HCA Department of Mathematics

Differential Equations

Further Mathematics 9649 (2025 onward)

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Syllabus requirements

- Analytical solution of first order and second order linear differential equations of the form
  - $\frac{dy}{dx} + p(x)y = q(x)$ using an integrating factor
  - $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ for $a, b \in \mathbb{R}$
  - $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$ for $a, b \in \mathbb{R}$ where $f(x)$ is a polynomial or pe^{kx} or $p\cos(kx)$ or $q\sin(x)$ including those that can be reduced by means of a given substitution
- Relationship between the solution of a non-homogenous equation and the associated homogenous equation
- Family of solution curves
- Phase lines and slope fields (non-examinable)
- Exponential growth model
- Logistic growth model, equilibrium points and their stability, and harvesting

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1 Introduction

"Among all the mathematical disciplines, the theory of differential equations is the most important; it furnishes the explanation of all those elementary manifestations of nature which involve time."

Thus declared Sophus Lie, a famed mathematician responsible for advances in linear and abstract algebra, and most importantly here, differential equations.

Differential equations are responsible for virtually every single interaction in this universe (this is meant literally): it is the language of Physics and all the other physical sciences. Indeed, these kinds of equations allow us to model *ongoing* processes that evolve with time, like radioactive decay and population growth.

So, what exactly is a differential equation? Differential equations are really just equations where the variable – usually a number in elementary algebra – is replaced by a derivative. It is quite similar to, but not exactly a polynomial. Here is an example:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0$$

If we compare this to a polynomial, it would look something like:

$$k^2 + 3k + 4 = 0$$

As we can see, differential functions are made up of derivatives of varying orders. The main difference between regular, algebraic equations and differential equations is that instead of solving for a *value*, we solve for a *function*. Hence we have the following:

Definition 1.1. A *linear ordinary differential equation* of order $n \in \mathbb{Z}^+$ in y can be written in the form

$$\sum_{0 \le k \le n} f_k(x) \frac{d^k y}{dx^k} = g(x)$$

where y is a function of x and $\frac{d^0y}{dx^0} = y$. Furthermore, the differential equation is called *homogenous* when $g(x) = 0$ and *non-homogenous* otherwise.

Do note the mention of the word *ordinary*. This is for accuracy purposes; in fact, it is completely possible to form differential equations using partial derivatives. These equations are called *partial differential equations*, and they are outside the scope of the syllabus and this document.

In H2 Mathematics, you will have dealt with differential equations of the form

$$\frac{dy}{dx} = f(x)g(y)$$

which can be solved by separating variables and integrating afterwards. If $g(y)$ is not a linear polynomial (i.e. degree 1), then the equation is not linear. The Further Mathematics syllabus concerns itself with differential equations which are strictly linear, albeit harder to solve compared to the aforementioned method.

2 First order linear differential equations

We recall that using Definition 1.1, a first order linear differential equation may be expressed in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x.

Example 2.1. Consider the following equation:

$$\frac{dy}{dx} - 4x^2y - 4 = 0 \iff \frac{dy}{dx} - 4y = 4$$

This is a first order linear differential equation with $P(x) = 4x^2$ and $Q(x) = 4$. This is a non-homogenous equation since $Q(x) \neq 0$.

2.1 Analytical solution for homogenous equations

We now proceed to find a solution to any equation in the form above. For starters, let us consider the general form of this type of equation:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Clearly, we need some term in the left-hand side to contain $Q(x)$. Hence y must contain $Q(x)$ or some function of it. Now, let us integrate both sides to get rid of the derivative:

$$\int \left(\frac{dy}{dx} + P(x)y\right)\, dx = \int Q(x)\, dx$$
$$y + \int P(x)y\, dx = \int Q(x)\, dx$$

We have y standing alone. Hence $\int P(x)y\, dx$ yields $Q(x)$ in some way. Recall that integrating $f'(x)e^{f(x)}$ with respect to x gives $e^{f(x)} + C$. We could now think that maybe, $P(x)y$ integrates to give a mere $-y$, when $y = e^{-\int P(x)\, dx}$. If only we could have $Q(x) = 0$, then indeed,

$$y + \int P(x)y\, dx = y - \int (-P(x))\, e^{-\int P(x)\, dx}\, dx = y - y = 0$$

2.2 Analytical solution for non-homogenous equations

However, $Q(x)$ isn't always 0 for most equations. In that case, we need y to contain $Q(x)$ as part of some factor, while containing $e^{-\int P(x)\, dx}$ as a factor.

