right) \right] + \left[\frac{a^3}{3} - 2\ln a - \frac{3a^2}{4} - \left(\frac{8}{3} - 2\ln 2 - 3 \right) \right]$$

$$= 2\ln 2 - \frac{1}{12} + \left[\frac{a^3}{3} - 2\ln a - \frac{3a^2}{4} + \frac{1}{3} + 2\ln 2 \right]$$

$$= 4\ln 2 - 2\ln a + \frac{a^3}{3} - \frac{3a^2}{4} + \frac{1}{4}$$

$$\therefore \int_1^a \left| x^2 - \frac{2}{x} - \frac{3}{2}x \right| \mathrm{d}x = \frac{a^3}{3} - \frac{3a^2}{4} + \frac{1}{4}$$

$$\Rightarrow \quad 4\ln 2 - 2\ln a + \frac{a^3}{3} - \frac{3a^2}{4} + \frac{1}{4} = \frac{a^3}{3} - \frac{3a^2}{4} + \frac{1}{4}$$

$\Rightarrow \quad \ln 2^4 = \ln a^2$

$\Rightarrow \; a = 2^2 = 4.$

[Turn Over

4

(i) $|iz+3|\le 3$

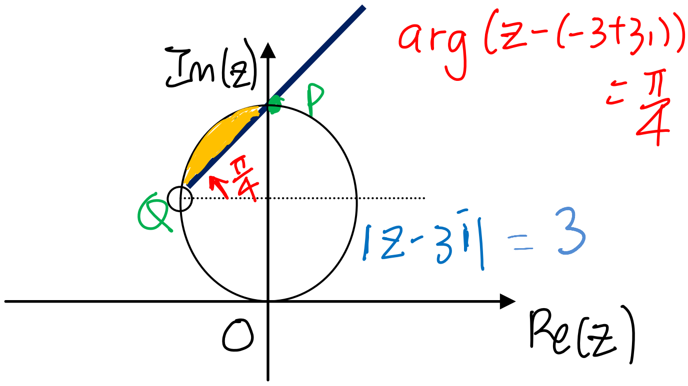
$\Rightarrow |i||z+\frac{3}{i}|\le 3$

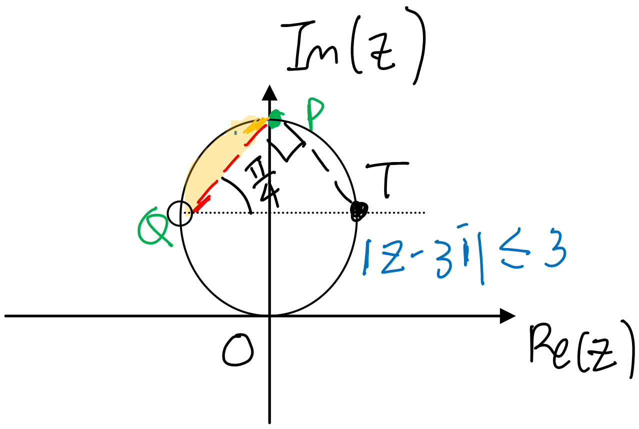
$\Rightarrow |z-3i|\le 3$

$$\arg\left(z+\left(3+\frac{3}{i}\right)\right)\ge\frac{\pi}{4}$$

$\Rightarrow \arg(z+(3-3i))\ge\frac{\pi}{4}$

$\Rightarrow \arg(z-(-3+3i))\ge\frac{\pi}{4}$





(ii)

(a)

Min |z - (3+3i)|=PT

$\frac{PT}{6}=\sin\frac{\pi}{4}\Rightarrow PT=3\sqrt{2}$

Max possible |z - (3+3i)|=QT = 6 units, but Q is not to be included.

