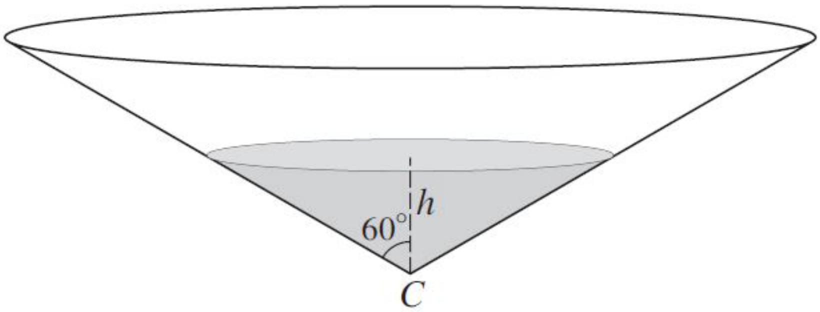
2021 Year 6 H2 Math Preliminary Paper 1: Solutions with Comments

1 Given the polynomial x^4+ax^2+bx+c has a factor $(x-2)$ and gives remainders 12 and 26 when divided by $(x-3)$ and $(x-4)$ respectively, find the values of *a*, *b* and *c*. [4]

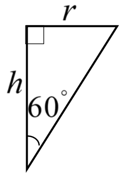
Solution		**Comments**
[4]	$\mathrm{f}(x)=x^4+ax^2+bx+c$ $\mathrm{f}(2)=0 \quad \Rightarrow 16+4a+2b+c=0 \;\Rightarrow 4a+2b+c=-16$ ...(1) $\mathrm{f}(3)=12 \quad \Rightarrow 81+9a+3b+c=12 \;\Rightarrow 9a+3b+c=-69$ ...(2) $\mathrm{f}(4)=26 \quad \Rightarrow 256+16a+4b+c=26 \Rightarrow 16a+4b+c=-230$...(3) Solving (1), (2) and (3), $a=-54,\ b=217,\ c=-234$	The majority of the students did well for this question. However, there were some students who inefficiently did long division to arrive at the 3 equations, with some making errors along the way. This should not be an unfamiliar question as it makes use of factor and remainder theorems which students are assumed to have prior knowledge of.

2



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is 60°, as shown in the diagram. At time $t = 0$, the tank is filled with 94π cm^3 of water. At this instant, a tap at C is turned on and water begins to flow out at a constant rate of 2π cm^{3}s^{-1}. Denoting h cm as the depth of water at time t s, find the rate of decrease of h when $t = 15$, leaving your answer in exact form. [4]

[The volume V of a cone of vertical height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.]

Solution		**Comments**
[4]	Volume of water in the conical container at time t seconds, $V = \frac{1}{3}\pi r^2 h$ $\tan 60^{\circ} = \frac{r}{h} \Rightarrow r = h\sqrt{3}$  Therefore $V = \pi h^3$ $\Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = 3\pi h^2 \frac{\mathrm{d}h}{\mathrm{d}t}$ When t =15, $V = 94\pi - (2\pi)(15) = 64\pi$, $\pi h^3 = 64\pi \Rightarrow h = 4$ $\therefore \frac{\mathrm{d}h}{\mathrm{d}t} = -2\pi \div 3\pi(4)^2 = -\frac{1}{24}$ $\therefore$Rate at which h is decreasing at the instant when t =15 is $\frac{1}{24}$ cms^{-1}.	Most students were able to handle the question with ease as it was something familiar that they had done before with some standard steps outlined in their lecture or revision notes. However, there were students who made the following **common errors**: 1. $h^3 = 64 \Rightarrow h = 8$ 2. $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{-2\pi}{3\pi h^2} = \frac{-2}{3}h^2$ 3. $\frac{\mathrm{d}V}{\mathrm{d}t} = 2\pi$ 4. Differentiating V without realising that both h and r are variables:

		Eg: $V=\frac{1}{3}\pi r^2 h$ $\Rightarrow \frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi r^2$ or $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{2}{3}\pi rh$ 5. Wrote either "$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{24}$" or "Rate of decrease of h $=-\frac{1}{24}$"

3 **(a)** Find $\int x\tan^{-1}x\,\mathrm{d}x$. [3]

(b) **(i)** Using the substitution $u=\frac{1}{x}$, or otherwise, find $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x$. [2]

(ii) Given that n is a positive integer, evaluate the integral $\pi\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x$, giving your answer in the form $a\pi$, where the possible values of a are to be determined. [3]

Solutions		**Comments**
(a) **[3]**	$\int x\tan^{-1}x\,\mathrm{d}x=\left(\frac{x^2}{2}\tan^{-1}x\right)-\int\frac{x^2}{2}\left(\frac{1}{1+x^2}\right)\mathrm{d}x$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int 1-\frac{1}{1+x^2}\,\mathrm{d}x$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\left[x-\tan^{-1}x\right]+c$ $=\frac{x^2+1}{2}\tan^{-1}x-\frac{x}{2}+c$	Most students were able to obtain the 1st line, but a number had difficulty proceeding on.

(b) (i) [2]	Given the substitution $u=\frac{1}{x}$, we have $\frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{1}{x^2}$ $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=-\int \sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\mathrm{d}x$ $=-\int \sin u\,\mathrm{d}u$ $=\cos u+c$ $=\cos\left(\frac{1}{x}\right)+c$ OR $\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=-\int \left(-\frac{1}{x^2}\right)\sin\left(\frac{1}{x}\right)\mathrm{d}x$ $=\cos\left(\frac{1}{x}\right)+c$	Generally this part was ok.
(b) (ii) [3]	$\pi\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=\pi\left[\cos\left(\frac{1}{x}\right)\right]_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}}$ $=\pi\left[\cos(n\pi)-\cos((n+1)\pi)\right]$ If n is even, $\pi\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=\pi\left[1-(-1)\right]=2\pi \quad a=2$ If n is odd, $\pi\int_{\frac{1}{(n+1)\pi}}^{\frac{1}{n\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x=\pi\left[-1-1\right]=-2\pi \quad a=-2$ *OR* $a=2(-1)^n$	Quite a number of students wrote down $a=\pm 2$.