Much earlier before we began this chapter, we have learnt about the product rule. We can now start with $y = e^{-\int P(x)\,dx}u(x)$ for some unknown function $u(x)$, and get

$$\frac{dy}{dx} = e^{-\int P(x)\,dx}\frac{du}{dx} - P(x)e^{-\int P(x)\,dx}u(x)$$

We observe a common factor and thus factorize:

$$\frac{dy}{dx} = e^{-\int P(x)\,dx}\left(\frac{du}{dx} - P(x)u(x)\right)$$

We substitute this inside our general equation:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$e^{-\int P(x)\,dx}\left(\frac{du}{dx} - P(x)u(x)\right) + P(x)e^{-\int P(x)\,dx}u(x) = Q(x)$$

$$e^{-\int P(x)\,dx}\frac{du}{dx} = Q(x)$$

Rearranging, we get

$$\frac{du}{dx} = Q(x)e^{\int P(x)\,dx} \implies u = \int Q(x)e^{\int P(x)\,dx}\,dx$$

Hence, we have the following:

Theorem 2.2. *The solution of the equation*

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is given by

$$y = e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)\,dx}\,dx$$

for any $P(x)$ and $Q(x)$.

Proof. Consider the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Differentiating y, one gets

$$\begin{aligned}\frac{dy}{dx} &= -P(x)e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)\,dx}\,dx + e^{-\int P(x)\,dx}Q(x)e^{\int P(x)\,dx}\\ &= -P(x)e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)\,dx}\,dx + Q(x)\end{aligned}$$

Also,

$$P(x)y = P(x)e^{-\int P(x)\,dx} \int Q(x)e^{\int P(x)\,dx}\,dx$$

Therefore we have

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and the proof is complete. $\square$

Make sure to not forget your constant of integration!

Example 2.3. Experiments indicate that the rate at which glucose is absorbed by the body is λG, where $\lambda \in \mathbb{R}$ and G is the amount of glucose present in the bloodstream. Glucose is injected into a patient's bloodstream at a constant rate of r units per unit time. Write a differential equation modelling the amount of glucose present in the patient's bloodstream at time t. Hence, given that the initial amount of glucose present is G_0, solve for G.

Solution. We are given that the rate r is constant, and we have λG. We need to find a differential equation in G with respect to t. Furthermore, we are told that glucose is removed since it is absorbed into the body out of the bloodstream, so we must account for that. Hence we have

$$\frac{dG}{dt} = r - \lambda G$$

and so

$$\frac{dG}{dt} + \lambda G = r$$
$$G = e^{-\lambda t}\left(\frac{r}{\lambda}e^{\lambda t} + C\right)$$
$$G = Ce^{-\lambda t} + \frac{r}{\lambda}$$

When $t = 0$, $G = G_0$ so $C = \left(G_0 - \frac{r}{\lambda}\right)$. Hence $G = \left(G_0 - \frac{\lambda}{r}\right)e^{-\lambda t} + \frac{r}{\lambda}$ and we are done. $\square$

2.3 Alternate derivation of the solution

`@garbageskill` remarks that there is a faster way with less guessing of what the solution y could be. In the above approach, we *guessed* that y had a factor of $e^{-\int P(x)\,dx}$. However, we can do away with this guess and instead derive the solution systematically using something known as an *integrating factor*.

Definition 2.4. Consider any differential equation, for example

$$\frac{dy}{dx} + P(x)y = Q(x).$$

An *integrating factor* is an expression which all terms are multiplied by, so that it is easier to solve the equation by integration.

Integrating factors can be used for nonlinear equations (for example, one with a term in y^2). However, we do not discuss it here; rather, we will use this idea of an *integrating factor* to see how we can obtain y.

