Full Name	Class Index No	Class
Answers		

Anglo-Chinese School (Barker Road)

PRELIMINARY EXAMINATION 2024
SECONDARY FOUR EXPRESS / FIVE NORMAL (ACADEMIC)

ADDITIONAL MATHEMATICS
4049
PAPER 2

2 HOURS 15 MINUTES

Candidates answer on the Question Paper.

READ THESE INSTRUCTIONS FIRST

Write your index number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}bc\sin A$$

1 A particle moves along the curve $y = \frac{3(x+6)}{x+4}$, where $x \neq -4$, in such a way that the y-coordinate of the particle is increasing at a constant rate of $\frac{4}{27}$ units per second. Find the x-coordinates of the particle at the instant that the x-coordinate of the particle is decreasing at a rate of 2 units per second. [5]

Given $\frac{dy}{dt} = \frac{4}{27}$ units/s , $\frac{dx}{dt} = -2$ units/s

$$y = \frac{3x+18}{x+4}$$

$$\frac{dy}{dx} = \frac{(x+4)(3) - (3x+18)\cdot 1}{(x+4)^2}$$

$$= \frac{3x+12-3x-18}{(x+4)^2}$$

$$= \frac{-6}{(x+4)^2}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$\frac{4}{27} = -\frac{6}{(x+4)^2} \times -2$$

$$\frac{4}{27} = \frac{12}{(x+4)^2}$$

$$(x+4)^2 = 81$$

$$x+4 = \pm 9$$

$$x = 5 \text{ or } -13$$

2 The number, v, of a certain virus present in a sample collected by a vaccine laboratory is given by $v = me^{2t} + n$, where m and n are constants and t is measured in days. Initially, the number of virus present was 2000. It increased to 5000 after 1 day.

(a) Find the value of m and of n. [4]

$v = me^{2t} + n$

$t = 0,\ v = 2000$

$2000 = me^{0} + n$

$m + n = 2000 \cdots (1)$

$t = 1,\ v = 5000$

$5000 = me^{2} + n \cdots (2)$

$(2) - (1),\quad 3000 = me^{2} - m$

$3000 = m(e^{2} - 1)$

$m = \dfrac{3000}{e^{2} - 1}$

$= 469.5529 \approx 470 \text{ (to 3sf)}$

$n = 2000 - 469.5529$

$= 1530.4470$

≈ 1530 (to 3sf)

(b) Find the number of days in which the number of virus present first reach 1 million. [2]

$1 \times 10^{6} = 469.5529e^{2t} + 1530.4470$

$e^{2t} = \dfrac{1 \times 10^{6} - 1530.4470}{469.5529}$

$e^{2t} = 2126.426$

$2t \ln e = \ln 2126.426$

$t = \dfrac{\ln 2126.426}{2}$

$= 3.83109$

≈ 3.83 days (to 3sf)

$\rightarrow$ show $D \geq 0$

$\rightarrow D \geq 0$

3 **(a)** Show that the roots of the equation $ax^2 + (3a+b)x + 3b = 0$ are real for all real values of a and b. [3]

$ax^2 + (3a+b)x + 3b = 0$

Discriminant $= (3a+b)^2 - 4a(3b)$

$= 9a^2 + 6ab + b^2 - 12ab$

$= 9a^2 - 6ab + b^2$

$= (3a-b)^2$

real & distinct roots $D>0$

real roots $D \geq 0$

For all real values of a and b, $(3a-b)^2 \geq 0$.

Since discriminant ≥ 0, hence roots of $ax^2 + (3a+b)x + 3b = 0$ are real.

(b) Find the range of values of m for which the line $y = mx - 3$ will never cut the curve $y^2 = 4x - 6y - 34$. [4]

