H2 Further Mathematics

2023 JPJC JC2 Prelim Examination Paper 1 Solutions

1

(a) $\mathbf{Ax}=\lambda\mathbf{x}$ and $\mathbf{Bx}=\mu\mathbf{x}$.

$$\begin{aligned}(\mathbf{A}+\mathbf{B})\mathbf{x} &= \mathbf{Ax}+\mathbf{Bx}\\ &= \lambda\mathbf{x}+\mu\mathbf{x}\\ &= (\lambda+\mu)\mathbf{x}\end{aligned}$$

$\therefore\ \lambda+\mu$ is an eigenvalue of $\mathbf{A}+\mathbf{B}$ with **x** as a corresponding eigenvector. (Shown)

(b)

$$\begin{aligned}\mathbf{Ax} &= \lambda\mathbf{x}\\ \begin{pmatrix}3 & -1 & 0\\ -4 & -6 & -6\\ 5 & 11 & 10\end{pmatrix}\begin{pmatrix}1\\ -1\\ 1\end{pmatrix} &= \lambda\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}\\ \lambda &= \underline{\underline{4}}\text{ (satisfying all 3 eqns)}\end{aligned}$$

(c)

	Eigenvalue		
Eigenvector	**A**	**B**	$\mathbf{A}+\mathbf{B}$
$\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$	4	2	6
$\begin{pmatrix}1\\ 2\\ -3\end{pmatrix}$	1	3	4
$\begin{pmatrix}1\\ 1\\ -2\end{pmatrix}$	2	1	3

Since $\mathbf{A}+\mathbf{B}$ is a 3×3 matrix with 3 distinct eigenvalues, so $\mathbf{A}+\mathbf{B}$ is diagonalisable.

$$\mathbf{A}+\mathbf{B} = \mathbf{PEP}^{-1}$$

where

$$\mathbf{P} = \underline{\underline{\begin{pmatrix}1 & 1 & 1\\ -1 & 2 & 1\\ 1 & -3 & -2\end{pmatrix}}}$$

and

$$\mathbf{E} = \begin{pmatrix}6 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 3\end{pmatrix}$$

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^3 &= \mathbf{PE}^3\mathbf{P}^{-1} = \mathbf{PDP}^{-1}\text{, where}\\ \mathbf{D} &= \mathbf{E}^3\\ &= \begin{pmatrix}6^3 & 0 & 0\\ 0 & 4^3 & 0\\ 0 & 0 & 3^3\end{pmatrix} = \underline{\underline{\begin{pmatrix}216 & 0 & 0\\ 0 & 64 & 0\\ 0 & 0 & 27\end{pmatrix}}}\end{aligned}$$

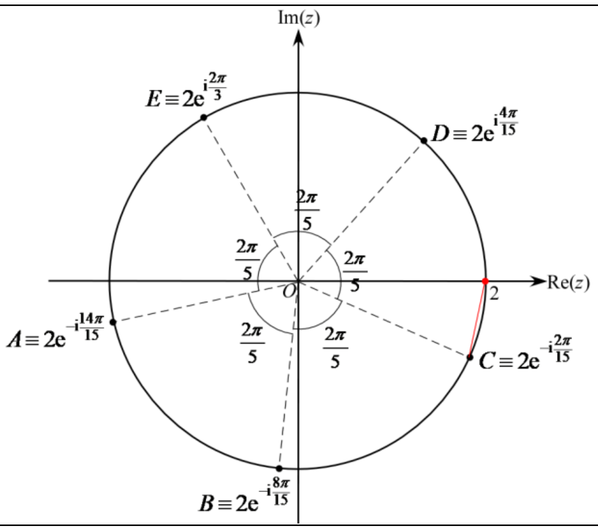
2	
(a)	$p_{n+1}=0.6\times1.2p_n+14000$ $p_{n+1}=0.72p_n+14000$
(b)	$p_n=c\left(0.72\right)^n+d$ $d=\dfrac{14000}{1-0.72}=50000$ $p_n=c\left(0.72\right)^n+50000$ Given that $p_0=220000$, $220000=c\left(0.72\right)^0+50000$ $c=170000$ Hence, $p_n=170000\left(0.72\right)^n+50000$.
	$p_n\le60000$ $170000\left(0.72\right)^n+50000\le60000$ Using GC, $p_8=62277>60000$ $p_9=58840<60000$ $p_{10}=56365<60000$ Hence, should consider closing after **<u>9</u>** months of business.