4 **(a)** Show that $(2r+1)^3-(2r-1)^3=12kr^2+k$, where k is a constant to be determined.

Use this result to find $\sum_{r=1}^{n} r^2$, giving your answer in the form $pn(qn+1)(2qn+1)$ where p and q are constants to be determined. [5]

(b) Raabe's test states that a series of positive terms of the form $\sum_{r=1}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]>1$, and diverges when $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]<1$.

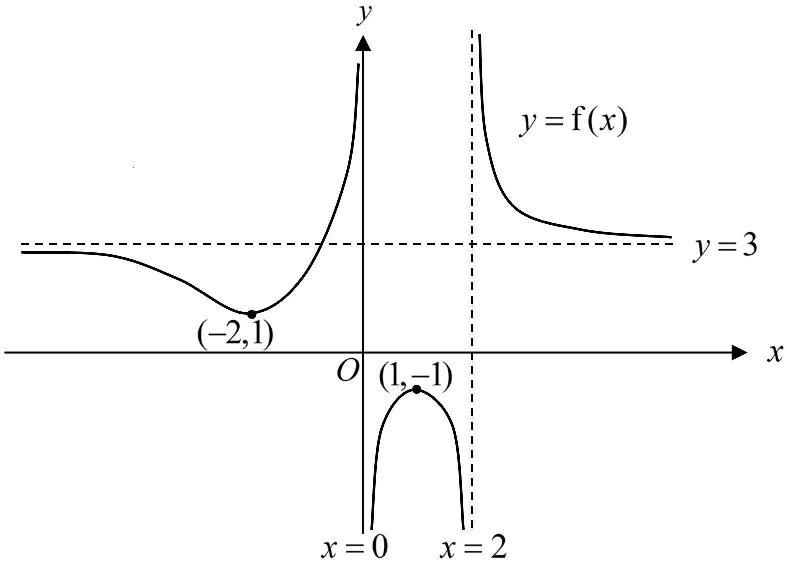
When $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right]=1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=1}^{\infty}\frac{1}{r^3}$ converges. [3]

Solutions		**Comments**
(a) [5]	$(2r+1)^3-(2r-1)^3$ $=\left[8r^3+12r^2+6r+1\right]-\left[8r^3-12r^2+6r-1\right]$ $=24r^2+2$ $\therefore k=2$ OR $(2r+1)^3-(2r-1)^3$ $=\left[(2r+1)-(2r-1)\right]\left[(2r+1)^2+(2r+1)(2r-1)+(2r-1)^2\right]$ $=2\left[4r^2+4r+1+4r^2-1+4r^2-4r+1\right]$ $=2\left(12r^2+1\right)$ $=24r^2+2$ $\therefore k=2$	Very well done.

	$\sum_{r=1}^{n}\left(24r^2+2\right)=\sum_{r=1}^{n}\left(\left(2r+1\right)^3-\left(2r-1\right)^3\right)$ $\begin{aligned}24\sum_{r=1}^{n}r^2+2\sum_{r=1}^{n}1 &= 3^3 - 1^3\\ &\quad +5^3 - 3^3\\ &\quad +7^3 - 5^3\\ &\quad + \ \ldots\\ &\quad +\left(2n-1\right)^3-\left(2n-3\right)^3\\ &\quad +\left(2n+1\right)^3-\left(2n-1\right)^3\\ &= \left(2n+1\right)^3 - 1\end{aligned}$ $24\sum_{r=1}^{n}r^2+2n=\left(2n+1\right)^3-1$ $\begin{aligned}\sum_{r=1}^{n}r^2 &= \frac{1}{24}\left[\left(2n+1\right)^3-\left(2n+1\right)\right]\\ &= \frac{\left(2n+1\right)}{24}\left[\left(2n+1\right)^2-1\right]\\ &= \frac{\left(2n+1\right)}{24}\left(2n+2\right)\left(2n\right)\\ &= \frac{\left(2n+1\right)}{24}\left(2n+2\right)\left(2n\right)\\ &= \frac{1}{6}n\left(n+1\right)\left(2n+1\right)\\ &\therefore p=\frac{1}{6},\quad q=1\end{aligned}$	Generally well done. Some common mistakes include having cubes of even numbers in the cancellation and wrongly concluding $\sum_{r=1}^{n}2=2$.

(b) **[3]**	Let $a_n = \frac{1}{n^3}$. $n\left(\frac{a_n}{a_{n+1}}-1\right) = n\left[\frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{(n+1)^3}\right)}-1\right]$ $= n\left[\frac{(n+1)^3}{n^3}-1\right]$ $= n\left(\frac{n^3+3n^2+3n+1-n^3}{n^3}\right)$ $= \frac{3n^2+3n+1}{n^2}$ $= 3+\frac{3}{n}+\frac{1}{n^2}$ $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left(3+\frac{3}{n}+\frac{1}{n^2}\right) = 3 > 1$. Therefore $\sum_{r=1}^{\infty}\frac{1}{r^3}$ converges.	Many students attempted to show that the expression $n\left(\frac{a_n}{a_{n+1}}-1\right) > 1$ and hence the limit is more than 1. However, this is not true in general. Some solutions had the n and r mixed up like $n\left(\frac{\frac{1}{r^3}}{\frac{1}{(r+1)^3}}-1\right)$. Better solutions come from those who worked out the value of the limit which is 3 by considering $n \to \infty$

5 **(a)** The graph of $y=\mathrm{f}(x)$ is shown below.