Recall the product rule

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Using prime notation,

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Hence, we are left with the objective of expressing the differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

in the form

$$M(x)\frac{dy}{dx} + \underbrace{M(x)P(x)}_{M'(x)}y = Q(x)M(x)$$

for some unknown, unique $M(x)$, where

$$M'(x) = M(x)P(x) \implies \int P(x)\,dx = \int \frac{M'(x)}{M(x)}\,dx = \ln(M(x))$$

This allows us to use the product rule (backwards). We have the fact that $M(x) = e^{\int P(x)\,dx}$ and

$$\frac{d}{dx}(M(x)y) = Q(x)M(x) \implies y = \frac{1}{M(x)}\int Q(x)M(x)\,dx$$

Finally, we have

$$y = e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)\,dx}\,dx$$

and we have successfully derived the solution. It is worth knowing that this method of finding some factor, multiplying it and integrating afterwards after applying some differentiation rules can prove very useful in solving other differential equations, especially nonlinear ones (beyond the syllabus).

3 Second order linear differential equations

A second order linear differential equation is of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

for any functions $P(x)$, $Q(x)$, $R(x)$.

Definition 3.1. If $P(x)$ and $Q(x)$ are constants, the differential equation is said to have *constant coefficients*. Otherwise, it is said to have *variable coefficients*.

The syllabus concerns itself only with the constant coefficient case, for this order of differential equations. As such, we will be discussing equations of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \qquad a, b \in \mathbb{R}.$$

In this chapter, we begin with the homogenous case, and then show how it relates to the non-homogenous case through the concept of a *complementary equation*.

3.1 Analytical solution for homogenous equations

Recall that an equation is homogenous when

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0.$$

Using the same idea of an exponential in Section 2.1, we notice that $y = e^{\lambda x}$ for some unknown λ. Substituting this in,

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

Since $e^{\lambda x}$ is a common factor, and is nonzero because all exponentials are nonzero, we obtain

$$\lambda^2 + a\lambda + b = 0$$

Definition 3.2. Consider the second order homogenous linear differential equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0.$$

The *characteristic equation* of this differential equation is defined as

$$\lambda^2 + a\lambda + b = 0$$

Math ever generalizes, so if we had a third order linear differential equation

$$\frac{d^3y}{dx^3} + a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

its characteristic equation is a cubic, which can also easily be solved.

Returning back to the discussion of second order equations, we are required to find λ such that the equation holds true. Quadratic equations have three possible cases where either $\lambda \in \mathbb{C}$, the equation has real and distinct roots, or the equation is a perfect square with repeated roots. While the complex case and the real, distinct cases have fairly similar treatment, we must deal with the repeated root case specially.

3.1.1 Distinct roots

Suppose that $a^2 - 4b \neq 0$. If the roots are complex, then $\lambda = \alpha \pm \beta i$ for some $\alpha, \beta \in \mathbb{R}$. We have $C_1 e^{\alpha+\beta i}$ as a solution, as well as $C_2 e^{\alpha-\beta i}$, for $C_1, C_2 \in \mathbb{R}$. This is true for any C_1, C_2 because they will all cancel out when substituted into the equation. Also, since $Ce^{a \pm bi} = Ce^a e^{\pm bi}$, the solution can be expressed in polar form. Thus we have found two distinct solutions.

The same applies for the real distinct roots case; it is merely the case where $\beta = 0$.

However, we are not done yet. Let us recall about this property of differentiation as a linear operation:

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

So if we had $y = u + v$ where $u = e^{\lambda_1 x}$ and $v = e^{\lambda_2 x}$ where λ_1 and λ_2 are the roots of the characteristic equation, we have

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

if and only if

$$\left(\frac{d^2u}{dx^2} + \frac{d^2v}{dx^2}\right) + a\left(\frac{du}{dx} + \frac{dv}{dx}\right) + b(u+v) = 0$$

Therefore, we can conclude the following:

Theorem 3.3. *A second-order homogenous linear differential equation*

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

admits the solution

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

where $C_1, C_2 \in \mathbb{R}$ and $\lambda^2 + a\lambda + b = 0$, and λ_1 and λ_2 are distinct roots of the quadratic equation.

