National Junior College
2016 – 2017 H2 Further Mathematics
Topic F7: Further Complex Numbers (Tutorial Set 2)

Basic Mastery Questions

1. Describe and sketch the locus of z in each of the following cases:

(a) $|z| = 3$

(b) $|z+1| = 2$

(c) $|4-2\mathrm{i}z| = 6$

(d) $|z+3+4\mathrm{i}| = |3-4\mathrm{i}|$

(e) $|z+2| = |z+1-3\mathrm{i}|$

(f) $|z-\mathrm{i}| = |z+\mathrm{i}|$

(g) $|2z+1-2\mathrm{i}| = |1-2z|$

(h) $\left|5(z+1)^2\right| = 150$

(i) $\arg(z-4) = \dfrac{\pi}{2}$

(j) $\arg\sqrt{(z+3-\mathrm{i})} = -\dfrac{\pi}{3}$

(k) $\arg(1-\mathrm{i}z) = \dfrac{3\pi}{4}$

(l) $\arg\left(\dfrac{1-z}{1-\mathrm{i}}\right) = -\dfrac{\pi}{6}$

(m) $\sqrt{zz^*} = |z-6|$

(n) $\left|\dfrac{2z-\mathrm{i}}{z-2\mathrm{i}}\right| = 1$

2. Shade, in separate diagrams, the region represented by each of the following inequalities:

(a) $\mathrm{Im}(z) \le 2$

(b) $|z^2| \ge 36$

(c) $1 \le |2z-1| \le \left|\sqrt{15}\mathrm{i}+1\right|$

(d) $|z+2| \le |z+1-3\mathrm{i}|$

(e) $\dfrac{\pi}{2} < \arg(2z-2) \le \dfrac{2\pi}{3}$

(f) $\arg(z-1-\mathrm{i})^2 \le -\dfrac{\pi}{3}$ and $z+z^* \ge -1$

(g) $|z| < |z-1|$ and $-\dfrac{\pi}{4} < \arg z < \dfrac{\pi}{4}$

(h) $\mathrm{Im}(z) \le 3$ and $\dfrac{\pi}{4} \le \arg(z+1-\mathrm{i}) \le \tan^{-1} 2$

(i) $|z-\mathrm{i}| \le 2$ and $|z-1| \le |z-1+\mathrm{i}|$

(j) $\dfrac{\pi}{4} \le \arg(z-\mathrm{i}) < \dfrac{\pi}{2}$ and $0 < \arg(z-2\mathrm{i}) < \dfrac{\pi}{4}$

3. The point P in an Argand diagram represents the variable complex number z, and the point A in the first quadrant represents the fixed complex number a. Sketch, on separate diagrams, the locus of P in the following cases, making clear the relationship between the locus and A.

(i) $|z| = |a|$ (ii) $|z-a| = 2|a|$ (iii) $|z-a| = |z|$ (iv) $\arg(z-a) = \arg(a)$

Practice Questions

1. The complex numbers z_1 and z_2 are given by $1+\mathrm{i}\sqrt{3}$ and $-1-\mathrm{i}$ respectively.
   (i) Express each of z_1 and z_2 in polar form $r(\cos\theta+\mathrm{i}\sin\theta)$ where $r>0$ and $-\pi<\theta\le\pi$. Give r and θ in exact form.
   (ii) Find the complex conjugate of $\frac{z_1}{z_2}$ in exact polar form.
   (iii) On a single Argand diagram, sketch the loci
   (a) $|z-z_1|=2$, (b) $\arg(z-z_2)=\frac{\pi}{4}$
   (iv) Find where the locus $|z-z_1|=2$ meets the positive real axis. (GCE 2010/P1/Q8)

2. The point P in an Argand diagram represents the variable complex z where $\arg(z-2+3\mathrm{i})=\frac{\pi}{3}$. Sketch the locus of P.
   Give a geometrical description of the locus given by $|z-2-\mathrm{i}|=k$, where $k\in\mathbb{R}^+$.
   (i) If the locus $|z-2-\mathrm{i}|=k$ just touches the locus of P, show that $k=2$.
   (ii) Find the set of values of k for which the two loci intersect at two points exactly.

3. The complex number z satisfies the equation $|z|=|z+2|$.
   (i) Show that the real part of z is –1.

   The complex number z also satisfies the equation $|z|=3$. The two possible values of z are represented by the points P and Q in an Argand diagram.

   (ii) Draw a sketch showing the positions of P and Q, and calculate the two possible values of arg z, leaving the answers in radians correct to 3 significant figures.

   It is given that P and Q lie on the locus $|z-a|=b$ where a and b are real, and $b>0$.

   (iii) Give a geometrical description of this locus, and hence find the least possible value of b and the corresponding value of a.

4. The complex number z satisfies $|z-2-5\mathrm{i}|\le 3$.
   (i) On an Argand diagram, sketch the region in which the point representing z can lie. [3]
   (ii) Find exactly the maximum and minimum possible values of $|z|$. [2]
   (iii) It is given that $0\le\arg z\le\frac{\pi}{4}$. With this extra information, find the maximum value of $|z-6-\mathrm{i}|$. Label the point(s) that correspond to this maximum value on your diagram with the letter P. [3]
   (GCE 2011/ P2/Q1)

5. Sketch in an Argand diagram, the set of points representing all complex numbers satisfying both the inequalities:

$$|z-1-\mathrm{i}| \le 2 \quad \text{and} \quad -\frac{\pi}{2} \le \arg(z-1) \le \frac{\pi}{2}.$$

Find the greatest and least values of $\tan(\arg z)$.

