EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2023

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME			
CIVICS GROUP		INDEX NO.	

FURTHER MATHEMATICS

9649/02

Paper 2 [100 marks]

14 September 2023

3 hours

Additional Materials: Answer Booklet

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

Section A: Pure Mathematics [50 Marks]

1 Using mathematical induction, prove that

$$\sum_{n=1}^{N} \frac{n+3}{(n+1)(n+2)2^n} = \frac{1}{2} - \frac{1}{(N+2)2^N}$$

for all positive integers N. [5]

Deduce the value of $\sum_{n=1}^{\infty} \frac{n+3}{(n+1)(n+2)2^n}$. [1]

2 **(i)** Let $V = \mathbb{R}^2$ and $\mathbf{u}, \mathbf{v} \in V$ such that $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Define addition and scalar multiplication as follow:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 v_1 \\ u_2 v_2 \end{pmatrix} \text{ and } k\mathbf{u} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$$

where k is a real number.

Explain why V is not a vector space with the stated operations. [2]

(ii) Let M be the vector space of all real $n \times n$ matrices with the standard addition and scalar multiplication for matrices.

Determine if the following are subspaces of M.

(a) All $n \times n$ matrices $\mathbf{A}$ such that $\mathbf{AB} = \mathbf{BA}$ for a fixed $n \times n$ matrix $\mathbf{B}$.

(b) All $n \times n$ matrices $\mathbf{A}$ such that $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$. [5]

3 It is given that $I = \int_{-2}^{2} 3^x \, \mathrm{d}x$.

(i) Use the trapezium rule with five ordinates to approximate I. [2]

(ii) Explain whether the approximation to I found in part **(i)** is an under-estimate or over-estimate to I. [2]

(iii) Use Simpson's rule with five ordinates to approximate I. [2]

(iv) Find the exact value of I. [2]

(v) Explain clearly which of the two rules would give a more accurate approximation to I when used with the same number of ordinates. [2]

(vi) Calculate the absolute percentage error in the approximation obtained in part **(iii)**. [1]

4 The variables x, y and z are related by the simultaneous differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - 2z = 8, \qquad (1)$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} - y + 5z = 16x. \qquad (2)$$

When $x = 0$, $y = z = 0$.

(a) By first differentiating (1), show that the simultaneous differential equations can be reduced to the second-order differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9\frac{\mathrm{d}y}{\mathrm{d}x} + 18y = 32x + 40. \qquad [2]$$

(b) Hence solve the differential equations to find y and z in terms of x. [10]

5 Consider the equation $2v^4 = 1 + \sqrt{3}\mathrm{i}$.

(a) Find the roots of the equation in the form $re^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

Let w denote the root with the smallest positive argument p.

(b) Show that

$$w^n + \frac{1}{w^n} = 2\cos np$$

for any positive integer n, and hence show that

$$\cos^4 p = \frac{7 + 4\sqrt{3}}{16}. \qquad [4]$$

On an Argand diagram, the point representing the complex number w is denoted by A.

(c) On the same Argand diagram, sketch the loci of points representing the complex number z given by

$$|z| = |w| \quad \text{and} \quad \arg(w - z) = \frac{\pi}{3},$$

indicating clearly where A is. [3]

The point of intersection of the two loci is denoted by P.

(d) Find the complex number represented by P and determine if it is a root of the equation in part **(a)**. [4]

Section B: Probability and Statistics [50 marks]

6 A news agency wishes to predict the results of an election. To do so, they interviewed a random sample of voters one day before the election. 700 voters out of the sample of 900 voters said they would vote for Candidate *T*.

(a) Find a 90% confidence interval for the proportion of the population who would vote for Candidate *T*. [3]

(b) In order to obtain a more accurate estimate, the agency proposed to take a larger random sample. Assuming that the sample proportion remains unchanged, estimate the sample size that will be needed for the resulting 90% confidence interval to have a width of 0.04. [3]

7 A student carried out an experiment by tossing four similar coins on a table and recording how many coins landed on heads. He repeated the experiment 200 times and the results he obtained were as follows:

Number of heads	0	1	2	3	4
Frequency	5	35	64	66	30

Let p be the probability that a coin will land on head. It is thought that a Binomial distribution with $p = 0.6$ would be a suitable model for the above data. Carry out a χ^2-test of goodness of fit at the 5% level of significance. [4]

This experiment is repeated 1000 times instead of 200, and a similar χ^2-test is performed on this new set of data. Given that the proportion for each number of heads remains the same, comment on whether the conclusion also remains the same. [2]

8 To assess the difference in wear on the two tyres of a motorcycle, 8 motorcycles, each initially with new tyres, were ridden for 1000 km. After this time, the depth of tread on each tyre was measured and recorded. A tyre that has lesser wear has a deeper depth of tread on it. The results are shown in the table below.