Since we have an infinite choice of what C_1 and C_2 can be, we have infinitely many solutions. This leads to the next definition:

Definition 3.4. A *particular solution* of a differential equation is one without any arbitrary constants (for example, C_1 or C_2).

Indeed, it is possible to represent this *family* of solutions diagramatically. We will see this using the concept of a *slope field*.

We have discussed the case for real and complex distinct roots, so now we are left with the case of real, repeated roots. As seen in the other cases, we always had two terms which do not add up into one; we will call them *linearly independent*, although this is not the formal definition. However, does this also hold for our real, repeated case? *Yes*, it does. A formal proof of this will not be provided as it is beyond the scope of this document. Currently, we can only *assume* that there is a second solution which is linearly independent of the first one.

3.1.2 Repeated roots

To start off, suppose that we have $y = Ce^{\lambda x}$ for $C \in \mathbb{R}$ and λ the only distinct real root of the characteristic equation. We return to the product rule:

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Hence if $y = uv$ for some unknown u and v which are functions of x, we have the following equations:

$$\frac{d^2y}{dx^2} + a\left(\frac{du}{dx}v + u\frac{dv}{dx}\right) + buv = 0$$

$$\frac{d^2y}{dx^2} = \left(\frac{d^2u}{dx^2}v + \frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{dv}{dx}\right) = \left(\frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right)$$

Substituting in, we have

$$\left(\frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right) + a\left(\frac{du}{dx}v + u\frac{dv}{dx}\right) + buv = 0$$

We know that $y = uv$ must have a factor of $e^{\lambda x}$, because how it differentiates is the key to solving our equation. Suppose that $u = e^{\lambda x}$ then. We require that

$$\left(\lambda^2 e^{\lambda x}v + 2\lambda e^{\lambda x}\frac{dv}{dx} + e^{\lambda x}\frac{d^2v}{dx^2}\right) + a\left(\lambda e^{\lambda x}v + e^{\lambda x}\frac{dv}{dx}\right) + be^{\lambda x}v = 0$$

Again, exponentials are never zero so we obtain, after dividing by $e^{\lambda x}$:

$$\left(\lambda^2 v + 2\lambda\frac{dv}{dx} + \frac{d^2v}{dx^2}\right) + a\left(\lambda v + \frac{dv}{dx}\right) + bv = 0$$

By the characteristic equation, $\left(\lambda^2 + a\lambda + b\right) v = 0$. Hence

$$\left(2\lambda\frac{dv}{dx} + \frac{d^2v}{dx^2}\right) + a\left(\frac{dv}{dx}\right) = 0$$

$$\frac{d^2v}{dx^2} + (a + 2\lambda)\frac{dv}{dx} = 0$$

By the definition of λ using the quadratic formula,

$$\lambda = \frac{-a}{2} \implies 2\lambda = -a$$

$$\frac{d^2v}{dx^2} + (a + 2\lambda)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} = 0 \implies v = C_1x + C_2 \qquad C_1, C_2 \in \mathbb{R}$$

Finally, we have found that the second linearly independent solution takes on the form $y = Cxe^{\lambda x}$. Hence we have the following:

Theorem 3.5 (Theorem 3.3 generalized)**.** *A second-order homogenous linear differential equation*

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

admits the solution

$$y = C_1y_1 + C_2y_2$$

where $C_1, C_2 \in \mathbb{R}$ and y_1, y_2 are linearly independent particular solutions to the differential equation.

Theorem 3.6. *A second-order homogenous linear differential equation*

$$\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + k^2y = 0$$

admits the solution

$$y = C_1e^{-k/2} + C_2xe^{-k/2}$$

where $C_1, C_2, k \in \mathbb{R}$.

3.2 Analytical solution for non-homogenous equations

A constant-coefficient second order linear non-homogenous equation takes on the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

for any $a, b \in \mathbb{R}$ and any function $f(x)$.

