COMPLEX NUMBERS [part FM]

1 HCI/2014/I/3

One of the roots of the equation $z^3 - 2z^2 + az + 1 + 3\text{i} = 0$ is $z = \text{i}$. Find the complex number a and the other roots. [5]

2 MI/2014/I/10

(i) By using de Moivre's theorem, or otherwise, show that

$$(1-i)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right).$$ [2]

(ii) Using the result in **(i)** or otherwise, find the least positive integer n for which $(1-i)^n$ is real and negative. Solution by trial and error will not be accepted. [3]

(iii) For the equation $z^4 + az^3 + bz^2 + cz + d = 0$ where a, b, c and d are real, give a brief explanation and determine the possible number of complex roots the equation can have. [2]

(iv) Solve the equation $z^4 + 4 = 0$, expressing the solutions in the form $x + iy$ where x and y are real. [4]

3 ACJC/2014/II/2

The complex number z satisfies the relations $\arg(z + 3 - 3\text{i}) = -\frac{1}{4}\pi$ and $|z - 3 + 3\text{i}| \le b$, where b is a constant and $1 < b < 3$.

(i) Illustrate each of the above relations on a single Argand diagram. [2]

(ii) Find the exact least possible value of $|z + 5\text{i}|$. [1]

(iii) Given that the least possible value of $|z|$ is $\sqrt{18} - 2$,

(a) find the value of b, [1]

(b) hence find an exact expression for z, in the form $x + \text{i}y$. [2]

(c) State the cartesian equation of the locus of the point representing complex variable w such that $|w| = |w - z_1|$, where z_1 is the complex number found in part **(b)**. [1]

4 **HCI/2014/II/3**

(a) **(i)** Find the fifth roots of –32, expressing the roots in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

(ii) The roots representing z_1 and z_2 are such that $0 < \arg(z_1) < \arg(z_2) < \pi$.

State the complex number w in the form $re^{i\theta}$ where $z_2 = wz_1$. [1]

(b) The complex number z satisfies $|z-3-3i| \ge |z-1-i|$ and $\frac{1}{6}\pi < \arg(z) \le \frac{1}{3}\pi$.

(i) On an Argand diagram, sketch the region in which the point representing z can lie. [3]

(ii) Find the area of the region in part **(b)(i)**. [3]

(iii) Find the range of values of $\arg(z-5+i)$. [2]

5 **RI/2014/II/4**

Do not use a calculator in answering this question.

The complex number z satisfies both the relations $\left|z+2\sqrt{3}-i\right| \le 4$ and $\frac{5}{6}\pi \le \arg(z+i) \le \pi$.

(i) On an Argand diagram, shade the region in which the point representing z can lie. [4]

(ii) Find the least possible value of $|z|$. [2]

(iii) State the cartesian form of the complex number z when $|z+i|$ is greatest. [1]

(iv) Find the range of values of $\arg\left(z+4\sqrt{3}+i\right)$. [2]

6 RVHS/2014/II/2

(i) Solve the equation $z^6+64=0$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\alpha}$, where $r>0$ and $-\pi<\alpha\le\pi$. [3]

(ii) Show the roots on an Argand diagram. [2]

The roots denoted by z_1 and z_2 are such that $0<\arg(z_1)<\arg(z_2)\le\frac{1}{2}\pi$. The complex numbers z_1 and z_2 are represented by the points Z_1 and Z_2 in the Argand diagram respectively.

(iii) Explain why the locus of all points z such that $|z|=|z-z_1|$ passes through the point Z_2. [1]

(iv) The complex number w satisfies the relation $\arg(w-z_1)=\arg(z_2-z_1)$. Sketch the locus of the points which represent w in the same Argand diagram. [2]

(v) Find the range of values of $\arg(w)$. [3]

7 CJC/2017/FM/Promo/7

It is given that the complex number $z=1+\cos\theta+\mathrm{i}\sin\theta$, where $-\pi<\theta\le\pi$.