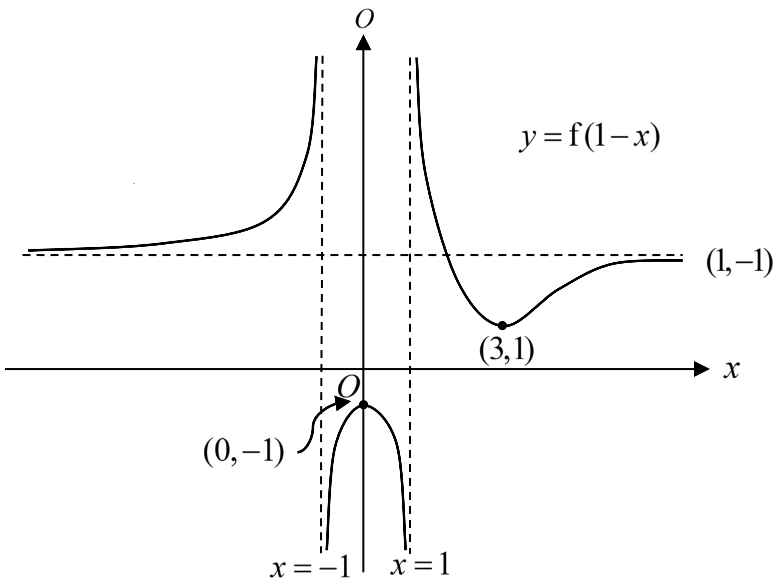


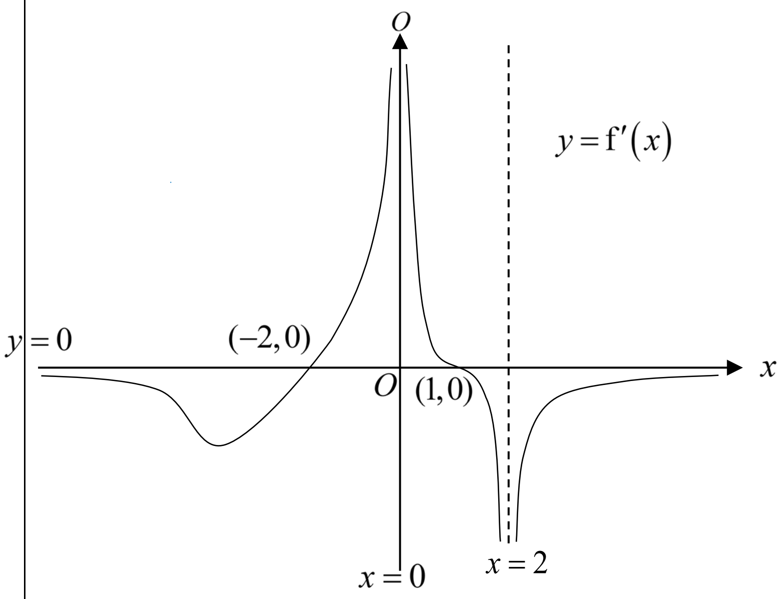
On separate diagrams, sketch the following graphs, indicating clearly the key features.

(i) $y=\mathrm{f}(1-x)$, [3]

(ii) $y=\mathrm{f}'(x)$. [3]

(b) State a sequence of transformations which transform the graph of $y=\ln\left(1-\frac{x}{2}\right)$ onto the graph of $y=\ln\left(\frac{2}{1-x}\right)$. [3]

Solutions		**Comments**
(a) **(i)** **[3]**	 	Most students correctly identify the two transformations involved, translation and reflection. However, a number of them did not transform the graph using the **correct sequence**. The correct method involves either “a translation of 1 unit in the **negative** *x*-direction followed by a reflection in ***y*-axis**”

		or "a reflection in ***y*-axis** followed by a translation of 1 unit in the **positive** *x*-direction". Some did not translate the graph in the correct direction or reflected in the wrong axis.
(a) (ii) [3]		Most students managed to derive the correct asymptotes and *x*-intercepts. However, a number of them drew part(s) of the graph in the wrong region(s) or had difficulties deriving the minimum turning point. Students need to label all the asymptotes including the axes.
(b) [3]	$y=\ln\left(1-\frac{x}{2}\right)\to y=\ln(1-x)\to y=-\ln(1-x)=\ln\left(\frac{1}{1-x}\right)$ $\to y=\ln\left(\frac{1}{1-x}\right)+\ln 2=\ln\left(\frac{2}{1-x}\right)$ Scale the graph of $y=\ln\left(1-\frac{x}{2}\right)$ parallel to the *x*-axis by factor $\frac{1}{2}$, followed by a reflection about the ***x*-axis**, followed by a translation of ln 2 units in the positive *y*-direction.	Most students managed to work out the correct "replacement" using algebraic manipulations but interpreted the corresponding transformation wrongly. A number of them used much longer method. Take note that students need to use

	OR $y=\ln\left(1-\frac{x}{2}\right)=\ln\left(\frac{2-x}{2}\right)\rightarrow y=-\ln\left(\frac{2-x}{2}\right)=\ln\left(\frac{2}{2-x}\right)$ $\rightarrow y=\ln\left(\frac{2}{1-x}\right)$ Reflect the graph of $y=\ln\left(1-\frac{x}{2}\right)$ about the ***x*-axis**, followed by a translation of 1 unit in the **negative** x-direction.	the correct terms/phrases to describe the transformations.

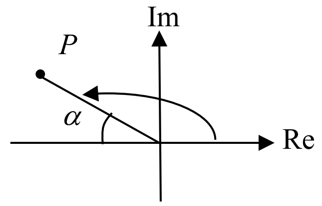
6 Do not use a calculator in answering this question.

(a) Show that $z = 2\text{i}$ is a root of the equation $z^3 + 2z + 4\text{i} = 0$. [2]
Hence find the other roots. [3]

(b) Let $w_1 = -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}\text{i}$ and $w_2 = 1 + \text{i}$.