Therefore $3\sqrt{2}\le|z-3-3i|<6$ since Q is not included. (Ans)

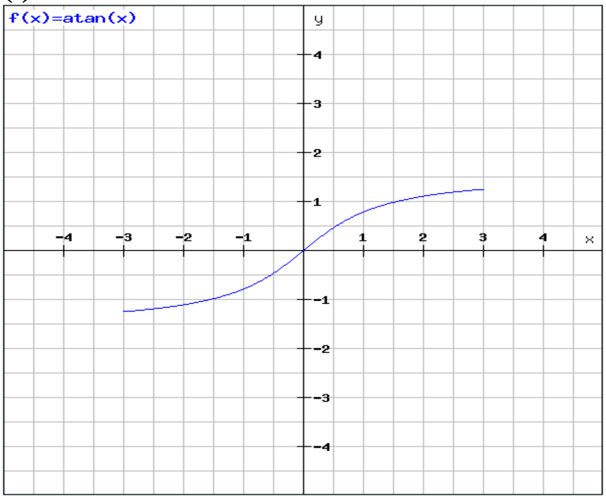
(b)

Max $\arg(z-(3+3i))$ occurs at point Q (not included) and min $\arg(z-(3+3i))$ occurs at point P.

$\frac{3\pi}{4}\le\arg(z-(3+3i))<\pi$ (Ans)

5

(i)



(ii)

$$f(x) = \tan^{-1} x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$(1+x^2)f'(x) = 1$$

Differentiating w.r.t. x :

$$(1+x^2)f''(x) + 2xf'(x) = 0$$

Differentiating w.r.t x:

$$(1+x^2)f'''(x) + 2xf''(x) + 2xf''(x) + 2f'(x) = 0$$

$$(1+x^2)f'''(x) + 4xf''(x) + 2f'(x) = 0$$

When

$$x = 0$$

$$f'(0) = 1$$

$$f''(0) = 0$$

$$f'''(0) = -2f'(0) = -2$$

Hence

$$\tan^{-1}(x) = x - \frac{2}{3!}x^3 + ...$$

$$» \; x - \frac{1}{3}x^3$$

(iii)

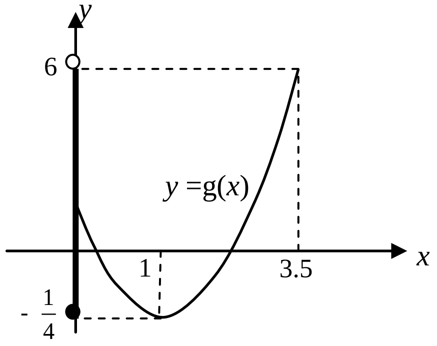
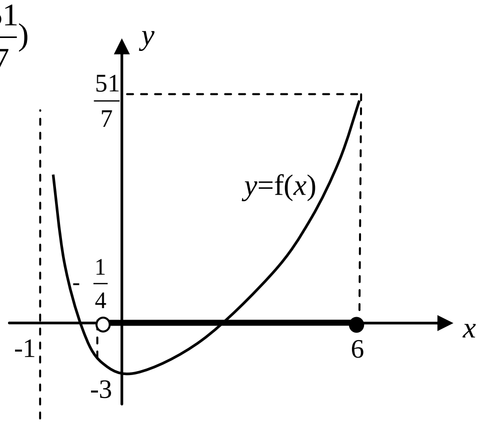
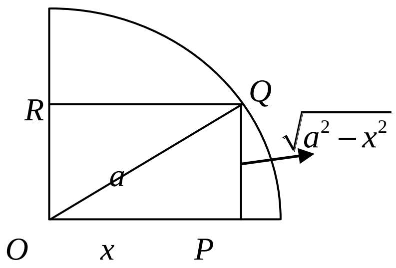
$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx \left(\frac{1}{\sqrt{3}}\right) - \frac{\left(\frac{1}{\sqrt{3}}\right)^3}{3} = \frac{8}{9\sqrt{3}}$$

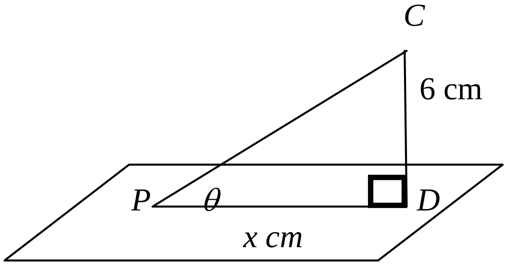
$$\frac{\pi}{6} \approx \frac{8}{9\sqrt{3}}$$

