ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS **9758**

JC2 Prelim Paper 1 (100 marks)

9 Sept 2024

3 hours

Additional Material(s): List of Formulae (MF 26)

CANDIDATE NAME

CLASS | | | / | |

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of 23 printed pages and 5 blank pages.

[Turn Over

1 **(a)** Sketch the graphs of $y = 3e^x$ and $y = x + 3$ on the same diagram. Indicate clearly the coordinates of the points of intersection between the 2 graphs.

Solve the inequality $3e^x > x + 3$. [3]

(b) Hence find $\int_{-2}^{2} \left|3e^x - x - 3\right| \, dx$, giving your answer in an exact form. [2]

2 **(i)** Find $\dfrac{d}{dx}\left(e^{\sin^{-1} x}\sqrt{1-x^2}\right)$. [1]

(ii) Hence using integration by parts, find $\displaystyle\int x \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx$. [3]

3 The curve C has parametric equations

$$x = 2t - \frac{1}{t^2}, \quad y = 2t + \frac{1}{t}, \qquad t \in \mathbb{R}, t \neq 0.$$

The point P on the curve has parameter $t = 1$.

(i) Find the equation of tangent and normal to C at the point P. [4]

(ii) The tangent at P meets the y-axis at B. The normal at P meets the x-axis at A. If O is the origin, find the area of the quadrilateral $OAPB$. [2]

4 A sequence is such that $u_0 = 2$ and $u_n = u_{n-1} + n^3 + \left(\frac{1}{2}\right)^n$ for $n \geq 1$.

(a) It is given that $\displaystyle\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$. By considering $\displaystyle\sum_{r=1}^{n}\left(u_r - u_{r-1}\right)$, find a formula for u_n in terms of n. [4]

(b) Hence, using the formula of u_n found in **(a)**, find $\displaystyle\sum_{r=9}^{n}\left((r+2)^3 + \left(\frac{1}{2}\right)^{r+2}\right)$ exactly. [3]

5 **(a)** It is given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors.

If $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$, show that the two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other. [4]

(b) (i) Explain why the result of

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$$

is a vector. [1]

(ii) Simplify $\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$. Show your workings clearly. [3]

6 **(i)** The variables *x* and *y* are related by

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+ky=2 \text{ and } y=1 \text{ at } x=0,$$

where *k* is a constant. Show that $(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. [1]

(ii) Given that *x* is small, find the series expansion of $\mathrm{g}(x)=\dfrac{1}{\sin^2\left(2x+\frac{\pi}{2}\right)}$ in ascending powers of *x*, up to and including the term in x^2.

If the coefficient of x^2 in the expansion of $\mathrm{g}(x)$ is equal to twice the coefficient of x^2 in the Maclaurin series for *y* in **(i)**, find the value of *k*. [5]

(iii) By further differentiation of the result found in **(i)**, and taking *k* =1, find the Maclaurin series for *y*, up to and including the term in x^3. [3]

7 **(a)** State a sequence of transformations that will transform the curve with equation $y^2-x^2=1$ on to the curve with equation

$9y^2-54y-x^2-2x+79=0$. [4]

(b) A curve *C* has equation

$$9y^2-54y-x^2-2x+79=0.$$

(i) For real values *x*, use a non-graphical method to determine that *y* cannot lie between *a* and *b*, where *a* and *b* are exact real constants to be determined. [3]

(ii) Sketch the curve *C*, indicating clearly the equations of all asymptotes and the coordinates of the turning points. [3]

(iii) By adding a suitable curve, determine the number of real roots of the equation,

$$9\left[(x+1)^2+3\right]^2-54\left[(x+1)^2+3\right]-x^2-2x+79=0.$$ [2]

[Turn Over

8 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \left|4+2x-x^2\right|,\ x \in \mathbb{R},\ x \geq 3.5,$$

$$\mathrm{g}: x \mapsto 4+\mathrm{e}^{ax},\ x \in \mathbb{R},\ x \geq -1,$$

where $a > 0$.