3.2.1 Relationship with homogenous equations

In order to solve non-homogenous equations, we must ourselves solve a particular homogenous equation. It has a name, as we see below:

Definition 3.7. The *complementary equation* to a second order linear non-homogenous differential equation with constant coefficients of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

is defined to be

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

for any $a, b \in \mathbb{R}$ and any function $f(x)$.

Now, suppose that y_1 and y_2 solve the complementary equation, and Y_1, Y_2 and Y_P (where Y_P is a particular solution) solve the main (non-homogenous) equation. Indeed, Y must be of the form $Y = y + Y_P$ because differentiation is linear over addition. Hence we can argue that $Y_1 - Y_2 = C_1y_1 + C_2y_2$ for arbitrary $C_1, C_2 \in \mathbb{R}$, because we want to get rid of Y_P.

Theorem 3.8. *Suppose that Y_1 and Y_2 solve a second order non-homogenous linear differential equation. Then $Y_2 - Y_1$ is of the form $Y_1 - Y_2 = C_1y_1 + C_2y_2$ for arbitrary $C_1, C_2 \in \mathbb{R}$.*

Proof. This is left as an exercise to the reader. □

By Theorem 3.8, we can set $Y_1 = y$ to be the general solution and $Y_2 = Y_P$ to be some kind of particular solution. We get $y = C_1y_1 + C_2y_2 + Y_P$. Hence, we are left to solve for *any* Y_P because as said earlier, all we need is a particular solution to find the general solution of the differential equation.

3.2.2 Method of undetermined coefficients

For equations of the form above, we can use the *method of undetermined coefficients* because we are dealing with constant coefficients. (There is another method called the *variation of parameters* which is much more powerful; this can be found in [2].)

This method only works for cases when $f(x)$ is a polynomial, or of the form pe^{kx}, $p\cos(kx) + q\sin(kx)$ where $p, q \in \mathbb{R}$. This allows us to make certain guesses about what our particular solution could be; Table 1 is due to [2].

Example 3.9. Consider the differential equation

$$y'' + 4y' + y = 5\sin(2x).$$

Our guess will be of the form $A\sin(2x) + B\cos(2x)$ (even though $b = 0$ in the above equation).

Form of $f(x)$	**Guess for the form of Y_P**
k	A
$a_0x^n + a_1x^{n-1} + ... + a_n$	$A_0x^n + A_1x^{n-1} + ... + A_n$
ae^{kx}	Ae^{kx}
$a\sin(kx) + b\cos(kx)$	$A\sin(kx) + B\cos(kx)$

Table 1: Guesses for Y_P.

Example 3.10. Consider the differential equation

$$y'' - 3y = 5e^{-2x}.$$

Our guess will be of the form $A\sin(2x) + B\cos(2x)$ (even though $b = 0$ in the above equation).

Once we have a guess for Y_P, we set $y = Y_P$ and substitute it into the differential equation, in order to find any unknown constants.

Example 3.11. Solve the differential equation

$$y'' - 4y = 5e^x.$$

Solution. The characteristic equation of the complementary (homogenous) equation is $\lambda^2 - 4$ so $\lambda = \pm 2$. Hence $y = C_1e^{2x} + C_2e^{-2x} + Y_P$ where Y_P is of the form Ae^x, as seen in Table 1. Substituting $y = Y_P$, one obtains

$$Ae^x - 4Ae^x = 5e^x \implies -3A = 5 \implies A = -\frac{5}{3}$$

Finally, our solution is $y = C_1e^{2x} + C_2e^{-2x} - \frac{5}{3}e^x$. $\square$

However, if we had $f(x) = 5e^{2x}$ instead, we would run into a problem: $f(x)$ is a solution of the complementary equation, so our guess Ae^{2x} solves the complementary equation $y'' - 4y = 0$. To fix this, we can consider Axe^{2x}, similar to the approach in Section 3.1.2. If that still does not work, we can consider Ax^2e^{2x}, Ax^3e^{2x} and so on.