(i) By considering appropriate trigonometric identities, or otherwise, show that the argument of z is $\frac{\theta}{2}$ and find the modulus of z in terms of θ. [3]

(ii) Hence, find the real and imaginary parts of $(1+\cos\theta+\mathrm{i}\sin\theta)^n$, where $n\in\mathbb{Z}^+$. [3]

(iii) By considering the binomial expansion of $\left[1+(\cos\theta+\mathrm{i}\sin\theta)\right]^n$, show that

$$1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos(2\theta)+\cdots+\binom{n}{n}\cos(n\theta)=\left[2\cos\left(\frac{\theta}{2}\right)\right]^n\cos\left(\frac{n\theta}{2}\right),$$

where $\binom{n}{r}=\frac{n!}{r!(n-r)!}$. [3]

8 **RI/2017/FM/Promo/6a**

On the same Argand diagram, sketch the loci of points given by each of the following equations:

$$L_1: \ |z+2-\mathrm{i}| = \sqrt{5},$$

$$L_2: \ \arg(z+3+\mathrm{i}) = \alpha, \text{ where } \alpha = \tan^{-1} 2.$$

Find, in the form $x+\mathrm{i}y$, the complex number which represents the point in the Argand diagram which is on both L_1 and L_2, giving the exact values of x and y. [5]

9 **SAJC/2017/FM/Promo/4**

(i) Solve the equation $z^3 - 3z^2 + 5z = -9$, giving your answers in exact form. [4]

(ii) Show the 3 roots of the equation in **(i)** on an Argand diagram and find the area of the triangle formed by joining the 3 points. [3]

(iii) Write down two roots of the equation $z^{300} - 3z^{200} + 5z^{100} = -9$ in polar form. [1]

10 **SAJC/2017/FM/Promo/9**

(ai) Sketch, on a single Argand diagram, the locus of z which satisfies both $|z+2-\mathrm{i}| \le 2$ and $|z+4-6\mathrm{i}| \le |z+2\mathrm{i}|$. [3]

(ii) It is given that $-\pi < \arg(z+2\mathrm{i}) \le \pi$. Find the complex numbers v and w that give the greatest and least values of $\arg(z+2\mathrm{i})$ respectively. [4]

(bi) The complex number w has modulus 6 and argument $-\frac{5\pi}{6}$, and the complex number z has modulus $4\sqrt{2}$ and argument $\frac{3\pi}{4}$. Find the modulus and argument of $\frac{z}{w}$, giving each answer exactly. [3]

(ii) Given that the Cartesian forms of w and z are $-3\sqrt{3}-3\mathrm{i}$ and $-4+4\mathrm{i}$ respectively, find the exact real part of $\frac{z}{w}$ and deduce that $\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$. [4]

11 TJC/2017/FM/Promo/6

(i) Show that if $z = e^{i\theta}$, then

$$z^k - \frac{1}{z^k} = 2i\sin k\theta,$$

where k is a positive integer. [1]

(ii) Show that $\sin^5\theta$ can be expressed in the form

$$A\sin\theta + B\sin 3\theta + C\sin 5\theta,$$

where the values of A, B and C are to be determined. [4]

(iii) Find the particular solution of the differential equation $\frac{dy}{dx} = \left(e^x \operatorname{cosec} y\right)^5$, given that $y = 0$ when $x = 0$. [3]

12 ACJC/2018/FM/Prelim/I/Q6

Let

$$C = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + \ldots + \cos 20\theta,$$
$$S = \sin 2\theta + \sin 4\theta + \sin 6\theta + \ldots + \sin 20\theta.$$

(i) Show that $C + iS = \frac{\sin 11\theta}{\sin\theta} e^{i10\theta}$ for all $\theta \neq n\pi, n \in \mathbb{Z}$. [3]

(ii) Hence, show that $\cos 2\theta + \cos 4\theta + \cos 6\theta + \ldots + \cos 20\theta = \frac{\sin 21\theta}{2\sin\theta} - \frac{1}{2}$. [3]

(iii) Deduce that $\sum_{r=1}^{10} r\sin 2r\theta = \frac{\sin 21\theta\cos\theta}{4\sin^2\theta} - \frac{21\cos 21\theta}{4\sin\theta}$. [3]

13 TJC/2018/FM/Prelim/I/7

(a) The complex numbers z and w are such that

$$|z - 4i| = 2 \text{ and } |w + 2i| = 1.$$

By considering an Argand diagram, find

(i) the least value of $|z - w|$,

(ii) the greatest value of $\arg(z - w)$. [4]

(b) The points P_1 and P_2 represent the complex numbers z_1 and z_2 respectively in an Argand diagram with origin O. Given that

$$z_1^2 - z_1 z_2 + z_2^2 = 0,$$

show that $$z_1 = z_2\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right).$$.