3

(a)

$$
\begin{aligned}
z^5+16+16\sqrt{3}\,\mathrm{i} &= 0 \\
z^5 &= -16\left(1+\sqrt{3}\,\mathrm{i}\right) \\
\left|-16\left(1+\sqrt{3}\,\mathrm{i}\right)\right| &= 16\sqrt{1+3}=32 \\
\arg\left(-16\left(1+\sqrt{3}\,\mathrm{i}\right)\right) &= -\left(\pi-\tan^{-1}\frac{\sqrt{3}}{1}\right) \qquad (\, z^5 \text{ lies in Q3}) \\
&= -\frac{2\pi}{3} \\
\therefore\ z^5 &= 32\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} \\
&= 32\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}+2k\pi\right)} \\
z &= \sqrt[5]{32}\ \mathrm{e}^{\mathrm{i}\left(\frac{-\frac{2\pi}{3}+2k\pi}{5}\right)}, \quad k=0,\ \pm1,\ \pm2 \\
&= 2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{15}+\frac{2k\pi}{5}\right)} \\
&= 2\mathrm{e}^{\mathrm{i}\left(\frac{(6k-2)\pi}{15}\right)} \\
&= \underline{\underline{2\mathrm{e}^{-\mathrm{i}\frac{14\pi}{15}},\ 2\mathrm{e}^{-\mathrm{i}\frac{8\pi}{15}},\ 2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{15}},\ 2\mathrm{e}^{\mathrm{i}\frac{4\pi}{15}},\ 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}}}
\end{aligned}
$$

(b)



For $|2-z|$ to be the least, look for the root that is closest to the complex number represented by $(2,0)$.

$$z=\underline{\underline{2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{15}}}}$$

4

Ellipse: $x^2+2y^2=2$ (1)

$P(x_1, y_1)$ is external to the ellipse.

(a) The equation of the line that has gradient m and passing through $P(x_1, y_1)$ is

$\underline{\underline{y-y_1=m(x-x_1)}}$.

(b) Line: $y= m(x-x_1)+y_1$ (2)

For points of intersection, sub. (2) into (1),

$$\begin{aligned}
x^2+2\left[m(x-x_1)+y_1\right]^2 &= 2\\
x^2+2\left[m^2(x^2-2x_1x+x_1^2)+2m(x-x_1)y_1+y_1^2\right] &= 2\\
x^2+2m^2x^2-4m^2x_1x+2m^2x_1^2+4my_1x-4mx_1y_1+2y_1^2 &= 2\\
(1+2m^2)x^2+4m(y_1-mx_1)x+2y_1^2+2m^2x_1^2-4mx_1y_1-2 &= 0 \quad \text{(Shown)}
\end{aligned}$$

(c) For the line to be tangent to the ellipse,

$$\begin{aligned}
\text{discriminant} &= 0\\
\left[4m(y_1-mx_1)\right]^2-4(1+2m^2)(2y_1^2+2m^2x_1^2-4mx_1y_1-2) &= 0\\
16m^2(y_1^2-2mx_1y_1+m^2x_1^2)-8(1+2m^2)(y_1^2+m^2x_1^2-2mx_1y_1-1) &= 0\\
2m^2(y_1^2-2mx_1y_1+m^2x_1^2)-(1+2m^2)(y_1^2+m^2x_1^2-2mx_1y_1-1) &= 0\\
2m^2y_1^2-4m^3x_1y_1+2m^4x_1^2-y_1^2-m^2x_1^2+2mx_1y_1+1-2m^2y_1^2-2m^4x_1^2+4m^3x_1y_1+2m^2 &= 0\\
-y_1^2-x_1^2m^2+2x_1y_1m+1+2m^2 &= 0\\
(x_1^2-2)m^2-2x_1y_1m+y_1^2-1 &= 0
\end{aligned}$$

(Shown)

(d) Let the gradients of the two tangents be m_1 and m_2 respectively.

For the two tangents to be perpendicular,

$$\begin{aligned}
m_1m_2 &= -1\\
\frac{y_1^2-1}{x^2-2} &= -1\\
y_1^2-1 &= 2-x^2\\
x_1^2+y_1^2 &= 3
\end{aligned}$$

As $P(x_1, y_1)$ varies, the locus of P at which the two tangents are perpendicular is $\underline{\underline{x^2+y^2=3}}$, which is a circle with centre $\underline{\underline{O}}$ and radius $\underline{\underline{\sqrt{3}}}$.