To generalize, suppose that for our particular solution Y_P we guess that $Y_P = g(x)$ where $g(x)$ takes on one of the forms in Table 1. If $Y_P = x^n g(x)$ where $n \geq 0$ solves the complementary equation, try $Y_P = x^{n+1}g(x)$. Clearly, **there must not be any term in Y_P solving the complementary equation.**

4 Family of solution curves

From the previous chapters, we know that there are infinitely many solutions of differential equations (varying the arbitrary constants). Since they all ultimately have similar shapes, we can call them a *family* of solution curves.

4.1 Sketching solution curves

Some questions may ask you to sketch the graph of some kind of solution to a differential equation, possibly with further conditions.

Example 4.1 (NJC 2017 Preliminary P2 Q4)**.** Find the general solution of the differential equation

$$(1+x)y - x\frac{dy}{dx} = x^3 - x^2.$$

Hence sketch and label clearly the equations of 2 **distinct** members of the family of solution curves where their nature of stationary points differ from each other.

Solution. The solution of the equation is $y = x^2 + Cxe^x$. Consider the cases $C = 0$ (yielding a parabola with only one minimum point) and $C = 1$. Your sketch (and your graphing calculator's sketch) should be similar to the one below:

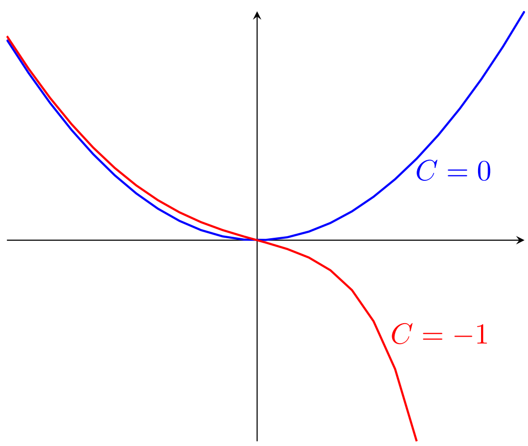

Figure 1: The graph when $C = 0$ and $C = -1$.

As required, the stationary points have different natures. □

4.2 Phase lines

A *phase line* is a vertical line which tells us the gradient of *any* solution curve to a first order *autonomous differential equation* defined as follows:

Definition 4.2. A differential equation of order n in y (where $n \geq 1$ and y is a function of t) is said to be *autonomous* when it can be expressed in the form

$$\frac{d^n y}{dt^n} = f\left(y, \frac{dy}{dt}, ..., \frac{d^{n-1}y}{dt^{n-1}}\right)$$

for any function f. In other words, the differential equation does not explicitly depend on t.

For purposes seen later, we introduce another definition:

Definition 4.3. $c \in \mathbb{R}$ is said to be an *equilibrium point* of an autonomous differential equation if

$$f\left(y, \frac{dy}{dt}, ..., \frac{d^{n-1}y}{dt^{n-1}}\right) = f\left(y, 0, ..., 0\right) = 0$$

when $y = c$. It is also called a 'critical point' and a 'stationary point'.

To simplify our discussion, we only consider first order equations of the form

$$\frac{dy}{dx} = f(y).$$

In order to construct a phase line of an autonomous differential equation, we follow the procedure below:

1. Draw a vertical line.
2. Find the equilibrium points of the differential equation.
3. By drawing filled-in circles on the line to mark all the equilibrium points, separate the line into different regions.
4. For each region where $f(y) > 0$ (y is increasing), draw an arrow pointing upwards.
5. For each region where $f(y) < 0$ (y is decreasing), draw an arrow pointing downwards.

This will help us sketch solution curves because we know about their behavior (and their gradients) in the different regions.

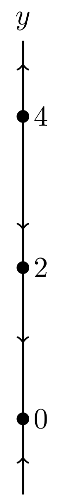

Figure 2: A sample phase line.

Figure 2 is an example of a phase line. The equilibrium points are at $y = 0$, $y = 2$ and $y = 4$.

We can also tell the regions where y is increasing and y is decreasing, and thus sketch a possible graph of y when we put this diagram and the graph of y side-by-side.