$\Rightarrow D<0$

$(mx-3)^2 = 4x - 6(mx-3) - 34$

$m^2x^2 - 6mx + 9 = 4x - 6mx + 18 - 34$

$m^2x^2 - 4x + 25 = 0$

$b^2 - 4ac < 0$

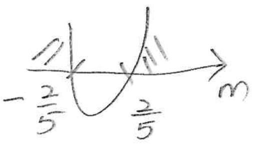
$(-4)^2 - 4m^2(25) < 0$

$16 - 100m^2 < 0$

$100m^2 - 16 > 0$

$4(25m^2 - 4) > 0$

$(5m-2)(5m+2) > 0$

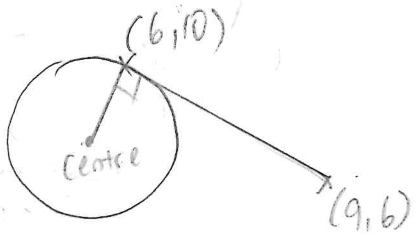


$m < -\frac{2}{5}$ or $m > \frac{2}{5}$

4 A tangent to a circle at the point (6, 10) passes through the point (9, 6). The centre of the circle lies on the line $3y = 4x + 13$.
Showing all your working, find the equation of the circle. [7]

gradient of tangent $= \frac{6-10}{9-6}$

$= -\frac{4}{3}$



gradient of radius $= \frac{3}{4}$

$(6, 10)$, $m = \frac{3}{4}$

$y = \frac{3}{4}x + c$

$10 = \frac{3}{4}(6) + c$

$c = \frac{26}{5}$

Equation of radius: $y = \frac{3}{4}x + \frac{11}{2}$ $\quad\ldots(1)$

$3y = 4x + 13$ $\quad\ldots(2)$

sub (1) into (2)

$3\left(\frac{3}{4}x + \frac{11}{2}\right) = 4x + 13$

$\frac{9}{4}x + \frac{33}{2} = 4x + 13$

$-\frac{7}{4}x = -\frac{7}{2}$

$\therefore x = 2$

$y = \frac{3}{4}(2) + \frac{11}{2}$

$= 7$

centre $(2, 7)$

radius $= \sqrt{(6-2)^2 + (10-7)^2}$

$= 5$ units

Equation of circle is $(x-2)^2 + (y-7)^2 = 25$ //

$\overset{\times}{\sin 2A} = 2\sin A \cos A$

5 **Do not use a calculator in this question.**

(a) Express sin 22.5° in the form of $\dfrac{\sqrt{a-\sqrt{a}}}{a}$, where a is an integer. [3]

$\cos 2A = 1 - 2\sin^2 A$

$\cos 45^\circ = 1 - 2\sin^2 22.5^\circ$

$\dfrac{\sqrt{2}}{2} = 1 - 2\sin^2 22.5^\circ$

$\dfrac{\sqrt{2}}{2} - 1 = -2\sin^2 22.5^\circ$

$\dfrac{\sqrt{2}-2}{2} = -2\sin^2 22.5^\circ$

$\sin^2 22.5^\circ = \dfrac{2-\sqrt{2}}{4}$

$\sin 22.5^\circ = \pm\dfrac{\sqrt{2-\sqrt{2}}}{\sqrt{4}}$

Since $\sin 22.5^\circ > 0$,

$\therefore \sin 22.5 = \dfrac{\sqrt{2-\sqrt{2}}}{2}$ (shown)

$\left(-\dfrac{\sqrt{2-\sqrt{2}}}{2}\right.$ is rejected.)

(b) Show that $\tan 15^\circ = 2 - \sqrt{3}$. [5]

$\tan 15^\circ = \tan(60^\circ - 45^\circ)$

$= \dfrac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$

$= \dfrac{\sqrt{3}-1}{1+\sqrt{3}} \times \dfrac{1-\sqrt{3}}{1-\sqrt{3}}$

$= \dfrac{\sqrt{3} - 3 - 1 + \sqrt{3}}{1^2 - \sqrt{3}^2}$

$= \dfrac{-4 + 2\sqrt{3}}{-2}$

$= 2 - \sqrt{3}$ (shown)

not! $\tan 15^\circ \neq \tan 65^\circ - \tan 15^\circ$ wrong!

$\tan(45^\circ - 30^\circ)$

$\dfrac{1}{\sqrt{3}}$

$\tan 15^\circ = \tan 45^\circ - \tan 30^\circ$ ×

→ to help, not give in u! → final answer in terms of x!

6 **(a)** **(i)** Using the substitution $u = x^3$ or otherwise, express $x^6 - 1$ as the product of two factors. [1]

$x^6 - 1 = (x^3)^2 - 1$ or $u^2 - 1$

$= (x^3+1)(x^3-1)$ // $= (u+1)(u-1)$

$= (x^3+1)(x^3-1)$

Do not leave answer in terms of u.