Find the smallest positive integer n such that $\arg\left(\frac{w_2}{w_1}\right)^n = -\frac{\pi}{2}$. [4]

Solutions		**Comments**
(a) **[2]**	Let $\text{f}(z) = z^3 + 2z + 4\text{i}$ $\text{f}(2\text{i}) = (2\text{i})^3 + 2(2\text{i}) + 4\text{i}$ $= 8\text{i}^3 + 4\text{i} + 4\text{i}$ $= -8\text{i} + 8\text{i}$ $= 0$ $z = 2\text{i}$ is a root of the equation $z^3 + 2z + 4\text{i} = 0$.	As the question indicates a calculator is not to be used, all working and calculations need to be shown clearly. For (a), you should show the steps as shown in the solution.
[3]	$z^3 + 2z + 4\text{i} = 0$ $(z - 2\text{i})(z^2 + 2\text{i}z - 2) = 0$ $z = 2\text{i}$ or $z = \frac{-2\text{i} \pm \sqrt{-4+8}}{2}$ $z = 2\text{i}$ or $z = -\text{i} \pm 1$ The other roots are $1 - \text{i},\ -1 - \text{i}$.	Do note that the equation $z^3 + 2z + 4\text{i} = 0$ does not have real coefficients (4i). Hence to obtain the other 2 roots (cubic equation has 3 roots), you should do long division or compare coefficients using a quadratic factor. Splitting into 3 linear factors is tedious and unnecessary. After obtaining the quadratic equation with complex coefficients, you can complete squares or simply use the quadratic formula.
(b) **[4]**	We have $\arg(w_1) = \frac{5\pi}{6}$ and $\arg(w_2) = \frac{\pi}{4}$.	Please be careful as many swapped the two complex numbers w_1, w_2. Many students still cannot find the argument of a complex number not in the first quadrant (in this case w_1) correctly.

	Hence $\arg\left(\frac{w_2}{w_1}\right)=\arg(w_2)-\arg(w_1)$ $=\frac{\pi}{4}-\frac{5\pi}{6}$ $=-\frac{7\pi}{12}$ $\arg\left(\frac{w_2}{w_1}\right)^n=n\arg\left(\frac{w_2}{w_1}\right)$ $=-\frac{7n\pi}{12}$ Hence we need to find the least positive integer n such that $\frac{-7n\pi}{12}=-\frac{\pi}{2}+m(2\pi)=\frac{(4m-1)\pi}{2}$, $m\in\mathbb{Z}$. Rearranging, $n=\frac{6-24m}{7}=\frac{6(1-4m)}{7}$. **Method 1** Therefore we need to have an integer m such that $1-4m$ is positive (and thus negative m) and a multiple of 7. Checking through the negative integer values of m, we have $1-4m=5,9,13,17,\underline{21},\ldots$. The corresponding least value of n is therefore 18.	For those who made this mistake, please follow the steps below: 1) Sketch quickly an argand diagram and locate the point representing the complex number w_1. It is in the 2nd quadrant.  arg(z) = $\pi-\alpha$ 2) Find the basic angle $\tan\alpha=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{6}}{2}}=\frac{1}{\sqrt{3}}\ \Rightarrow\alpha=\frac{\pi}{6}$ 3) $\arg(w_1)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$. Note that since we are looking for the argument to be $-\frac{\pi}{2}$, the possible candidates are all angles that differ from it by a multiple of 2π. As the question said a calculator is not to be used, to justify that the smallest n is 18, you will need to explain why all the smaller n cannot work with clear working. (someone listed 18 such n, which while correct, is time consuming) Excellent responses often used some divisibility argument to reduce the cases to check, and subsequently calculating for each of the remaining cases the corresponding n, verifying clearly that they are **all** not integers.

7	**Method 2** m | $n=\dfrac{6(1-4m)}{7}$ -1 | $\dfrac{30}{7}$ -2 | $\dfrac{54}{7}$ -3 | $\dfrac{78}{7}$ -4 | $\dfrac{102}{7}$ -5 | 18 Hence smallest $n=18,$ corresponding to when $m=-5$.	

m	$n=\dfrac{6(1-4m)}{7}$
-1	$\dfrac{30}{7}$
-2	$\dfrac{54}{7}$
-3	$\dfrac{78}{7}$
-4	$\dfrac{102}{7}$
-5	18

7 A curve C has parametric equations

$$x=\sin t\ ,\quad y=\frac{1}{3}\cos t\ ,\ \text{ for } -\pi \le t \le \frac{\pi}{4}.$$

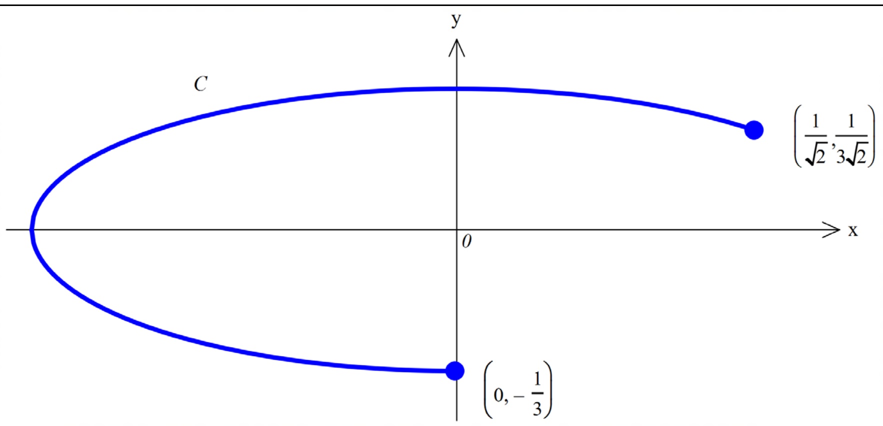
(i) Find the equation of the normal to C at the point P with parameter $t=p$. [3]