$$\pi \approx \frac{16}{3\sqrt{3}} = \frac{16\sqrt{3}}{9}$$

	(iv) $\tan^{-1}(\sqrt{3}) \approx (\sqrt{3}) - \frac{(\sqrt{3})^3}{3} = 0$ $\frac{\pi}{3} \approx 0$ $\pi \approx 0$ (v) The approximation in **(iii)** is better than that in **(iv)** because the value of x substituted in **(iii)** is closer to zero as compared to the value of x substituted in **(iv)**.
6	(i) $x = at^2$ $\quad y = at^3$ $\frac{dx}{dt} = 2at$ $\quad \frac{dy}{dt} = 3at^2$ $\quad \frac{dy}{dx} = \frac{3at^2}{2at} = \frac{3}{2}t$ When $x = \frac{25}{4}a = at^2$, $t = \pm\frac{5}{2}$ When $y = \frac{-125}{8}a = at^3$, $\quad t = -\frac{5}{2}$. $\quad \therefore t = \frac{-5}{2}$ for $\left(\frac{25}{4}a, \frac{-125}{8}a\right)$ Note that $t = \frac{5}{2}$ does not give the correct point. When $t = \frac{-5}{2}$, gradient of tangent $= \frac{-15}{4}$, Eqn of tangent : $y - \left(\frac{-125}{8}a\right) = \frac{-15}{4}\left(x - \frac{25}{4}a\right)$ $16y + 250a = -60x + 375a$ $60x + 16y = 125a$ ------(1) (ii) Subst $x = at^2$ and $y = at^3$ into (1): $60at^2 + 16at^3 = 125a$ $16t^3 + 60t^2 - 125 = 0$ $(4t-5)(2t+5)(2t+5) = 0$ $\therefore t = \frac{5}{4}$ or $\frac{-5}{2}$ Note that $t = \frac{-5}{2}$ is rejected. When $t = \frac{5}{4}$, $\quad x = a\left(\frac{5}{4}\right)^2 = \frac{25}{16}a$, $\quad y = a\left(\frac{5}{4}\right)^3 = \frac{125}{64}a$, Hence the coordinates of the point where the tangent meets the curve again is at $\left(\frac{25}{16}a, \frac{125}{64}a\right)$

	(iii) $\frac{dy}{dx}=\frac{3}{2}t$, gradient of normal $=\frac{-2}{3t}$ Eqn of normal : $y-0=\frac{-2}{3t}\left(x-\frac{21}{2}a\right)$ $y=\frac{-2}{3t}x+\frac{7a}{t}$ -------(2) Subst $x=at^2$ and $y=at^3$ into (2): $at^3=\frac{-2}{3t}\left(at^2\right)+\frac{7a}{t}$ $3t^4+2t^2-21=0$ $\left(3t^2-7\right)\left(t^2+3\right)=0$ $\therefore t^2=\frac{7}{3}$ or $t^2=-3$ (rejected) $\therefore t=\pm\sqrt{\frac{7}{3}}$ When $t=\sqrt{\frac{7}{3}}$, $x=\frac{7}{3}a$, $y=\left(\frac{7}{3}\right)^{\frac{3}{2}}a$, When $t=-\sqrt{\frac{7}{3}}$, $x=\frac{7}{3}a$, $y=-\left(\frac{7}{3}\right)^{\frac{3}{2}}a$, $\left(\frac{7}{3}a,\left(\frac{7}{3}\right)^{\frac{3}{2}}a\right)$ and $\left(\frac{7}{3}a,-\left(\frac{7}{3}\right)^{\frac{3}{2}}a\right)$
7	$\frac{dy}{dx}=\frac{(x+1)(4x+a)-(2x^2+ax-3)}{(x+1)^2}$ $\frac{dy}{dx}=0$, $(x+1)(4x+a)-(2x^2+ax-3)=0$ $4x^2+4x+ax+a-2x^2-ax+3=0$ $2x^2+4x+(a+3)=0$ For stationary points to exist: $4^2-4(2)(a+3)\geq 0$ $a<-1$ (since $a\neq -1$) For f^{-1} to exist, there should not be any stationary point in the given domain, hence $a>-1$. Therefore, the set of values: $\{a\in\mathbb{R} : a>-1\}$.
	(i) $f(x)=\frac{2x^2-3x-3}{x+1}$