(ii) Hence express $x^6 - 1$ as the product of four factors with integer coefficients. [1]

$x^3+1 = (x+1)(x^2-x+1)$ ← a^3+b^3

$x^3-1 = (x-1)(x^2+x+1)$ ← a^3-b^3

$\therefore (x^3+1)(x^3-1) = (x+1)(x^2-x+1)(x-1)(x^2+x+1)$ //

6 **(b)** **(i)** Find the remainder when $f(x) = 3x^3 - 5x^2 + 7x - 4$ is divided by $x - 1$. [1]

$f(1) = 3(1)^3 - 5(1)^2 + 7(1) - 4$
$= 1$
Remainder $= 1$

(ii) Hence show that $h = -1$ for which $g(x) = f(x) + h$ is divisible by $x - 1$. [2]

$g(x)$ is divisible by $x-1 \Rightarrow g(1) = 0$
$g(1) = 0$
$f(1) + h = 0$
$1 + h = 0$
from (b)(i) $f(1) = 1$
$h = -1$ (shown)

(iii) Explain why the equation $g(x) = 0$ has only one real root. [4]

$g(x) = f(x) + h = 3x^3 - 5x^2 + 7x - 4 + (-1)$
$= 3x^3 - 5x^2 + 7x - 5$
$g(x)$ is divisible by $x-1$, $\therefore x-1$ is a factor of $g(x)$

$$\begin{array}{r} 3x^2 - 2x + 5 \\ x-1 \overline{\big) 3x^3 - 5x^2 + 7x - 5} \\ -(3x^3 - 3x^2) \\ \hline -2x^2 + 7x \\ -(-2x^2 + 2x) \\ \hline 5x - 5 \\ 5x - 5 \\ \hline 0 \end{array}$$

$\therefore (x-1)(3x^2 - 2x + 5) = 0$
$x - 1 = 0$, $\therefore x = 1$
or $3x^2 - 2x + 5 = 0$
discriminant $= (-2)^2 - 4(3)(5)$
$= -56 < 0$
hence $3x^2 - 2x + 5 = 0$ has no real roots.
$\therefore$ when $g(x) = 0$, $x = 1$ is the only real root

$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$
$= \dfrac{2 \pm \sqrt{-56}}{6}$
since x is undefined, $3x^2 - 2x + 5 = 0$ has no real root

$x-1$ is not a root
$x-1$ is a factor.
root $\Rightarrow$ solution $\Rightarrow x = \underline{1}$

7 **(a)** Two variables x and y are related by the equation $xy^2 = ax + by$. Explain how a straight line graph can be drawn to represent the given equation. [2]

$xy^2 = ax + by$

$y^2 = a\left(\frac{x}{x}\right) + b\left(\frac{y}{x}\right)$

$y^2 = b\left(\frac{y}{x}\right) + a$ ← no x or y multiply to a!

