RAFFLES INSTITUTION
H2 Mathematics (9758)
2024 Year 6

<u>2024 Year 6 Timed Practice Revision Practice Paper 1</u>
<u>(Source: 2019 Year 6 Term 3 Common Test)</u>

The solution will be released in Ivy on 27 May (Monday)

Total Marks: 100 **Duration: 3h**

**** **Note that Qn 2 (Complex Numbers) will NOT be examined in the coming Timed Practice. So, you should complete this paper within 2 hrs 47 mins.**

Section A: Pure Mathematics [59 marks]

1 The curve C_1 with equation $\frac{x^2}{a^2}+\frac{y^2}{4a^2}=1$ is transformed to curve C_2 by a translation of a units in the positive x-direction, followed by a stretch with scale factor 2 parallel to the x-axis, and followed by a reflection in the y-axis, where a is a positive constant.

(i) Find the equation of C_2. [3]

(ii) Describe the shape of C_2 geometrically. [2]

2 **Do not use a calculator in answering this question.**

The equation $z^2-2\sqrt{3}\,z+4=0$ has two complex roots z_1 and z_2, where $0<\arg(z_1)<\frac{\pi}{2}$.

(i) Find z_1 and z_2 in polar form. [3]

(ii) Show that $\frac{z_1^{\,4}}{z_2^{\,2}}=-4$. [1]

(iii) Find the set of possible values of n, $n\in\mathbb{Z}$, for which $\frac{z_1^{\,n}}{1+\sqrt{3}\,\mathrm{i}}$ is a real number. [3]

3 The function f is defined as follows.

$$\mathrm{f} : x \mapsto x^2 - 3x, \text{ for } |x| < a.$$

(i) State the largest value of a for which the function f^{-1} exists. Hence find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1} for this value of a. [3]

In the rest of the question, use $a = 1$.

The function g is defined on an interval A as follows.

$$\mathrm{g} : x \mapsto \ln(3x+2), \text{for } x \in A.$$

(ii) A student suggests 2 possible intervals for A as follows.

(a) $\left(-\dfrac{2}{3}, -\dfrac{1}{3}\right)$

(b) $\left(-\dfrac{1}{2}, 0\right)$

Determine which of the above intervals will result in the existence of the composite function fg, justifying your answer. In the case(s) that fg exists, find the range of fg. [4]

4 **(a)** A geometric series has first term $2\sqrt{2}\sin\theta$ and second term $\sin 2\theta$.

(i) Show that the series is convergent for $\dfrac{\pi}{2} < \theta < \pi$. [2]

(ii) It is given that $\theta = \dfrac{3\pi}{4}$ and S_n denotes the sum of the first n terms of the series. Find S_n and hence determine the exact value of S_∞. [3]

(b) **(i)** Show that $\dfrac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1} - \sqrt{n}$, where $n \in \mathbb{Z}^+$. [1]

(ii) Hence find the least possible value of N such that $\displaystyle\sum_{n=4}^{N} \frac{1}{\sqrt{n+1}+\sqrt{n}} > 100$. [2]

5 The parametric equations of a curve are

$$x = t + 2, \quad y = \frac{t}{2t-1},$$

where $t \in \mathbb{R}$, $t \neq \frac{1}{2}$.

(i) Sketch the curve, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the axes. [3]

(ii) The tangent to the curve at the point $\left(p+2, \frac{p}{2p-1}\right)$ intersects the x-axis at point A and the y-axis at point B. Find, in terms of p, an expression for the area of the triangle OAB. [5]

6 **(a)** Given $\mathbf{a}$ and $\mathbf{b}$ are two non-zero and non-parallel vectors and $\mathbf{c} = |\mathbf{b}|\mathbf{a} + |\mathbf{a}|\mathbf{b}$, show that the length of projection of $\mathbf{c}$ onto $\mathbf{a}$ and the length of projection of $\mathbf{c}$ onto $\mathbf{b}$ have the same magnitude. [3]

(b) The equations of line l_1, planes π_1 and π_2 are

$$l_1 : \frac{x-3}{2} = \frac{1-y}{2} = -z,$$
$$\pi_1 : 2x + 3y - z = 5,$$
$$\pi_2 : -ax - 3y + 2z = b,$$

respectively.

(i) If l_1 lies on π_2, find the values of a and b. [2]

For the rest of the question, l_1 does not lie on π_2 and l_1 intersects π_1 at point F.

(ii) Find the coordinates of F. [3]

(iii) Find, in terms of a, a direction vector of the line of intersection l_2 between π_1 and π_2. [2]

(iv) Find the relationship between a and b if F also lies on π_2. State, in terms of a, a vector equation of l_2. [2]

7 The S-I model is used to study the spread of infectious ~~diseases~~ across different scenarios. In this question, it is assumed that we are studying a homogeneous population in a closed community of constant size N throughout the period of consideration. The population is divided into two groups – one group of infected individuals who have the diseases and another group of healthy and susceptible individuals who becomes infected when they come into some form of contact with the other group of infected individuals. Using x and y to represent the number of infected individuals and number of healthy and susceptible individuals at time t respectively, we can model the situation using the following differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kxy, \qquad \textbf{(I)}$$

where k is a positive constant.