	$D_g:(0,3.5) \rightarrow R_g:[-0.25,6) \rightarrow R_{fg}:[-3,\frac{51}{7})$  
	(ii) $\mathrm{h}(x)=\mathrm{f}(\mathrm{f}^{-1}\mathrm{h}(x))$ $=\dfrac{2e^{2x}-3e^{x}-3}{e^{x}+1}$ Therefore, $\mathrm{h}: x \mapsto \dfrac{2e^{2x}-3e^{x}-3}{e^{x}+1}, \qquad x \in \mathbb{R}, x>0$
8(a)	 $A=x\sqrt{a^2-x^2}$ $A^2=x^2(a^2-x^2)=a^2x^2-x^4$ $2A\dfrac{\mathrm{d}A}{\mathrm{d}x}=2a^2x-4x^3$ $A\dfrac{\mathrm{d}A}{\mathrm{d}x}=a^2x-2x^3$ $A\dfrac{\mathrm{d}^2A}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}A}{\mathrm{d}x}\right)^2=a^2-6x^2$ For max A, $\dfrac{\mathrm{d}A}{\mathrm{d}x}=0$

	$x(a^2-2x^2)=0$ Since $x\neq 0,\ x^2=\frac{a^2}{2}$ $x=\frac{a}{\sqrt{2}}\ (\because x>0)$ When $x=\frac{a}{\sqrt{2}}, \frac{\mathrm{d}A}{\mathrm{d}x}=0,$ $A\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=a^2-6\left(\frac{a^2}{2}\right)=-2a^2<0$ Hence $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}<0$ $\therefore x=\frac{a}{\sqrt{2}}$ gives a max A Perimeter of $OPQR$ = $2x + 2\sqrt{a^2-x^2}$ When $x=\frac{a}{\sqrt{2}}$, Perimeter of $OPQR$ $=2\left(\frac{a}{\sqrt{2}}\right)+2\sqrt{a^2-\frac{a^2}{2}}$ $=\sqrt{2}a+\sqrt{2}a=2\sqrt{2}a$ $=4\left(\frac{a}{\sqrt{2}}\right)=4x=4OP$
(b)	 Let PD = x cm and $\angle CPD=\theta$ rads at any time t. Given: $\frac{\mathrm{d}x}{\mathrm{d}t}=2\ \text{cms}^{-1}$, To find $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when x = $6\sqrt{3}$ cm $\tan\theta=\frac{6}{x}$ $x=6\cot\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=-6\operatorname{cosec}^2\theta$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{2}{-6\operatorname{cosec}^2\theta} = -\frac{1}{3}\sin^2\theta$$

When $x = 6\sqrt{3}, \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$$\therefore \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{3}\sin^2\frac{\pi}{6} = -\frac{1}{12} \text{ rad/s} = -0.0833 \text{ rad/s (3 s.f.)}$$

9

(i)

$$\overrightarrow{OA} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix};\ \overrightarrow{OB} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix};\ \overrightarrow{OP} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix};\ \overrightarrow{OR} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix};$$

$$\overrightarrow{PR} = \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix}$$

Hence the vector equation of line PR is $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

(ii)

$$\overrightarrow{OQ} = \frac{\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}}{4} = \frac{1}{4}\begin{pmatrix} 18 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 1 \\ 0 \end{pmatrix}$$

(iii)

$$l : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix},\ \mu \in \mathbb{R}$$

Equate: $\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$

$\lambda = -2 \quad \mu = 3$

$$\overrightarrow{OX} = \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix} + 3\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ 11 \end{pmatrix}$$

(iv)

$$\overrightarrow{QX} = \begin{pmatrix} -9/2 \\ -9 \\ 11 \end{pmatrix}$$

$$|\overrightarrow{QX} \times \mathbf{m}| = \frac{\left|\begin{pmatrix} -9/2 \\ -9 \\ 11 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}\right|}{\sqrt{9+4}} = \frac{\left|\begin{pmatrix} -22 \\ 33 \\ -36 \end{pmatrix}\right|}{\sqrt{13}} = 14.855$$

$|\overrightarrow{QX} \times \mathbf{m}|$ is the perpendicular/shortest distance from point X to line AB.