Plot y^2 against $\frac{y}{x}$ to draw the straight line graph //

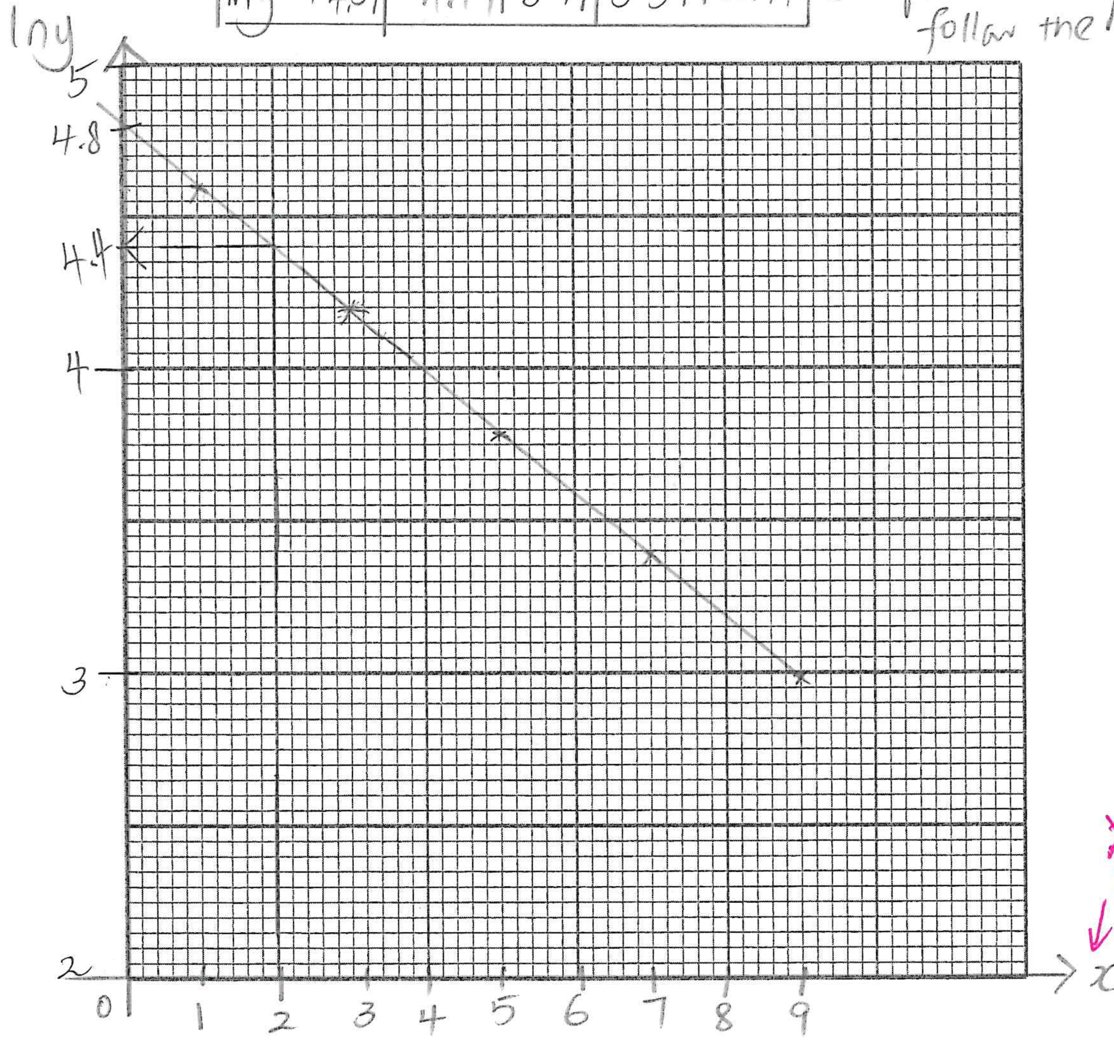
(b) The table shows experimental values of two variables x and y. It is known that x and y are related by the equation $y = pe^{-qx}$ where p and q are constants.

x	1	3	5	7	9
y	98.2	65.9	44.1	29.6	19.8

(i) On the grid below plot $\ln y$ against x and draw a straight line graph. [2]

x	1	3	5	7	9
$\ln y$	4.59	4.19	3.79	3.39	2.99

← give to 3 sf follow the table.



X-axis must start from 0 to read Y-intercept

7 **(b)** Use your graph to estimate

(ii) the value of p and of q, [3]

No sub. X & Y values into equation

$$y = pe^{-qx}$$

$$\ln y = \ln(pe^{-qx})$$

$$\ln y = \ln p + (-qx)\ln e$$

$$\ln y = -qx + \ln p$$

$-q$ = gradient.

using $(2, 4.4)$ and $(8.4, 3.1)$

$$-q = \frac{4.4 - 3.1}{2 - 8.4}$$

$$= -0.203125$$

$$\therefore q = 0.203125$$

$$\text{or } q = \frac{13}{64}$$

$\ln p$ is the vertical intercept

$$\therefore \ln p = 4.8$$

$$p = e^{4.8}$$

$$\approx 121.5104$$

$$\approx 122$$

$p = 4.8$ X

$\ln p = 4.8$

$p = e^{4.8}$

(iii) the value of y when $x = 2$. [1]

No. sub. of X into eq$^{\underline{n}}$.