(a) Find $\mathrm{f}^{-1}(x)$ and state its domain. [3]

(b) Find the value of x for which $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. [2]

(c) Show that the composite function fg exists and express the exact range of fg in the form of $A + B\mathrm{e}^{-a} + C\mathrm{e}^{-2a}$, where A, B and C are real constants. [4]

(d) Without the use of a graphing calculator, solve the inequality $\dfrac{\mathrm{g}(x)}{x^2-2x-2} \geq 0$.

Leave your answer in exact form. [3]

9 **(a)** The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}} \text{ and } z_2 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right).$$

(i) Find $z_1 + z_2$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r is an exact real constant in trigonometric form such that $r > 0$, and θ is in the form $k\pi$ where k is an exact real constant such that $-1 < k \leq 1$. [3]

(ii) Find also $z_1 + z_2$ in the form $x + \mathrm{i}y$, where x and y are exact real constant.

Hence show that $\tan\left(\dfrac{3\pi}{8}\right) = 1 + \sqrt{2}$. [2]

(b) The complex number w is given by $w = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \dfrac{\pi}{2}$.

(i) Show that $1 - w^2 = -2\mathrm{i}w\sin\theta$. [2]

(ii) Hence find the modulus and argument of $1 - w^2$ in terms of θ. [2]

(iii) Given that $\left(\dfrac{1-w^2}{\mathrm{i}w^*}\right)^n$ is real and negative and that $\theta = \dfrac{\pi}{5}$, find the three smallest positive integer values of n. [3]

10 A rice retailer pledges to donate a bowl of rice for every kilometre run by participants in a service-learning project. Donations will be made in complete bowls, based on the cumulative distance each individual ran by the end of the 28-day period. Distances ran by multiple individuals will not be combined. For example, if person A runs 18.8 km and person B runs 11.2 km, the retailer will donate a total of 29 bowls. Two such participants, athlete A and B, will each accumulate the distance they run for a total of 28 days via a plan each devised.

- Athlete A plans to run 5 km on the first day and then increase the distance by a fixed 0.65 km more than the previous day.
- Athlete B plans to run 7 km on the first day and then increase the distance by 4% more than the previous day.

(a) Determine the least number of days required for the cumulative distance of athlete A to exceed that of athlete B. [3]

(b) How many bowls of rice will both athletes contribute, in total, at the end of the 28-day period? [3]

(c) Suppose athlete A plans to cover at least 400 km by the end of the 28-day period, what is the minimum distance he should run in day 1 if the plan to increase by 0.65 km more than the previous day remains the same. Give your answer to the nearest metres. [3]

(d) On days where the distance athlete B is supposed to run exceeds 10 km based on his own plan, he will limit it to exactly 10 km instead. Given this change, how many bowls of rice will he contribute at the end of the 28-day period? [3]

11 In a large town, the number of people infected by a particular virus t days after the virus was first discovered is x. It is assumed that the rate of infection is proportional to x. Initially there are 5 people who are infected by the virus, and there are 5120 people who are infected by the virus 30 days after the virus was first discovered.

(i) Show that $x = 5(2)^{\frac{t}{3}}$. [5]

A cure and vaccine for the virus were discovered and administered to the population 30 days after the virus was discovered. Individuals who were cured are not at risk of reinfection. The number of people infected by the virus p days after the cure and vaccine were administered is represented by y. It is believed that the new rate of infection from then on is proportional to $6400y - y^2$.

It is given that 30 days after the cure or vaccine was administered, 3200 people remain infected with the virus.

(ii) Show that $y = \dfrac{6400}{1+2^{\left(\frac{p}{15}+H\right)}}$, where H is a constant to be determined. [6]

(iii) By finding the number of people in the town that will be infected by the virus in the long term, comment on the effectiveness of the cure and vaccine administered. [2]

End of Paper