Additionally, we also introduce some new terminology that classifies equilibrium points.

Definition 4.4. Consider a first-order autonomous differential equation of the form

$$\frac{dy}{dt} = f(y)$$

and let $y = k$ be a zero of $f(y)$, in other words an equilibrium point of the differential equation. If solution curves on both sides of $y = k$ converge to $y = k$ as $t \to \infty$, $y = k$ is said to be a *stable* equilibrium point. Otherwise, if the solution curves diverges, $y = k$ is said to be *unstable*. If one side of the equilibrium point has solution curves diverging, and the other side has solution curves converging, $y = k$ is said to be *semistable*.

Although we don't know the reasons behind this terminology, we could probably assume that the word *stable* comes from the fact that when we change a particular solution curve slightly, we still converge towards $y = k$. However, when it's *unstable*, the path of the curve significantly changes, as seen below. Finally, *semistable* points may have gotten their name because they possess both properties of stable and unstable equilibrium points to some extent.

The easiest way to classify these equilibrium points is to draw a phase line of some differential equation with equilibrium points. If the arrows point

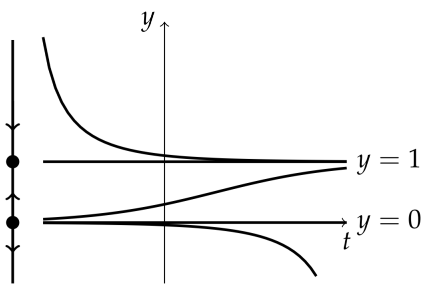

Figure 3: A phase line and the graphs of some solution curves from [1].

away from the region, the point is said to be *unstable*. Otherwise, the point is said to be *stable* when the arrows are pointing into the region. Lastly, the point is *semistable* if one arrow points into the region and the other arrow points away.

Example 4.5. Classify the equilibrium points in Figure 2.

Solution. The point $y = 4$ is unstable, the point $y = 0$ is stable, and the point $y = 2$ is semistable. □

4.3 Slope fields

Besides phase lines, we can obtain information about a first-order differential equation of the form

$$y' = f(x, y)$$

where the differential equation need not be autonomous by a different diagram. Sometimes, it is impossible to explicitly find a solution for y (try solving $y' + y^2 \sin(xy) + 1 = 0$; even WolframAlpha can't). As an alternative to solving this, we can use a *slope field*. By computing $f(x, y)$ at different points, and drawing lines slanted to reflect the gradient, we get a diagram like the one in Figure 4. We now have a diagram which tells us the gradient of *any* solution curve at a particular point, and so we can just trace along the arrows to sketch a solution curve (as seen in red in Figure 4).

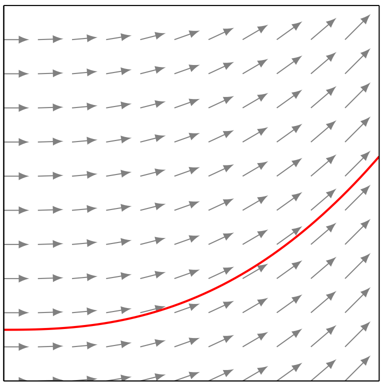

Figure 4: The slope field of $\dfrac{dy}{dx} = x\sqrt{x}$.

5 Population models

Differential equations, as Sophus Lie said, 'furnish the explanation of all those elementary manifestations of nature which involve time'. Indeed, they make for excellent models in predicting how processes evolve as time passes.

One such process is population change (growth or decay). The syllabus discusses two such models: the *exponential growth* model and the *logistic growth* model which builds on the previous model.

5.1 Exponential growth

The exponential growth model appears in the form

$$\frac{dP}{dt} = kP$$

where P is the size of the population, dependent on the time t. This assumes that the rate of change of P is *proportional* to its total population at some particular time t, which is not always the case.

This is a first order autonomous linear differential equation which can be solved by separating variables.

$$\int \frac{1}{P}\,dP = \int k\,dt \implies P = Ce^{kx}$$

When $k > 0$, there is growth, Otherwise, when $k < 0$, P is decreasing and thus there is decay.