Hence or otherwise, prove that the triangle OP_1P_2 is equilateral. [7]

AJC/2018/FM/Prelim/I/5

14 The complex number z can be expressed as $e^{i\theta}$ where $-\pi < \theta \le \pi$.

(a) Given that z satisfies the equation $|z-1| = 1$, find the possible values of θ by means of a geometrical argument or otherwise. [3]

(b) It is given, instead, that z satisfies the equation $\arg(1+z+z^2+...+z^{n-1}) = 0$ for some positive integer $n \ge 2$ and $z \neq 1$. Determine the set of possible values of θ, giving your answer in terms of n. [7]

Q	**Answers**
1	$a = -2+3\text{i}$, -1 or $3-\text{i}$
2	(ii) least $n = 4$ (iv) $z = 1+i,\ 1-i,\ -1+i,\ -1-i$
3	(ii) $\frac{5\sqrt{2}}{2}$, (iii)(a) $b = 2$, (b) $\left(3-\sqrt{2}\right)+\left(-3+\sqrt{2}\right)\text{i}$, (c) $y = x-3+\sqrt{2}$
4	(a)(i) $z = 2\text{e}^{\text{i}\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)}$, $k = 0, \pm 1, \pm 2$; (ii) $w = \text{e}^{\text{i}\left(\frac{2\pi}{5}\right)}$. (ii) $16-8\sqrt{3}$ or 2.14; (iii) $2.36 \le \arg\left(z-5+\text{i}\right) < 2.94$
5	(ii) $h = \sqrt{3}\sin(\frac{1}{6}\pi) = \frac{1}{2}\sqrt{3}$ (iii) $-4\sqrt{3}+3\text{i}$ (iv) $0 \le \arg\left(z+4\sqrt{3}+\text{i}\right) < \frac{2}{3}\pi$
6	(i) $z = 2\text{e}^{i\left(\frac{\pi}{6}+\frac{k\pi}{3}\right)}, k = 0, \pm 1, \pm 2, -3$ (v) $\frac{1}{6}\pi < \arg\left(w\right) < \frac{5}{6}\pi$
7	(i) $\lvert z\rvert = 2\cos\frac{\theta}{2}$ (ii) $\left(2\cos\frac{\theta}{2}\right)^n \cos\frac{n\theta}{2}$, $\left(2\cos\frac{\theta}{2}\right)^n \sin\frac{n\theta}{2}$
8	(a) $-1+3\text{i}$
9	(i) $-1,\ 2 \pm \sqrt{5}\text{i}$ (iii) $\cos\frac{\pi}{100}+\text{i}\sin\frac{\pi}{100}$ and $\cos\frac{-\pi}{100}+\text{i}\sin\frac{-\pi}{100}$
10	(a)(ii) $v = -4+\text{i}$, $w = -\frac{4}{5}+\frac{13}{5}\text{i}$; b (i) $\frac{2}{3}\sqrt{2}$, $-\frac{5}{12}\pi$, (ii) $\frac{1}{3}(\sqrt{3}-1)$
11	(ii) $A = \frac{5}{8}, B = -\frac{5}{16}, C = \frac{1}{16}$ (iii) $-\frac{5}{8}\cos y + \frac{5}{48}\cos 3y - \frac{1}{80}\cos 5y = \frac{\text{e}^{5x}}{5} - \frac{11}{15}$
13	(a) (i) 3 (ii) $\frac{2\pi}{3}$
14	(a) $\frac{\pi}{3}$ or $-\frac{\pi}{3}$ (b) $\left\{\theta:\ \theta = \frac{2k\pi}{n-1} \text{ where } k \in \mathbb{Z}\setminus\{0\}, \frac{1-n}{2} < k \le \frac{n-1}{2}\right\}$