5

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 7 & 8 \\ 1 & 7 & 11 & 13 \\ 1 & 2 & 5 & 5 \end{pmatrix} \text{ and } \mathbf{M}_2 = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 5 & 1 & -3 & -3 \\ 3 & -1 & -1 & -1 \\ 13 & -1 & -6 & -6 \end{pmatrix}$$

(a) Method 1 Use $\mathbf{M}_1$ to find column space of $\mathbf{M}_1$

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 7 & 8 \\ 1 & 7 & 11 & 13 \\ 1 & 2 & 5 & 5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for R_1 is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 11 \\ 5 \end{pmatrix} \right\}$.

Method 2 Use $\mathbf{M}_1^T$ to find row space of $\mathbf{M}_1^T$ first

$$\mathbf{M}_1^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 7 & 2 \\ 1 & 7 & 11 & 5 \\ 2 & 8 & 13 & 5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for R_1

= a basis for the **column** space of $\mathbf{M}_1$ = basis for the **row** space of $\mathbf{M}_1^T$

$$= \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

(b) Let $\mathbf{M}_2\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 2 & 0 & -1 & -1 \\ 5 & 1 & -3 & -3 \\ 3 & -1 & -1 & -1 \\ 13 & -1 & -6 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By GC (simultaneous eqn solver),

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

A basis for K_2 is $\underline{\underline{\left\{\begin{pmatrix}1\\1\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\2\end{pmatrix}\right\}}}$.

Let $$\begin{pmatrix}1\\1\\2\\0\end{pmatrix} = k_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+k_2\begin{pmatrix}1\\4\\7\\2\end{pmatrix}+k_3\begin{pmatrix}1\\7\\11\\5\end{pmatrix}$$

By GC (simultaneous eqn solver) (or any correct method, e.g., by inspection)

$$\begin{pmatrix}1\\1\\2\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\4\\7\\2\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\7\\11\\5\end{pmatrix}$$

Let $$\begin{pmatrix}1\\1\\0\\2\end{pmatrix} = m_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+m_2\begin{pmatrix}1\\4\\7\\2\end{pmatrix}+m_3\begin{pmatrix}1\\7\\11\\5\end{pmatrix}$$

By GC (simultaneous eqn solver),

$$\begin{pmatrix}1\\1\\0\\2\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-\begin{pmatrix}1\\4\\7\\2\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\7\\11\\5\end{pmatrix}$$

Since each basis vector for K_2 can be expressed as a linear combination of the basis vectors for R_1, $\therefore$ all vectors in K_2 are in R_1, hence K_2 is a subspace of R_1. (Shown)

(c) The set of vectors which belong to R_1 but do not belong to K_2 is denoted by W.

Since both R_1 and K_2 are linear spaces, both spaces contain **0**, which is in $R_1 \cap K_2 = K_2 \ (\because K_2 \subseteq R_1)$.

Hence $\mathbf{0} \notin W$. $\therefore$ W is <u>not a vector space</u>. (Shown)

(d) $T_3 : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by $\mathbf{M}_2\mathbf{M}_1$.

<u>Method 1 Find rref $\mathbf{M}_2\mathbf{M}_1$</u>

$$\mathbf{M}_2\mathbf{M}_1 = \begin{pmatrix}0 & -7 & -14 & -14\\0 & -18 & -36 & -36\\0 & -10 & -20 & -20\\0 & -45 & -90 & -90\end{pmatrix} \xrightarrow[\text{clearly}]{\text{rref}} \begin{pmatrix}0 & 1 & 2 & 2\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

$$\begin{aligned}\operatorname{rank}(\mathbf{M}_2\mathbf{M}_1) &= 1\\ \operatorname{rank}(T_3) + \operatorname{nullity}(T_3) &= 4 \ (= \text{dim of } \mathbb{R}^4\text{, the domain of } T_3\ )\\ [\text{Or: } \operatorname{rank}(\mathbf{M}_2\mathbf{M}_1)+\operatorname{nullity}(\mathbf{M}_2\mathbf{M}_1) &= 4 \ (= \text{number of columns of } \mathbf{M}_2\mathbf{M}_1)]\\ 1+\operatorname{nullity}(T_3) &= 4\end{aligned}$$

$$\text{nullity}\left(T_3\right) = \underline{\underline{3}}$$

<u>Method 2 Solve $\mathbf{M}_2\mathbf{M}_1\mathbf{x} = \mathbf{0}$</u>

$$\mathbf{M}_2\mathbf{M}_1 = \begin{pmatrix} 0 & -7 & -14 & -14 \\ 0 & -18 & -36 & -36 \\ 0 & -10 & -20 & -20 \\ 0 & -45 & -90 & -90 \end{pmatrix}$$