6. The complex number z satisfies the relations $|z| \le 6$ and $\left|1-\frac{8+6\mathrm{i}}{z}\right| = 1$.

(a) Illustrate both of these relations on a single Argand diagram. [3]

(b) Find the greatest and least possible values of arg z, giving your answers in radians correct to 3 decimal places. [4]

(GCE 2008/P2/Q3 (Modified))

7. A fixed complex number a is such that $0 < \arg(a) < \frac{\pi}{2}$. On a single Argand diagram, sketch the loci given by $|z-a| = |z-7a|$ and $|z-4a| = 3|a|$.

The two complex numbers that satisfy the above equations are represented by the complex numbers p and q.

Find the possible values of $\arg\left(\frac{p}{q}\right)$.

Find $|p+q|$ in terms of a. (HCI/2009/ P1/4)

8. (i) Solve the equation $z^7-(1+\mathrm{i})=0$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\alpha}$, where $r>0$ and $-\pi < \alpha \le \pi$. [3]

(ii) Show the roots on an Argand diagram. [4]

(iii) The roots represented by z_1 and z_2 are such that $0 < \arg(z_1) < \arg(z_2) < \frac{1}{2}\pi$. Explain why the locus of all points z such that $|z-z_1| = |z-z_2|$ passes through the origin. Draw this locus on your Argand diagram and find its exact Cartesian equation. [5]

(GCE 2009 / P1 / Q9)

9. Given that $z = 2(1+\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \le \pi$, show that the locus of the points representing z, as θ varies, is a circle. Sketch the locus.

Hence, or otherwise, find the greatest and the least value of $|z+\mathrm{i}|$, and find z in the exact form $x+\mathrm{i}y$ that gives these values of $|z+\mathrm{i}|$.

10. Two loci in the Argand diagram are given by the equations

$$|z^*-1+\mathrm{i}| = 2 \text{ and } \arg(z-2\mathrm{i}) = \frac{\pi}{5}.$$

Draw an Argand diagram to show both loci and find the value of z that corresponds to the point of intersection of these loci. [5]

(HCI/2015/P1/Q10b modified)

11. Solve the equation $z^5+32=0$, expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [2]

z_1, z_2 and z_3 are three of the roots of $z^5+32=0$ such that $0<\arg z_1<\arg z_2<\arg z_3\le\pi$.

(i) Find the smallest positive integer n such that $\left(\dfrac{z_1}{z_2^{\,*}}\right)^n$ is real and positive. [3]

(ii) The points A and B represent the roots z_1 and z_3 respectively in the Argand diagram. The line segment BA' is obtained by rotating the line segment BA through $\dfrac{\pi}{2}$ clockwise about the point B. Find the real part of the complex number represented by point A', giving your answer in exact trigonometric form. [4]

(AJC/2015/P2/Q2)

12. (i) Sketch, on an Argand diagram, the set of points representing the complex number z which satisfy both conditions: $|z+2\mathrm{i}|\le|z-4-2\mathrm{i}|$ and $0\le\arg(z+2\mathrm{i})<\tan^{-1}2$. Hence, find the greatest value of $|z-4\mathrm{i}|$ which satisfy the above conditions, giving your answer in exact form.

(ii) With the help of your sketch in (i), sketch, on another Argand diagram, the set of points representing the complex number z which satisfy both conditions:

$$|z^*+2\mathrm{i}|\le|z^*-4-2\mathrm{i}| \quad \text{and} \quad 0\le\arg(z+2\mathrm{i})<\tan^{-1}2$$

Numerical Answers to Practice Questions

1. (i) $2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right), \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$ (ii) $\sqrt{2}\left(\cos\frac{11\pi}{12}+\mathrm{i}\sin\frac{11\pi}{12}\right)$ (iv) $x=2$
2. (ii) $2<k<4$
3. (ii) ±1.91 radians (iii) least $b=2\sqrt{2}, a=-1$
4. (ii) $\sqrt{29}\pm3$ (iii) $\sqrt{17}$
5. Least -1, greatest 3
6. (b) Least 0.058, greatest 1.229
7. 1.29 or -1.29, $8|a|$
8. (i) $z=2^{\frac{1}{14}}\mathrm{e}^{\mathrm{i}\pi\frac{8k+1}{28}}, k=0,\pm1,\pm2,\pm3$ (ii) $y=x\tan\frac{5\pi}{28}$
9. $\sqrt{5}+2$ with $z=\left(2+\frac{4\sqrt{5}}{5}\right)+\left(\frac{2\sqrt{5}}{5}\right)\mathrm{i}$, $\sqrt{5}-2$ with $z=\left(2-\frac{4\sqrt{5}}{5}\right)+\left(-\frac{2\sqrt{5}}{5}\right)\mathrm{i}$
10. $z=2.71+3.97\mathrm{i}$
11. $z=2\mathrm{e}^{-\frac{3\pi}{5}\mathrm{i}}, 2\mathrm{e}^{-\frac{\pi}{5}\mathrm{i}}, 2\mathrm{e}^{\frac{\pi}{5}\mathrm{i}}, 2\mathrm{e}^{\frac{3\pi}{5}\mathrm{i}}, 2\mathrm{e}^{\pi\mathrm{i}}$ (i) 5 (ii) $-2+2\sin\frac{\pi}{5}$
12. (i) $2\sqrt{13}$