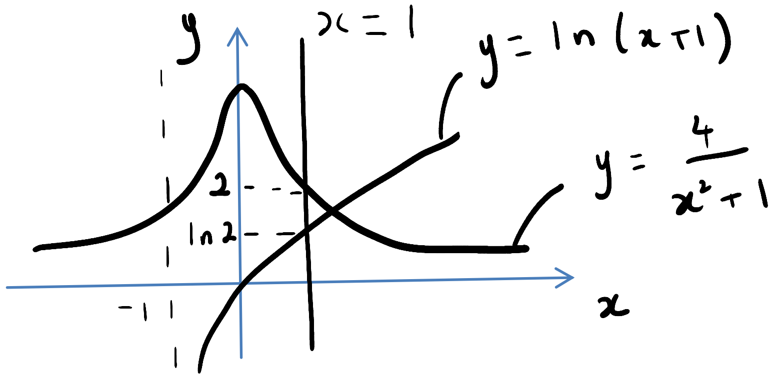
Area of triangle $AXB = \frac{1}{2}\left|\overrightarrow{AB}\right| 14.855$

$$= \frac{1}{2}\sqrt{36+16}(14.855) = 53.6 \text{ (3 s.f.)}$$

10	(i) $\Pi_1 : \boldsymbol{r}.\begin{pmatrix}-1\\2\\0\end{pmatrix} = -5$; Let the centre of centre be M. $\overrightarrow{OA} = \begin{pmatrix}-4\\3\\1\end{pmatrix} + \gamma\begin{pmatrix}-1\\2\\0\end{pmatrix}$ for a fixed γ. $\left[\begin{pmatrix}-4\\3\\1\end{pmatrix} + \gamma\begin{pmatrix}-1\\2\\0\end{pmatrix}\right].\begin{pmatrix}-1\\2\\0\end{pmatrix} = -5$ 4+ g+ 6+ 2g= - 5 Solving for γ gives $\gamma = -3$. $\overrightarrow{OA} = \begin{pmatrix}-1\\-3\\1\end{pmatrix}$ Radius= MA $= \left\|\begin{pmatrix}-1\\-3\\1\end{pmatrix} - \begin{pmatrix}-4\\3\\1\end{pmatrix}\right\| = \left\|\begin{pmatrix}3\\-6\\0\end{pmatrix}\right\| = \sqrt{45}$ units
	(ii) Direction vector of line l_1 is $=\begin{pmatrix}-4\\3\\1\end{pmatrix} - \begin{pmatrix}0\\0\\a\end{pmatrix} = \begin{pmatrix}-4\\3\\1-a\end{pmatrix}$ $\begin{pmatrix}-4\\3\\1-a\end{pmatrix}.\begin{pmatrix}-1\\2\\0\end{pmatrix} = \left\|\begin{pmatrix}-4\\3\\1-a\end{pmatrix}\right\|\left\|\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\|\cos(90° - 30°)$ $4 + 6 = \sqrt{16 + 9 + (1-a)^2}\sqrt{5}(0.5)$ $25 + (1-a)^2 = 80$ $(1-a) = \pm\sqrt{55}$ $a = 1 \mp \sqrt{55}$ As $a > 0$, $a = 1 + \sqrt{55}$
	(iii) A vector perpendicular to Π_2: $\begin{pmatrix}1\\2\\2\end{pmatrix} \times \begin{pmatrix}-2\\2\\-1\end{pmatrix} = -3\begin{pmatrix}2\\1\\-2\end{pmatrix}$ Π_2: $\boldsymbol{r}.\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}4\\3\\1\end{pmatrix}.\begin{pmatrix}2\\1\\-2\end{pmatrix} = 9$ To show that Π_1 and Π_2 are perpendicular we must show that the normal between Π_1 and Π_2 are perpendicular. Hence we need to show $\begin{pmatrix}2\\1\\-2\end{pmatrix}.\begin{pmatrix}-1\\2\\0\end{pmatrix}=0$ $\begin{pmatrix}2\\1\\-2\end{pmatrix}.\begin{pmatrix}-1\\2\\0\end{pmatrix} = -2 + 2 + 0 = 0$ Hence Π_1 and Π_2 are perpendicular.