Let $\mathbf{M}_2\mathbf{M}_1\mathbf{x} = \mathbf{0}$.

By GC (simultaneous eqn solver),

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

$\because$ the null space of T_3 has 3 basis vectors, $\therefore$ $\text{nullity}\left(T_3\right) = \underline{\underline{3}}$.

6

C: $x = at^2$, $\quad y = 2at$, $a > 0$ is a constant, $t \in \mathbb{R}$

Observe that $P\left(at^2, 2at\right)$ is on C. $F(a, 0)$.

(a)

$$\begin{aligned}
PF &= \sqrt{\left(at^2 - a\right)^2 + (2at)^2} \\
&= \sqrt{a^2t^4 - 2a^2t^2 + a^2 + 4a^2t^2} \\
&= \sqrt{a^2t^4 + 2a^2t^2 + a^2} \\
&= \sqrt{\left(at^2 + a\right)^2} \\
&= at^2 + a \quad \left(\because at^2 + a > 0\right)
\end{aligned}$$

(b)

Distance of P from the line $x = -a$

$$\begin{aligned}
&= \left|at^2 - (-a)\right| \\
&= at^2 + a \quad \left(\because at^2 + a > 0\right) \\
&= PF
\end{aligned}$$

$\therefore$ C is a parabola.

(with F as its focus and $x = -a$ as its directrix)

[Recall definition: A ***parabola*** is the set of all points in the plane that are equidistant from a fixed point (***focus***) and a fixed line (***directrix*** of the parabola) that does not contain the focus.]

Alternative method: find Cartesian equation of *C*

$$C: \quad x = at^2 \qquad (1)$$

$$y = 2at \qquad (2) \qquad a > 0 \text{ is a constant, } t \in \mathbb{R}$$

From (2), $\quad t = \dfrac{y}{2a}$

Sub. into (1), $\quad x = a\left(\dfrac{y}{2a}\right)^2$

$$= \frac{y^2}{4a}$$

$$y^2 = 4ax$$

$\therefore$ C is a parabola.

(with F as its focus and $x = -a$ as its directrix)

(c)

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = 2at, \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} = 2a$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{2a}{2at} = \frac{1}{t}$$

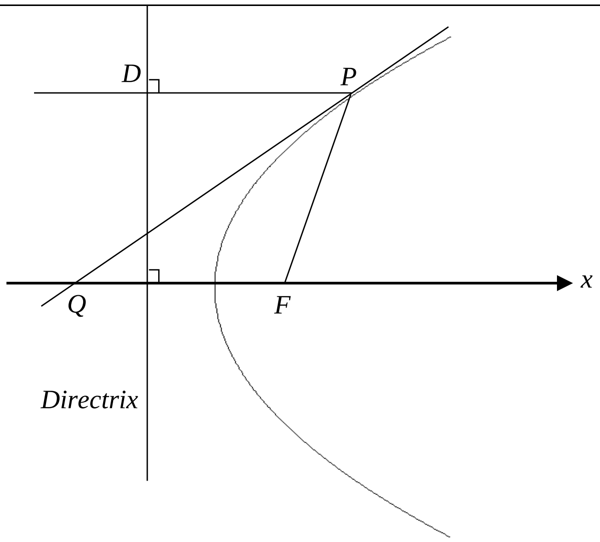
The equation of tangent at $P\left(at^2, 2at\right)$ is

$$\begin{aligned}
y - 2at &= \frac{1}{t}\left(x - at^2\right) \\
y &= \underline{\underline{\frac{1}{t}x + at}}
\end{aligned}$$

When $y = 0$, $$\frac{1}{t}x + at = 0$$
$$x = -at^2$$

$\therefore$ this tangent meets the x-axis at $Q\left(-at^2, 0\right)$. (Shown)

(d)