Read from graph

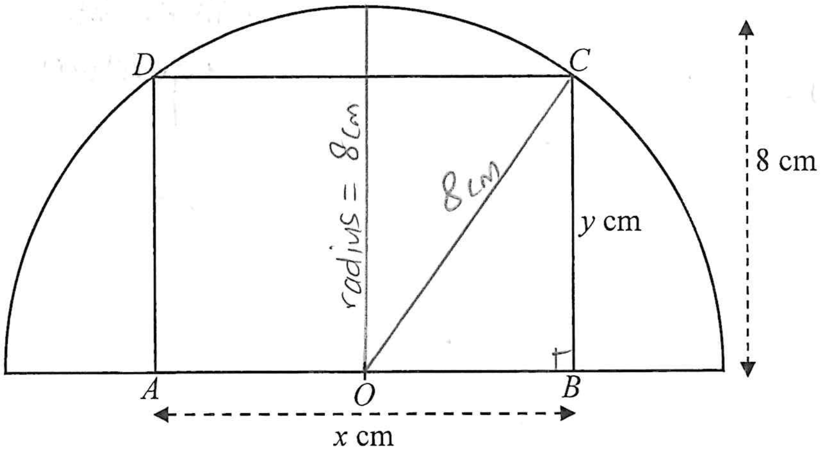
$$\ln y = 4.4$$

$$y = e^{4.4}$$

$$\approx 81.4508$$

$$\approx 81.5$$

8 $ABCD$ is a rectangle which fits inside a semicircle of radius 8 cm and centre O. It is given that $AB = x$ cm and $BC = y$ cm.



(a) Show that that A cm^2, the area of the rectangle, is given by $A = \frac{x}{2}\sqrt{256 - x^2}$. [2]

$OB = \frac{x}{2}$

In $\triangle OBC$,

$y^2 + \left(\frac{x}{2}\right)^2 = 8^2$

$y^2 = 64 - \frac{x^2}{4}$

$y^2 = \frac{1}{4}(256 - x^2)$

$y = \frac{1}{2}\sqrt{256 - x^2}$, $y > 0$, $\therefore -\frac{1}{2}\sqrt{256^2 - x^2}$ is rejected

Area of rectangle

$A = x \times y$

$= x \times \frac{1}{2}\sqrt{256 - x^2}$

$= \frac{x}{2}\sqrt{256 - x^2}$ (shown)

$\frac{dA}{dx} = 0$

8 (b) Given that x can vary, $\boxed{\text{find the value of } x}$ which gives a stationary value of A. [4]

$$A = \frac{x}{2}(256 - x^2)^{\frac{1}{2}}$$

$$\frac{dA}{dx} = \frac{1}{2}(256 - x^2)^{\frac{1}{2}} + \frac{x}{2}\left[\frac{1}{2}(256 - x^2)^{-\frac{1}{2}}(-2x)\right]$$

$$= \frac{1}{2}\sqrt{(256 - x^2)} - \frac{x^2}{2\sqrt{256 - x^2}}$$

$$\frac{dA}{dx} = 0$$

$$\frac{1}{\cancel{2}}\sqrt{256 - x^2} = \frac{x^2}{\cancel{2}\sqrt{256 - x^2}}$$

$$\sqrt{256 - x^2} \times \sqrt{256 - x^2} = x^2$$

$$256 - x^2 = x^2$$

$$256 = 2x^2$$

$$x = \sqrt{\frac{256}{2}}, \quad x > 0$$

$$= 11.3137$$

$$\approx 11.3 \text{ cm} \quad (\text{to } 3 \text{ sf})$$

(c) By $\boxed{\text{considering the sign of } \frac{dA}{dx}}$, determine whether the stationary value of A is maximum or minimum. [2]

→ use 1st derivative test.

do not use 11.31

x	11.2	11.3137	11.4
value of $\frac{dy}{dx}$	0.224	0	−0.175
sketch of tangent	/	—	\

$$\frac{dA}{dx} = \frac{1}{2}\sqrt{256 - x^2} - \frac{x^2}{2\sqrt{256 - x^2}}$$

$$= \frac{256 - x^2 - x^2}{2\sqrt{256 - x^2}}$$

$$= \frac{256 - 2x^2}{2\sqrt{256 - x^2}}$$

$$= \frac{2(128 - x^2)}{2\sqrt{256 - x^2}}$$

$$= \frac{128 - x^2}{\sqrt{256 - x^2}}$$

The stationary value of A is maximum.

$v=0,\ t=0$ (starting already at rest)

9 A particle starts from rest from a point O and moves in a straight line such that its velocity v m/s, is given by $v = 24t - 6t^2$, where t is the time in seconds after the start of its motion.