Example 5.1 (O-Level Additional Mathematics 2024 *modified*)**.** The manager of a coffee shop purchases a coffee machine for \$1800. The value of the machine \$$V$ can be modelled by a differential equation, which depreciates at a rate with respect to time t in months. The rate of change is proportional to V. When 12 months have passed, the value of the machine has dropped to \$1000. Find V in terms of t.

Solution. Because the coffee machine's value V depreciates at a rate proportional to V, one has $\frac{dV}{dt} = kV$. Obviously $V = Ce^{kt}$ and since the initial value is $V = 1800$ when $t = 0$, $C = 1800$. Now we are left to solve for k, and we leave this to the reader to do so. □

5.2 Logistic growth

Recall the equation $\frac{dP}{dt} = kP$. Although a decent model, it is largely unrealistic for most populations, because they will eventually decrease in growth due to internal competition to survive (with limited resources, assuming they grow in a closed environment). P. F. Verhulst, a Belgian demographer, adapted the exponential growth model to create the *logistic growth* model represented by the differential equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where r is the *growth rate* and K is the *carrying capacity.* Note that as the carrying capacity grows to infinity, one has

$$\lim_{K\to\infty} rP\left(1 - \frac{P}{K}\right) = rP$$

and this reduces to the exponential growth model. However, when K is sufficiently low, we have $\frac{dP}{dt} = rP - \frac{r}{K}P^2$ and $\frac{P^2}{K}$ generally grows faster than P.

This is another first order autonomous linear differential equation which can be solved using the same method to yield

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

where $P(0) = P_0$.

We also observe that

$$\lim_{t\to\infty} P(t) = \begin{cases} \dfrac{KP_0}{P_0 + (K - P_0)(0)} = K & \text{if } r > 0 \\ \\ 0 & \text{if } r < 0 \end{cases}$$

We assume that, of course, $P_0 > 0$.

5.3 Harvesting

Suppose that a house has a bedbug infestation problem, and over a period of time 35 bedbugs were caught. The process of catching these bedbugs is known as *harvesting* (bedbugs).

We can include this concept into our logistic equation by simply adding a term $H(t)$ into the equation. This has a special name:

Definition 5.2. Consider the logistic equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - H(t)$$

where r is the growth rate and K is the carrying capacity. Then $H(t)$ is called the *harvesting term* in the equation.

Normally, we let $H(t)$ be a constant and thus write only H. While solving this equation is doable, most applications of this model involve studying whether the population will reach extinction (i.e. $P = 0$ eventually) or survive. Here, we assume that $r, K > 0$.

Now, suppose we fix r and K such that we only vary H. Then we can find the equilibrium points:

$$rP\left(1 - \frac{P}{K}\right) - H = 0 \implies P = \frac{K}{2} \pm \sqrt{\frac{K^2}{4} - \frac{KH}{r}}$$

Because the coefficient of P on the left-hand side is negative, $\frac{dP}{dt}$ is a quadratic which has a maximum point. Hence as H increases, the graph of $\frac{dP}{dt}$ against P moves down (is translated in the direction of the negative y-axis), and the zeroes of the graph (the equilibrium points) get closer to each other.

In order to find the number of equilibrium points, we can use an expression $rK^2 - 4KH$ in a similar spirit to the discriminant of a quadratic equation. Hence we have the following:

1. When $H = 0$, the equilibrium points are $P = 0$ and $P = K$.
2. When $rK^2 - 4KH > 0$ or $H > \dfrac{rK}{4}$, there are two equilibrium points.
3. When $rK^2 - 4KH < 0$ or $H < \dfrac{rK}{4}$, there are no equilibrium points, because all the roots are complex.

For case 3, H is large enough to ensure that P decreases because $\frac{dP}{dt} < 0$ for all P. Hence P is guaranteed to reach $P = 0$ (i.e. extinction).

References

[1] Russell Herman. *A first course in differential equations for scientists and engineers*. Russell Herman, 2018.

[2] Gilbert Strang. *Calculus*. Wellesley-Cambridge Press, 1991.