Method 1

$Q\left(-at^2, 0\right)$, $F\left(a, 0\right)$

$QF = \left|a - (-at^2)\right| = a + at^2 = PF$

$\therefore$ ΔQFP is isosceles $\Rightarrow \angle PQF = \angle FPQ$

$\because$ DP // x-axis, $\therefore \angle DPQ = \angle PQF$

$\therefore \angle DPQ = \angle PQF = \angle FPQ$

Hence $\angle DPQ = \angle FPQ$. (Shown)

Method 2

$PD = PF$ (from **(b)** Method 1), so ΔPDF is isosceles.

$$m_{DF} = \frac{-2at}{2a} = -t = PF$$

$$m_{PQ} = \frac{1}{t} \quad \text{(from } \textbf{(c)}\text{)}$$

$$m_{DF} \times m_{PQ} = -t\left(\frac{1}{t}\right) = -1 \Rightarrow DF \perp PQ$$

Hence $\angle DPQ = \angle FPQ$. (Shown)

7

(a) $z = \cos\theta + \mathrm{i}\sin\theta$

(i)

$$z^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta \qquad \text{(de Moivre's Thm)}$$

$$\frac{1}{z^n} = (\cos\theta + \mathrm{i}\sin\theta)^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta) \qquad \text{(de Moivre's Thm)}$$

$$= \cos n\theta - \mathrm{i}\sin n\theta$$

$$\therefore\ z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \text{(Proved)}$$

(ii)

$$\left(z+\frac{1}{z}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z}\right)^2 + 20z^3\left(\frac{1}{z}\right)^3 + 15z^2\left(\frac{1}{z}\right)^4 + 6z\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$$

$$(2\cos\theta)^6 = z^6 + 6z^4 + 15z^2 + 20 + 15\left(\frac{1}{z^2}\right) + 6\left(\frac{1}{z^4}\right) + \frac{1}{z^6}$$

$$64\cos^6\theta = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$$= 2(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

$$\therefore\ \cos^6\theta = \underline{\underline{\frac{1}{32}(10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta)}}$$

(b) $z = \mathrm{e}^{i\theta}$

[Note that $\sum_{n=1}^{N}\cos(2n-1)\theta$ = Re$\left(\sum_{n=1}^{N} z^{2n-1}\right)$]

<u>Method 1 After obtaining GP sum expression, express in the form $z^n - z^{-n}$, then convert to trigo form, before obtaining the real part</u>

$$\sum_{n=1}^{N} z^{2n-1} = z + z^3 + z^5 + \ldots + z^{2N-1}$$

$$= \frac{z\left[\left(z^2\right)^N - 1\right]}{z^2 - 1} \qquad (\text{sum of GP, } a = z, r = z^2)$$

$$= \frac{z\left[z^N\left(z^N - z^{-N}\right)\right]}{z^2 - 1}$$

$$= \frac{z^N\left(z^N - z^{-N}\right)}{z - z^{-1}}$$

$$= \frac{z^N(2\mathrm{i}\sin N\theta)}{2\mathrm{i}\sin\theta}$$

$$= \frac{(\cos N\theta + \mathrm{i}\sin N\theta)(\sin N\theta)}{\sin\theta}$$

$$\sum_{n=1}^{N}\cos\left(2n-1\right)\theta$$

$$= \operatorname{Re}\left(\sum_{n=1}^{N}z^{2n-1}\right)$$

$$= \operatorname{Re}\frac{\left(\cos N\theta+\mathrm{i}\sin N\theta\right)\left(\sin N\theta\right)}{\sin\theta}$$

$$= \frac{\cos N\theta\sin N\theta}{\sin\theta}$$

$$= \frac{2\sin N\theta\cos N\theta}{2\sin\theta}$$

$$= \frac{\sin 2N\theta}{2\sin\theta},\ \sin\theta\neq 0\ \text{(Shown)}$$

Method 2 After obtaining GP sum expression, convert to exponential form then trigo form, before obtaining the real part

$$\sum_{n=1}^{N}z^{2n-1}$$

$$= z+z^3+z^5+\ldots+z^{2N-1}$$

$$= \frac{z\left[\left(z^2\right)^N-1\right]}{z^2-1} \qquad \text{(sum of GP, } a=z, r=z^2\text{)}$$