(a) Find the value of t at which the particle is instantaneously at rest. [2]

$\rightarrow v=0$

$v = 24t - 6t^2$

$V = 0$

$6t(4-t) = 0$

$t = 0$ or $t = 4$
rejected

$\therefore t = 4$ when it is instantaneously at rest.

(b) When will the particle return to its starting point? [3]

$\leftarrow s=0$

$S = \int v \, dt$

$= \int 24t - 6t^2 \, dt$

$S = 12t^2 - 2t^3 + C$

when $t=0$, $S=0$, $C=0$

$\therefore S = 12t^2 - 2t^3$

$12t^2 - 2t^3 = 0$

$2t^2(6-t) = 0$

$t^2 = 0$ or $6 - t = 0$

$\therefore$ After 6 second, the particle will return to the starting point.

9 **(c)** Determine if the particle is accelerating after 2 seconds. Explain your answer with clear workings. [3]

$a = \frac{dv}{dt}$

$= 24 - 12t$

After 2seconds, $t > 2$

$-12t < -24$

$24 - 12t < 24 - 24$

$24 - 12t < 0$

Since acceleration is negative, hence the particle is not accelerating after 2 seconds

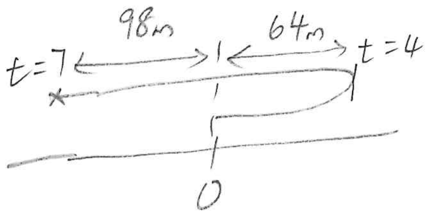
(d) Calculate the total distance travelled during the first 7 seconds. [4]

when $t = 0$, $s = 0$

when $t = 4$, $s = 12(4)^2 - 2(4)^3$

$= 64\text{m}$

when $t = 7$, $s = 12(7)^2 - 2(7)^3$



$= -98\text{m}$

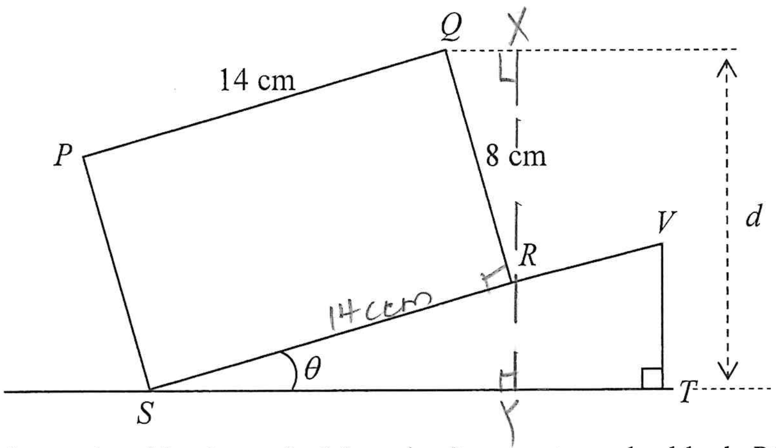
on left side of point O.

Total distance travelled

$= 64 \times 2 + 98$

$= 226\text{m}$

10



The diagram shows the side view of a 14 cm by 8 cm rectangular block *PQRS*, placed on a ramp, *VS*, tilted at an acute angle of $\theta°$.
The ramp is placed on a horizontal surface *ST* and *d* is the perpendicular distance from *Q* to *ST*. $\angle VTS = 90°$.

show the trigo ratios

(a) Show that $d = 8\cos\theta + 14\sin\theta$. [2]

Let $\angle SYR = \angle QXR = 90°$

In $\triangle SYR$, $\sin\theta = \frac{RY}{14}$

$RY = 14\sin\theta$

$\angle SRY = 180° - \theta - 90° = 90° - \theta$

$\angle QRY = 180° - 90° - (90° - \theta)$

$= \theta$

In $\triangle QXR$, $\cos\theta = \frac{XR}{8}$

$XR = 8\cos\theta$

$\therefore d = XR + RY$

$= 8\cos\theta + 14\sin\theta$ //

(b) Express *d* in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [4]