(i) By expressing y in terms of N and x, show that equation (**I**) can be rewritten in the form of a first order differential equation in terms of x and t. Given that $x = \alpha$ when $t = 0$, solve this differential equation and show that the solution can be expressed in the form

$$x = \frac{A}{1+(B-1)\mathrm{e}^{-Nkt}},$$

where A and B are constants expressed in terms of N and α. [8]

(ii) This model was tested using some data from the 2003 SARS epidemic in Singapore where $\alpha = 1$, $N = 206$, k was estimated to be 8.1835×10^{-4} and time was measured in days.

Write down the equation of the corresponding solution curve and sketch the part of the curve which is relevant to this context. (Your sketch should be suitably labelled on the axes.) State what happens to x for large values of t. [4]

Section B: Probability and Statistics [41 marks]

8 **(i)** A code consists of 10 digits which are either zeros or ones, for example, 1011011010. Calculate the number of such codes if there is no restriction. [1]

Given further that the 10 digits consists of 4 zeros and 6 ones, calculate the number of such codes if

(ii) there is no other restriction, [1]

(iii) all the zeros must be separated and the first and last digits must be different, [2]

(iv) no more than 4 ones are together. [2]

9 In an online shop, the time taken, in hours, to sell a watch is a normally distributed continuous random variable X. The standard deviation of X is 0.68 hours and the expected value of X is 1.75 hours. After an aggressive advertising campaign, the total time taken to sell 8 watches is found to be 11 hours. Test, at 5% level of significance, whether there is evidence that the mean time taken to sell a watch has decreased. State an assumption that you have used in your calculation. [6]

10 In this question you should state the parameters of any distributions that you use.

Crispy Cream Donut Shop sells 2 types of donuts: Ring Donuts and Filled Donuts. The masses in grams of Ring Donuts and Filled Donuts have normal distributions $\mathrm{N}\left(54,\ 1.5^2\right)$ and $\mathrm{N}\left(86,\ 2^2\right)$ respectively.

(i) Find the probability that the total mass of 3 randomly chosen Ring Donuts is more than twice the mass of a randomly chosen Filled Donut. [3]

12 donuts are packed into a paper box. The mass in grams of an empty paper box has a normal distribution $\mathrm{N}\left(80,\ 5^2\right)$.

(ii) The probability that the total mass of a box containing 6 Ring Donuts and 6 Filled Donuts is more than m grams is 0.95. Find m. [4]

(iii) State an assumption that you have used in your calculations in parts **(i)** and **(ii)**. [1]

11 **(a)** Two digits X and Y are chosen independently at random from the set of 10 digits $\{0, 1, 2, \ldots, 9\}$. Events A and B are defined as follows:

A: $X = Y + 1$, .

B: X and Y are both less than 6.

Find

(i) $\mathrm{P}(A)$, [1]

(ii) $\mathrm{P}(B)$, [1]

(iii) $\mathrm{P}(A \cup B)$. [2]

(b) On a particular afternoon in June, 5 girls and 4 boys were in the Shaw Library and 6 girls and 9 boys were in the Hullett Library. A teacher selects at random 2 students from each library to distribute 4 free concert tickets.

(i) Calculate the probability that 2 girls and 2 boys received the tickets. [3]

(ii) Given that 2 girls and 2 boys received the tickets, calculate the probability that the 2 students selected from the Hullett Library are of the same gender. [2]

12 The National Aeronautics and Space Administration (NASA) compiles data on space shuttle launches and publishes them on its website. The following table displays the frequency distribution for the number of crew members on each of the 135 missions from April 1981 to July 2011.

Crew Size	2	3	4	5	6	7	8	9	10
Frequency	4	0	3	36	24	48	14	0	6

(i) Let the random variable C denote the crew size of a randomly selected mission between April 1981 to July 2011. Obtain the probability distribution of C and find $\mathrm{E}(C)$. [3]

A group of students are doing research on the data collected by the crew members of these missions. Each student is randomly assigned one mission, and the student is required to write one report for each crew member in the mission.

(ii) Find the probability that the total number of reports written by 2 randomly chosen students is 8. [2]

(iii) Show that the probability of the total number of reports written by 2 randomly chosen students exceeding 8 is 0.971. [2]

There are n mentors and each mentor is randomly assigned 2 students to grade their reports. Let T denote the number of mentors who need to grade more than 8 reports in total.

(iv) Find the value of n if $\mathrm{E}(T) = 20.391$. [1]

(v) Use the value of n found in part **(iv)** to calculate the probability that more than 19 mentors need to grade more than 8 reports each. [2]

(vi) Find the value of n if the most probable value of T is 25. [2]

******* **End of Paper** *******