$$= \frac{z^{2N+1}-z}{z^2-1}$$

$$= \frac{\mathrm{e}^{\mathrm{i}(2N+1)\theta}-\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}2\theta}-1} \qquad \text{(For realisation below: Note that } \mathrm{e}^{\mathrm{i}2\theta}-1=\left(\cos 2\theta-1\right)+\mathrm{i}\sin 2\theta\text{)}$$

$$= \frac{\mathrm{e}^{\mathrm{i}(2N+1)\theta}-\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}2\theta}-1}\times\frac{\mathrm{e}^{-\mathrm{i}2\theta}-1}{\mathrm{e}^{-\mathrm{i}2\theta}-1}$$

$$= \frac{\mathrm{e}^{\mathrm{i}(2N-1)\theta}-\mathrm{e}^{\mathrm{i}(2N+1)\theta}-\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\theta}}{1-\left(\mathrm{e}^{\mathrm{i}2\theta}+\mathrm{e}^{-\mathrm{i}2\theta}\right)+1}$$

$$= \frac{\cos\left(2N-1\right)\theta+\mathrm{i}\sin\left(2N-1\right)\theta-\cos\left(2N+1\right)\theta-\mathrm{i}\sin\left(2N+1\right)\theta-\cos\theta+\mathrm{i}\sin\theta+\cos\theta+\mathrm{i}\sin\theta}{2-2\cos 2\theta}$$

(Optional step)

$$\sum_{n=1}^{N}\cos\left(2n-1\right)\theta$$

$$= \operatorname{Re}\left(\sum_{n=1}^{N}z^{2n-1}\right)$$

$$= \frac{\cos\left(2N-1\right)\theta-\cos\left(2N+1\right)\theta}{2-2\cos 2\theta}$$

$$= \frac{-2\sin 2N\theta\sin(-\theta)}{2\left(1-\cos 2\theta\right)} \qquad \text{(Factor Formula)}$$

$$= \frac{2\sin 2N\theta\sin\theta}{2\left(2\sin^2\theta\right)}$$

$$= \frac{\sin 2N\theta}{2\sin\theta},\ \sin\theta\neq 0\ \text{(Shown)}$$

8

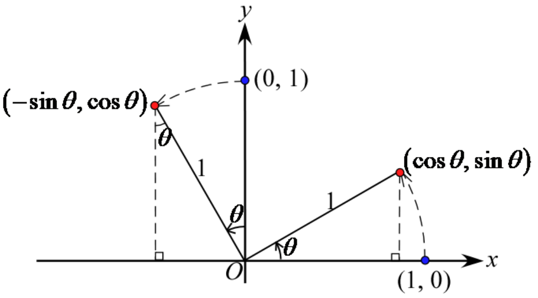
(a)

Let the matrix be **M**.

$$\mathbf{M}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

Hence $\mathbf{M}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. (Shown)



(b)

$C_1 : x^2+y^2-6xy-4=0$

<u>Method 1 Use part **(a)**</u>

The matrix representing a rotation of $45°$ anticlockwise about the origin is

$$\mathbf{M}=\begin{pmatrix}\cos 45° & -\sin 45°\\ \sin 45° & \cos 45°\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

Let (x, y) be any point on C_1 and (X, Y) be the corresponding image under the rotation.

<u>Method 1A Work with **M**</u>

$$\text{Then}\qquad \begin{pmatrix}X\\Y\end{pmatrix} = \mathbf{M}\begin{pmatrix}x\\y\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y\\ \frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y\end{pmatrix}$$

$$X = \frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y \qquad (1)$$

$$Y = \frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y \qquad (2)$$

(1) + (2), $\quad X+Y = \frac{2}{\sqrt{2}}x$

$$x = \frac{1}{\sqrt{2}}(X+Y) \qquad (3)$$

(1) − (2), $\quad X-Y = -\frac{2}{\sqrt{2}}y$

$$y = -\frac{1}{\sqrt{2}}(X-Y) \qquad (4)$$

Sub. (3) and (4) into eqn of C_1, $\qquad x^2+y^2-6xy-4 = 0$

$$\begin{aligned}
\left[\frac{1}{\sqrt{2}}(X+Y)\right]^2+\left[-\frac{1}{\sqrt{2}}(X-Y)\right]^2-6\left[\frac{1}{\sqrt{2}}(X+Y)\right]\left[-\frac{1}{\sqrt{2}}(X-Y)\right]-4 &= 0\\
\frac{1}{2}\left(X^2+2XY+Y^2\right)+\frac{1}{2}\left(X^2-2XY+Y^2\right)+3\left(X^2-Y^2\right)-4 &= 0\\
X^2+Y^2+3X^2-3Y^2-4 &= 0\\
4X^2-2Y^2-4 &= 0\\
X^2-\frac{Y^2}{2} &= 1
\end{aligned}$$