Let $8\cos\theta + 14\sin\theta = R\cos(\theta - \alpha)$

$R = \sqrt{8^2 + 14^2}$

$= \sqrt{260}$

$= 2\sqrt{65}$

$\alpha = \tan^{-1}\left(\frac{14}{8}\right)$

$= 60.2551°$

$\therefore d = 2\sqrt{65}\cos(\theta - 60.3°)$ //

10 **(c)** Find the smallest value of θ such that $d = 10\sqrt{2}$. [4]

$2\sqrt{65}\cos(\theta - 60.2551^\circ) = 10\sqrt{2}$

$\cos(\theta - 60.2551^\circ) = 0.877058$

basic $\angle = 28.7105^\circ$

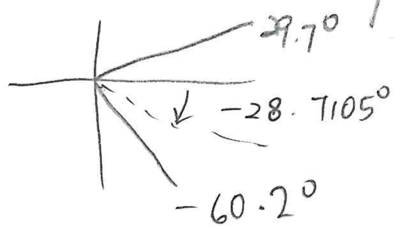
$0 < \theta < 90^\circ$

$-60.2551^\circ < \theta - 60.2551^\circ < 29.7$

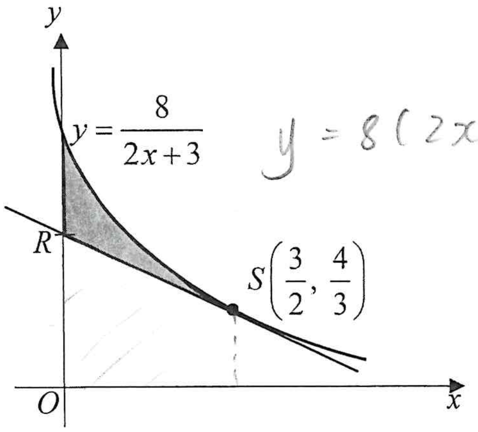
$\therefore \theta - 60.2551^\circ = -28.7105^\circ, 28.7105^\circ$

$\theta = 31.5446^\circ$ or 88.9656°

smallest value of $\theta = 31.5^\circ$ (to 1dp)



11 The diagram shows part of the curve $y = \dfrac{8}{2x+3}$. The tangent to the curve at the point $S\left(\dfrac{3}{2}, \dfrac{4}{3}\right)$ intersects the y-axis at R.



(a) Find the y-coordinate of R. [4]

$\dfrac{dy}{dx} = 8\left[-1(2x+3)^{-2} \cdot 2\right]$

$= \dfrac{-16}{(2x+3)^2}$

at $x = \frac{3}{2}$, gradient of RS $= \dfrac{-16}{(2\times\frac{3}{2}+3)^2}$

$= \dfrac{-16}{36} = -\dfrac{4}{9}$

$y = -\dfrac{4}{9}x + c$

$\dfrac{4}{3} = -\dfrac{4}{9}\left(\dfrac{3}{2}\right) + c$

$c = 2$

y-coordinate of $R = 2$ //

11 **(b)** Find the exact area of the shaded region. Express your answer in the form of $\left(\ln a - \frac{b}{c}\right)$ units^2, where a, b and c are integers. [6]

$$8\int_0^{1.5} \frac{1}{2x+3}\,dx = 8\left[\frac{\ln(2x+3)}{2}\right]_0^{1.5}$$
$$= 4\left[\ln(2x+3)\right]_0^{1.5}$$
$$= 4\left[\ln(2\times1.5+3) - \ln(0+3)\right]$$
$$= 4\left[\ln 6 - \ln 3\right]$$

$\log a - \log b = \log\left(\frac{a}{b}\right)$

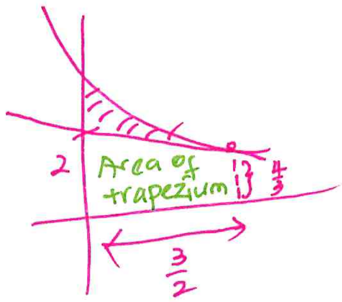
$$= 4\ln\left(\frac{6}{3}\right)$$
$$= 4\ln 2$$

$a\log b = \log b^a$

$$= \ln 2^4 = \ln 16$$

$\therefore$ Area of shaded region

$$= \ln 16 - \frac{1}{2}\left(2 + \frac{4}{3}\right)\times\frac{3}{2}$$
$$= \ln 16 - \frac{5}{2} \text{ units}^2$$



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