(ii) The normal to C at the point when $t=-\dfrac{\pi}{4}$ cuts the curve again at point A. Find the coordinates of point A, correct to 2 decimal places. [4]

(iii) Sketch the graph of C, giving the coordinates of the end points in exact form. [2]

(iv) Find the area of the region bounded by C, the x-axis and the lines $x=0$ and $x=\dfrac{1}{\sqrt{2}}$. [2]

Solutions		**Comments**
(i) **[3]**	$\dfrac{dx}{dt}=\cos t,\quad \dfrac{dy}{dt}=-\dfrac{1}{3}\sin t$ $\dfrac{dy}{dx}=-\dfrac{1}{3}\tan t$ At P, $\dfrac{dy}{dx}=-\dfrac{1}{3}\tan p$ Gradient of normal $=3\cot p$ Equation of normal at P: $y-\dfrac{1}{3}\cos p=(3\cot p)(x-\sin p)$ $y=(3\cot p)x-\dfrac{8}{3}\cos p$	Almost all candidates were able to differentiate the parametric equations to find $\dfrac{dy}{dx}$ but a significant number did not arrive at the correct expression for the gradient of the normal, either overlooking the negative sign in the relationship between the gradients of the tangent and normal or

<table>
<tr><td></td><td></td><td>writing $-\dfrac{1}{\left(-\dfrac{1}{3}\tan t\right)}$ as $3\tan t$.</td></tr>
<tr><td>(ii)
[4]</td><td>When $t=-\dfrac{\pi}{4}$, equation of normal: $y=(-3\cot\dfrac{\pi}{4})x-\dfrac{8}{3}\cos\dfrac{\pi}{4}$
$y=-3x-\dfrac{8}{3\sqrt{2}}$ ------ (*)
For normal to cut C again, substitute $x=\sin t$, $y=\dfrac{1}{3}\cos t$ into (*)
$\dfrac{1}{3}\cos t=-3\sin t-\dfrac{8}{3\sqrt{2}}$
$\dfrac{1}{3}\cos t+3\sin t=-\dfrac{8}{3\sqrt{2}}$
From GC, $t=-2.5775$ (4 d.p.)
$x=\sin(-2.5775)=-0.53$ (2 d.p.)
$y=\dfrac{1}{3}\cos(-2.5775)=-0.28$ (2 d.p.)
The coordinates of point A is $(-0.53, -0.28)$.</td><td>Those candidates who answered this part well, usually approached it with the method as given in the solution, although there were some attempts which did not take into account the given range of values of t when solving.</td></tr>
<tr><td>(iii)
[2]</td><td>
</td><td>Most candidates realised they had to adjust the window settings of the graphing calculator in order to obtain the part of the ellipse represented by the parametric equations, corresponding to the given range of values of t.</td></tr>
<tr><td>(iv)
[2]</td><td>Area of required region
$=\int_0^{\frac{1}{\sqrt{2}}} y\ \mathrm{d}x$
$=\int_0^{\frac{\pi}{4}} \dfrac{1}{3}\cos t\dfrac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t$
$=\dfrac{1}{3}\int_0^{\frac{\pi}{4}} \cos^2 t\ \mathrm{d}t$
$=0.214$ units^2 (3sf)</td><td>Most candidates used the parametric equations to find the area, but not all correctly substituted $\dfrac{\mathrm{d}(\sin t)}{\mathrm{d}t}\mathrm{d}t$ for $\mathrm{d}x$. The most common incorrect answers involved candidates evaluating the integrals without converting the limits.</td></tr>
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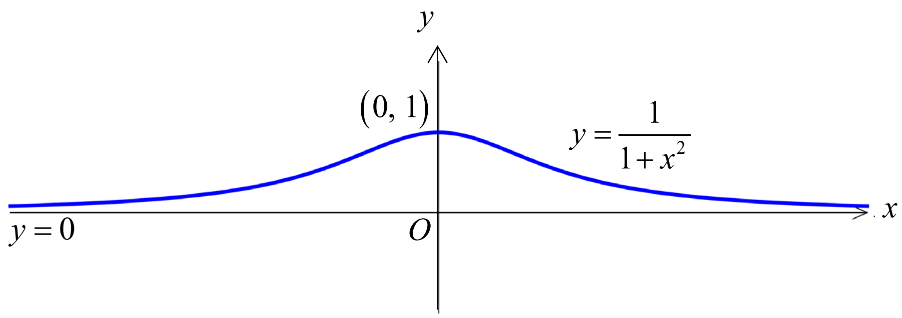
8 **(i)** The curve G has equation $y=\dfrac{1}{1+x^2}$. Sketch the graph of G, stating the equation(s) of any asymptote(s) and the coordinates of any turning point(s). [2]

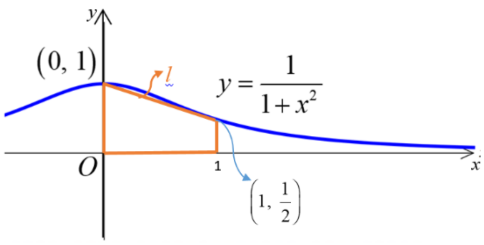
(ii) The line l intersects G at $x=0$ and is tangential to G at the point (c,d), where $c>0$. Find c and d, and determine the equation of l. [4]

Let R denote the region bounded by G, the x-axis and the lines $x=0$ and $x=1$.

(iii) By comparing the area of R and the area of the trapezoidal region between l and the x-axis for $0\le x\le 1$, show that $\pi>3$. [2]

(iv) By considering the volume of revolution of a suitable region rotated through 2π radians about the y-axis, show that $\ln 2>\dfrac{2}{3}$. [3]

Solutions		**Comments**
(i) [2]		Almost all the students are able to get this part correctly. Only a handful of students wrongly labelled the asymptote as $x=0$.
(ii) [4]	Now, $y=\dfrac{1}{1+x^2}\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-2x}{\left(1+x^2\right)^2}$. At (c,d), we have $d=\dfrac{1}{1+c^2}$ and $\dfrac{d-1}{c-0}=\dfrac{-2c}{\left(1+c^2\right)^2}\Rightarrow d=\dfrac{-2c^2}{\left(1+c^2\right)^2}+1$. Hence we have	Most students are able to do this part.