Motorcycle	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Depth of tread on front tyre (mm)	2.4	1.5	2.3	2.4	2.6	2.5	2.1	2.6
Depth of tread on rear tyre (mm)	2.3	1.9	2.1	1.8	1.8	2.8	1.4	2.1

(a) To test the hypothesis that there is no difference in the average wear for the front and rear tyres, explain why a paired sample *t*-test may not be appropriate. [1]

(b) Test if there is a difference in the average wear for the front and rear tyres at the 10% level of significance, using the Wilcoxon signed rank test. [4]

It was later discovered that there were two mistakes in the record. The first mistake was the depths of tread on the front and rear tyres of Motorcycle *F* were swapped. The second mistake was the depth of the front tyre of Motorcycle *B* should be *b* instead of 1.5.

(c) Given that there was no change to the ranks, and that the Wilcoxon signed rank test at the 5% level of significance concluded that there was sufficient evidence that the front tyre has lesser average wear than the rear tyre, find the range of values of *b*. [2]

9 Let *X* be a continuous random variable with probability density function

$\mathrm{f}(x) = 4x^3$ if $0 < x < 1$ and $\mathrm{f}(x) = 0$ otherwise.

Let $Y = 1 - 5X^2$.

(a) Find the probability density function of Y. [6]

(b) Find the expectation of Y. [2]

10 A marketing company wants to ensure it provides sufficient bandwidth to handle a certain number of the survey responses submitted on its website. The website receives an average of 20 responses per day.

(i) State the conditions under which the number of responses received per day can be well modelled by a Poisson distribution. [2]

Assume that these conditions in part **(i)** hold in a randomly chosen week.

(ii) Find the probability that the marketing company received the first response in the 3rd hour of the day. [2]

(iii) Find the probability that on one day, the marketing company receives 16 to 18 responses, given that it receives a total of 30 responses over 2 days. [4]

Given that T denotes the time, in minute, between the receipt of two consecutive responses.

(iv) State the probability density function of T. [1]

(v) Find the greatest integer value of n such that there is a probability of at least 0.3 that there are no responses received in n consecutive minutes. [2]

11 There are two weighing machines, *A* and *B*, which are used to measure the masses of packets of rice produced in a factory. An engineer discovers that machine A is faulty and suspects that it is over-stating the mass of the packets. He checks machine *B* and finds that it is working properly. We can assume that the masses of the packets are normally distributed.

To test machine *A*, the engineer randomly selects a small number of packets from a large batch produced on a certain day. These packets are weighed using machine *A* and then tested to see if the mean mass of the contents reported by machine *A* is greater than the stated mass on the packets.

(i) Identify the appropriate hypothesis test for the engineer's proposal. [1]

A sample of eight packets is selected at random from a large batch and measured using machine *A*. The masses of the packets of rice, *x* g, reported by machine *A*, are

1998.5	2000.4	1999.9	2005.8	2011.5	2007.6	2001.3	2002.4

(ii) A hypothesis test, performed at the $\alpha\%$ significance level, found that there was significant evidence that the mean mass of the packets reported by machine A is greater than 2000 g. Find the minimum value of α. [4]

Another sample of eight packets is selected at random from **another large batch** on a different day. Their masses were measured using machine B. The masses, *y* g, are summarised by

$$\sum(y-2000)=33.1 \text{ and } \sum(y-2000)^2=266.99.$$

(iii) State the appropriate hypothesis test to determine whether there is significant evidence, at the 5% significance level, that there is a difference between the mean masses of the packets in the two batches. Carry out the test and state any assumption required for the test. [6]

(iv) Give a possible explanation as to why the result of the test in part **(iii)** could contradict that the engineer's finding that machine A is faulty? [1]