11

(a)



Area of Q = $\int_0^1 \left(\frac{4}{x^2+1} - \ln(x+1)\right)dx$

$= 2.76$ units^2

(aii)

From the diagram in (a)

When $x = 1,\ y = \ln 2$ (for $y = \ln(x+1)$)

When $x = 1,\ y = 2$ (for $y = \frac{4}{x^2+1}$)

$$Volume = \pi\int_0^{\ln 2}(e^y-1)^2\,dy + \pi\int_2^4\left(\sqrt{\frac{4}{y}-1}\right)^2 dy + \pi(1)^2(2-\ln 2)$$

$$= \pi\int_0^{\ln 2}(e^{2y}-2e^y+1)\,dy + \pi\int_2^4\left(\frac{4}{y}-1\right)dy + \pi(2-\ln 2)$$

$$= \pi\left[\frac{e^{2y}}{2}-2e^y+y\right]_0^{\ln 2} + \pi\left[4\ln y - y\right]_2^4 + \pi(2-\ln 2)$$

$$= \pi\left[2-2(2)+\ln 2-\left(\frac{1}{2}-2\right)\right] + \pi\left[4\ln 2-2\right] + \pi(2-\ln 2)$$

$$= \pi\left(-\frac{1}{2}+4\ln 2\right)\text{units}^3$$

(b)

(i) Since the rectangles are an overestimation of the area under the curve, so

$A <$ Total Area of rectangles

$$< \frac{3}{n+1}\left(\frac{1}{0+1} + \frac{1}{\left(\frac{3}{n+1}\right)^2+1} + \frac{1}{\left(\frac{6}{n+1}\right)^2+1} + \dots + \frac{1}{\left(\frac{3(n-1)}{n+1}\right)^2+1} + \frac{1}{\left(\frac{3n}{n+1}\right)^2+1}\right)$$

	$< \dfrac{3}{n+1}\sum_{r=0}^{n}\dfrac{1}{\left(\dfrac{3r}{n+1}\right)^2+1}$ $< \dfrac{3}{n+1}\sum_{r=0}^{n}\dfrac{(n+1)^2}{9r^2+(n+1)^2}$ $< \sum_{r=0}^{n}\dfrac{3(n+1)}{9r^2+(n+1)^2}$ (ii) As $n\to\infty$, the area of the rectangles $= A$ $A=\int_0^3\dfrac{1}{1+x^2}\,dx=\left[\tan^{-1}(x)\right]_0^3=\tan^{-1}(3)$
12(a)	$u=\sqrt{x+1} \;\Rightarrow\; u^2=x+1 \;\Rightarrow\; x=u^2-1$ $\Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}u}=2u$ $\int \dfrac{2x}{\sqrt{x+1}}\,\mathrm{d}x$ $=\int \dfrac{2(u^2-1)}{u}\cdot 2u\,\mathrm{d}u$ $=4\int (u^2-1)\,\mathrm{d}u$ $=4\left[\dfrac{u^3}{3}-u\right]+C$ $=\dfrac{4}{3}(x+1)\sqrt{x+1}-4\sqrt{x+1}+C$
(b)	$\int \dfrac{\mathrm{d}x}{(1+x^2)\tan^{-1}x} \quad , \quad x>0$ $=\int \dfrac{\frac{1}{1+x^2}}{\tan^{-1}x}\,\mathrm{d}x$ $=\ln(\tan^{-1}x)+C$
(c)	(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}e^{\sqrt{1-x^2}}=e^{\sqrt{1-x^2}}\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-x^2}\right)$ $=e^{\sqrt{1-x^2}}\cdot\dfrac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)$ $=-\dfrac{xe^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$
	(ii) $\int_0^1 x\,e^{\sqrt{1-x^2}}\,\mathrm{d}x$ $=\int_0^1 \dfrac{x\,e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2}\,\mathrm{d}x$

	Let $u=\sqrt{1-x^2}$ $\quad \frac{dv}{dx}=\frac{xe^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}$ $\frac{du}{dx}=-\frac{x}{\sqrt{1-x^2}}$ $\quad v=-e^{\sqrt{1-x^2}}$ $\int_0^1 x\,e^{\sqrt{1-x^2}}\,\mathrm{d}x=\left[-\sqrt{1-x^2}\,e^{\sqrt{1-x^2}}\right]_0^1-\int_0^1\frac{x\,e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}\,\mathrm{d}x$ $=\left[-0-(-e)\right]-\left[-e^{\sqrt{1-x^2}}\right]_0^1=e+e^0-e$ $=1$