	$\frac{-2c^2}{\left(1+c^2\right)^2}+1=\frac{1}{1+c^2}$ $\Rightarrow -2c^2+\left(1+c^2\right)^2=\left(1+c^2\right)$ $\Rightarrow c^4-c^2=0$ $\Rightarrow c^2\left(c^2-1\right)=0$ $\Rightarrow c=0 \text{ or } c=\pm 1.$ Since $c>0$, $c=1$ and $d=\frac{1}{2}$. Equation of l is $y-\frac{1}{2}=\frac{-2(1)}{\left(1+1^2\right)^2}(x-1) \Rightarrow y=-\frac{x}{2}+1$.	
(iii) **[2]**	Area of $R=\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x=\left[\tan^{-1}x\right]_0^1=\frac{\pi}{4}$. Area of $R>$ Area of trapezium $\Rightarrow \frac{\pi}{4}>\frac{1}{2}\left(1+\frac{1}{2}\right)(1)=\frac{3}{4}$ $\Rightarrow \pi>3$ (Shown).  Note: Besides using the formula for area of trapezium, we can also use the following: $\int_0^1 -\frac{x}{2}+1\,\mathrm{d}x=\frac{3}{4}$.	Students need to realize that they need to show $\pi>3$, hence the answer for area must be exact.
(iv) **[3]**	$\pi\int_{\frac{1}{2}}^1 x^2\,\mathrm{d}y=\pi\int_{\frac{1}{2}}^1 \frac{1}{y}-1\,\mathrm{d}y$ $=\pi\left[\ln y-y\right]_{\frac{1}{2}}^1$ $=\pi\left[\left(\ln 1-1\right)-\left(\ln\frac{1}{2}-\frac{1}{2}\right)\right]$ $=\pi\left(-\ln\frac{1}{2}-\frac{1}{2}\right)=\pi\left(\ln 2-\frac{1}{2}\right)$.	Students who draw a diagram for (iii) will be able to see which volume to compare for (iv).

	Now, Volume obtained > Volume of cone with radius 1 and height $\frac{1}{2}$ $\Rightarrow \pi\int_{\frac{1}{2}}^{1} x^2 \ \mathrm{d}y > \frac{1}{3}\pi\left(1^2\right)\left(\frac{1}{2}\right)$ $\Rightarrow \pi\left(\ln 2 - \frac{1}{2}\right) > \frac{\pi}{6}$ $\Rightarrow \ln 2 > \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$ (shown). Note: Besides using the formula for volume of cone, can also consider $\pi\int_{\frac{1}{2}}^{1}\left(2-2y\right)^2 \ \mathrm{d}y = \frac{\pi}{6}$.	

9 The equations of a plane p_1 and a line l are shown below:

$$p_1: \quad \mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 10,$$

$$l: \quad \frac{x+1}{3} = z+4, \ y = 1.$$

Referred to the origin O, the position vector of the point A is $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$.

(i) Find the coordinates of the foot of perpendicular, N, from A to p_1. [4]

(ii) Find the position vector of the point B which is the reflection of A in p_1. [2]

(iii) Hence, or otherwise, find an equation of the line l', the reflection of l in p_1. [4]

(iv) Another plane, p_2, contains B and is parallel to p_1. Determine the exact distance between p_1 and p_2. [2]

Solutions		**Comments**
(i) **[4]**	Where $l_{AN}: \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ intersects $p_1: \mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 10$, $\left[\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 10$ $(2-1-3)+\lambda(1+1+1)=10 \Rightarrow 3\lambda = 12 \quad \Rightarrow \quad \lambda = 4$ $\overrightarrow{ON} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$ The coordinates of N are $(6,-3,1)$.	Most students did well for this question except some misinterpreted $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \overrightarrow{AN}$ instead of $\overrightarrow{ON}$ and did not indicate final answer in coordinates form.
(ii) **[2]**	$\overrightarrow{OB} = \overrightarrow{OA} + 2\overrightarrow{AN}$ $= \overrightarrow{OA} + 2\overrightarrow{ON} - 2\overrightarrow{OA}$ $= 2\overrightarrow{ON} - \overrightarrow{OA}$ $= 2\begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ -7 \\ 5 \end{pmatrix}$	Most students were able to either apply ratio theorem or use alternative vector form to find $\overrightarrow{OB}$, but were careless when computing the final answer.
(iii) **[4]**	Equation of line $l: \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -4 \end{pmatrix} + s\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}$	Most students were able to compute the equation of line l except those who made mistakes in identifying

	When $s=1$, $\mathbf{r}=\begin{pmatrix}-2\\1\\-3\end{pmatrix}=\overrightarrow{OA}$, so A lies on l Let point of intersection of l and p_1 be X. When l intersects p_1, $\left[\begin{pmatrix}-1\\1\\-4\end{pmatrix}+s\begin{pmatrix}3\\0\\1\end{pmatrix}\right].\begin{pmatrix}1\\-1\\1\end{pmatrix}=10 \Rightarrow s=4$ $\overrightarrow{OX}=\begin{pmatrix}11\\1\\0\end{pmatrix}$ Equation of reflected line $l':\mathbf{r}=\begin{pmatrix}10\\-7\\5\end{pmatrix}+\mu\left(\begin{pmatrix}10\\-7\\5\end{pmatrix}-\begin{pmatrix}11\\1\\0\end{pmatrix}\right)$ $l':\mathbf{r}=\begin{pmatrix}10\\-7\\5\end{pmatrix}+\mu\begin{pmatrix}-1\\-8\\5\end{pmatrix}, \quad \mu\in\mathbb{R}$	$\begin{pmatrix}-1\\1\\-4\end{pmatrix}$ and $\begin{pmatrix}3\\0\\1\end{pmatrix}$. Some students failed to verify point A lies on line l which is essential in using point B to find direction vector in the given solution. Some students went through part (i) and (ii) approach to find the direction vector of l' which is lengthy and prone to careless mistake in computation.
(iv) **[2]**	Distance between p_1 and $p_2=\frac{1}{2}AB=\frac{1}{2}\left\|\begin{pmatrix}8\\-8\\8\end{pmatrix}\right\|=4\sqrt{3}$ OR Distance between p_1 and $p_2=BN=AN=\left\|4\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\|=4\sqrt{3}$ OR $p_2:\mathbf{r}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}10\\-7\\5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=22$ Distance between p_1 and $p_2=\frac{22-10}{\sqrt{1^2+1^2+1^2}}=\frac{12}{\sqrt{3}}=4\sqrt{3}$ OR Distance between p_1 and p_2 $=\frac{\left\|\overrightarrow{XB}.\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\|}{\sqrt{1^2+1^2+1^2}}=\frac{\left\|\begin{pmatrix}-1\\-8\\5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\|}{\sqrt{1^2+1^2+1^2}}=\frac{12}{\sqrt{3}}=4\sqrt{3}$	Most students were able to compute the distance either via *BN* and *AN*. There were some who applied length of projection onto normal vector which is acceptable as well.

10 Bob purchases a house and takes a loan of \$$A$ from a bank. The sum of money owed to the bank t months after taking the loan is denoted by \$$x$. Both x and t are taken to be continuous variables. The sum of money owed to the bank increases, due to interest, at a rate proportional to the sum owed and decreases at a constant rate r as Bob repays the bank.
When $x = a$, interest and repayment balance. Write down a differential equation relating x and t, and solve it to give x in terms of t, r, a and A. [8]

State the condition under which the sum of money owed to the bank is repaid in a finite time T months, justifying your answer. Show that $T = \frac{a}{r}\ln\left(\frac{a}{a-A}\right)$. [4]

Solutions		Comments
[8]	Since the sum of money owed to the bank increases at a rate proportional to the sum owed and Bob repays the bank at a constant rate r, $\frac{dx}{dt} = kx - r$, where $k > 0$. When $x = a$, interest and repayment balance. Then $\frac{dx}{dt} = 0 = ka - r \Rightarrow k = \frac{r}{a}$ Therefore $\frac{dx}{dt} = \frac{r}{a}(x) - r = \frac{r}{a}(x-a)$	Most students were able to form the differential equation correctly. It is recommended to find k using the information that $\frac{dx}{dt} = 0$ at $x = a$ and simplify the differential equation before solving it.
	$\frac{dx}{dt} = \frac{r}{a}(x-a)$ $\int \frac{1}{x-a}\,dx = \int \frac{r}{a}\,dt$ $\ln\lvert x-a\rvert = \frac{rt}{a} + C \quad , C \in \mathbb{R}$ $\lvert x-a\rvert = e^{\frac{rt}{a}}e^{C}$ $x - a = Be^{\frac{rt}{a}} \quad$ where $B = \pm e^{C}$ $x = Be^{\frac{rt}{a}} + a$ When $t = 0, x = A$, $A = B + a \Rightarrow B = A - a$ $x = (A-a)e^{\frac{rt}{a}} + a$	A handful of students forgot to include the absolute sign on $x-a$ in the natural log when performing integration on LHS, or failed to consider $\pm$ when taking exponential on both side in the next line. There were some students who inappropriately used A to denote the arbitrary constant, overlooking the fact that the letter is used to denote the amount of loan \$$A$ in the question. This often resulted in the solution which is in terms of A to be wrong.

[4]	For the loan to be repaid in a finite time T, $x=\left(A-a\right)\mathrm{e}^{\frac{rt}{a}}+a$ must be a decreasing function as t increases. So $A-a<0 \Rightarrow A<a$ When the loan is repaid, $x = 0$. $0=\left(A-a\right)\mathrm{e}^{\frac{rT}{a}}+a$ $\Rightarrow \dfrac{a}{a-A}=\mathrm{e}^{\frac{rT}{a}}$ $\Rightarrow T=\dfrac{a}{r}\ln\left(\dfrac{a}{a-A}\right)$ (Shown)	Even though most students were able to state $x=0$ when loan is repaid, many were not able to prove the statement because they did not obtain the correct solution from the previous part.

11 $\left[\text{It is given that the volume of a sphere of radius } R \text{ is } \dfrac{4}{3}\pi R^3.\right]$

Craft drinks have been gaining popularity in the beverage industry in recent years. These drinks are usually freshly made and served cold, with much attention given to the ingredients that make up the drinks and the entire process of preparation.

Ice is a very important ingredient in the making of a craft drink as it affects two crucial components: the temperature and the dilution of the drink. Hence, great emphasis is placed on the shapes of the ice, as different shapes will offer different surface areas and thus have a direct impact on the taste of the drink.

An ice manufacturer, who specialises in producing cylindrical shaped ice suitable for craft drinks served in tall glasses, wants to find out information about the surface area of the cylindrical shaped ice he produces.

(i) A piece of cylindrical shaped ice has radius r, height h and a fixed volume V. Show that its surface area, S, is given by $2\pi r^2 + \dfrac{2V}{r}$. [2]

(ii) Use differentiation to find, in terms of V, the minimum value of S, proving that it is a minimum. You are to give your answer in the form $k\left(m\pi V^m\right)^{\frac{1}{k}}$, where k and m are positive integers to be found. Find also the ratio $r : h$ that gives this minimum value of S. [7]

There has been a growing trend to use one large piece of ice for craft drinks to create a better drinking experience for the customers. Spherical shaped ice is considered ideal as it can keep the drink at a constant cold temperature with minimal dilution.

(iii) For the minimum value of S found in part **(ii)**, show that the volume of the largest spherical shaped ice that can be carved out is $\dfrac{m}{k}V$, where k and m are the same integers found in part **(ii)**. [2]

(iv) State, giving a reason, whether the manufacturer should proceed to carve out spherical shaped ice from the existing cylindrical shaped ice produced. [1]

Solutions		**Comments**
(i) **[2]**	$V=\pi r^2 h \Rightarrow h=\dfrac{V}{\pi r^2}$. Now, $S=2\pi r^2+2\pi rh$ $=2\pi r^2+2\pi r\left(\dfrac{V}{\pi r^2}\right)$ $=2\pi r^2+\dfrac{2V}{r}$ (Shown).	Most students did well for part (i) and were able to prove the result. However some students did not show the substitution steps in detail.
(ii) **[7]**	$S=2\pi r^2+\dfrac{2V}{r} \Rightarrow \dfrac{\mathrm{d}S}{\mathrm{d}r}=4\pi r-\dfrac{2V}{r^2}$. $\dfrac{\mathrm{d}S}{\mathrm{d}r}=0 \Rightarrow 4\pi r-\dfrac{2V}{r^2}=0$ $\Rightarrow 4\pi r^3-2V=0$ $\Rightarrow r=\left(\dfrac{V}{2\pi}\right)^{\frac{1}{3}}$. $\dfrac{\mathrm{d}^2S}{\mathrm{d}r^2}=4\pi+\dfrac{4V}{r^3} \Rightarrow \left.\dfrac{\mathrm{d}^2S}{\mathrm{d}r^2}\right\|_{r=\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}}=4\pi+\dfrac{4V}{\left(\frac{V}{2\pi}\right)}=12\pi>0.$ So S is minimum when $r=\left(\dfrac{V}{2\pi}\right)^{\frac{1}{3}}$. $\therefore S=2\pi r^2+\dfrac{2V}{r}$ $=2\pi\left(\dfrac{V}{2\pi}\right)^{\frac{2}{3}}+\dfrac{2V}{\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}}$ $=(2\pi)^{\frac{1}{3}}V^{\frac{2}{3}}+2^{\frac{4}{3}}\pi^{\frac{1}{3}}V^{\frac{2}{3}}$ $=\left(2\pi V^2\right)^{\frac{1}{3}}(1+2)$ $=3\left(2\pi V^2\right)^{\frac{1}{3}}$, where $k=3$ and $m=2$. When $r=\left(\dfrac{V}{2\pi}\right)^{\frac{1}{3}}$, $\dfrac{r}{h}=\dfrac{r}{\left(\frac{V}{\pi r^2}\right)}=\dfrac{\pi r^3}{V}=\dfrac{\pi\left(\frac{V}{2\pi}\right)}{V}=\dfrac{1}{2}$.	Most students were able to perform differentiation well and find the minimum value of S and r. However, there were students who made the following common errors: 1. Letting $\dfrac{\mathrm{d}V}{\mathrm{d}r}=0$ or $\dfrac{\mathrm{d}S}{\mathrm{d}V}=0$ instead of $\dfrac{\mathrm{d}S}{\mathrm{d}r}=0$, when V is already stated clearly in the question as a **fixed** volume. 2. A handful of students did not prove that S is minimum by either 2nd or 1st derivative test. 3. Among students who used 1st derivative test to prove min, many did not factorize or break down $\dfrac{\mathrm{d}S}{\mathrm{d}r}$ before concluding that S is a min 4. Quite a handful of students did not state the final values of k and m Majority of students had difficulty finding the ratio of

	Therefore $r : h = 1: 2$.	$r : h$. Among those who found it correctly, many translated to the wrong ratio. Eg. $h=2r$ should obtain a ratio of $r : h = 1: 2$, however it was written as $r : h = 2: 1$. Many students gave the final answer in fraction form too when the question requested for ratio form.
(iii) **[2]**	Largest spherical shaped ice that can be carved out has radius, $R = r = \left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$, since when S is minimum, $h = 2r$. Hence the volume of the largest spherical ice is $\frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left(\frac{V}{2\pi}\right) = \frac{2V}{3}$ (Shown).	Students who attempted part (ii) correctly were mostly able to identify that $R = r = \left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$ before finding the volume of the largest spherical ice. However, majority of students failed to give a proper justification on why $R = r = \left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$ and show the relation between the spherical shaped ice which will be carved out from the existing cylindrical shaped ice. Some students also had the **misconception** that the spherical shaped ice shares the same surface area as the cylindrical shaped ice, which led to students equating the two surface areas to find the volume.
(iv) **[1]**	No, the manufacturer should not proceed as the spherical shaped ice has volume at least $\frac{2}{3}V$ and so $\frac{1}{3}$ of the volume of the cylindrical shaped ice will go to waste which is quite a lot. OR Yes, the manufacturer should proceed even though the spherical shaped ice has volume at least $\frac{2}{3}V$ as the $\frac{1}{3}V$	Most students were unable to give a reason on why the manufacturer should proceed to carve out spherical shaped ice from the existing cylindrical shaped ice produced. Students should note that the reason is to be given from the ice manufacturer's viewpoint,

	of crushed ice that is leftover during carving can be used for other drinks which require crushed ice.	NOT from the beverage industry. Some common misconceptions include students giving reasons that less ice will make the drink less diluted etc. However, the dilution of the drink is not of any concern to the ice manufacturer as they do not produce the drink. Another misconception which students have is that since the spherical shaped ice is smaller than the cylindrical shaped ice, the manufacturer will be able to save cost by producing less ice. However, it is not the case as the spherical shaped ice will STILL have to be carved out from the existing cylindrical shaped ice, which means that the manufacturer will have to produce the cylindrical shaped ice in full first before carving out the smaller spherical shaped ice.