$\therefore$ equation of C_2 is $x^2-\frac{y^2}{2}=1$. (Shown)

Method 1B Work with $\mathbf{M}^{-1}$

Then

$$\begin{aligned}
\begin{pmatrix} X \\ Y \end{pmatrix} &= \mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}\\
\begin{pmatrix} x \\ y \end{pmatrix} &= \mathbf{M}^{-1}\begin{pmatrix} X \\ Y \end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1}\begin{pmatrix} X \\ Y \end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}\\
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}}X+\frac{1}{\sqrt{2}}Y \\ -\frac{1}{\sqrt{2}}X+\frac{1}{\sqrt{2}}Y \end{pmatrix}\\
x &= \frac{1}{\sqrt{2}}(X+Y) \qquad (1)\\
y &= -\frac{1}{\sqrt{2}}(X-Y) \qquad (2)
\end{aligned}$$

Sub. (1) and (2) into eqn of C_1, $x^2+y^2-6xy-4 = 0$

$$\left[\frac{1}{\sqrt{2}}(X+Y)\right]^2+\left[-\frac{1}{\sqrt{2}}(X-Y)\right]^2-6\left[\frac{1}{\sqrt{2}}(X+Y)\right]\left[-\frac{1}{\sqrt{2}}(X-Y)\right]-4 = 0$$

(M1) sub

$$\begin{aligned}
\frac{1}{2}\left(X^2+2XY+Y^2\right)+\frac{1}{2}\left(X^2-2XY+Y^2\right)+3\left(X^2-Y^2\right)-4 &= 0\\
X^2+Y^2+3X^2-3Y^2-4 &= 0\\
4X^2-2Y^2-4 &= 0\\
X^2-\frac{Y^2}{2} &= 1
\end{aligned}$$

$\therefore$ equation of C_2 is $x^2-\dfrac{y^2}{2}=1$.(Shown)

Method 2 Use Polar Coordinates

$$\begin{aligned} C_1: \quad x^2+y^2-6xy-4 &= 0 \\ r^2-6(r\cos\theta)(r\sin\theta)-4 &= 0 \\ r^2-3r^2\sin 2\theta-4 &= 0 \end{aligned}$$

C_2 is obtained by rotating C_1 45° anticlockwise about O.

$$\begin{aligned} C_2: \quad r^2-3r^2\sin 2(\theta-45°)-4 &= 0 \\ r^2-3r^2\sin(2\theta-90°)-4 &= 0 \\ r^2+3r^2\sin(90°-2\theta)-4 &= 0 \\ r^2+3r^2\cos 2\theta-4 &= 0 \\ r^2+3r^2\left(\cos^2\theta-\sin^2\theta\right)-4 &= 0 \\ r^2+3\left(x^2-y^2\right)-4 &= 0 \\ x^2+y^2+3\left(x^2-y^2\right)-4 &= 0 \\ x^2-\frac{y^2}{2} &= 1 \end{aligned}$$

$\therefore$ equation of C_2 is $x^2-\dfrac{y^2}{2}=1$. (Shown)

(c) C_2 is a hyperbola.

(d)

(i) $C_2: x^2-\dfrac{y^2}{\left(\sqrt{2}\right)^2}=1,\ a=1, b=\sqrt{2}$

$$\begin{aligned} c &= \sqrt{1+2}=\sqrt{3} \\ e &= \sqrt{3} \end{aligned}$$

(ii) Coordinates of the foci are $\underline{\underline{\left(\sqrt{3},0\right) \text{ and } \left(-\sqrt{3},0\right)}}$.

(iii) Equations of the directrices are $\underline{\underline{x=\dfrac{1}{\sqrt{3}} \text{ and } x=-\dfrac{1}{\sqrt{3}}}}$.

9

(a)

$$y=\frac{e^{x}+e^{-x}}{2}$$

$$\frac{dy}{dx}=\frac{e^{x}-e^{-x}}{2}$$

$$\text{Arc length}=\int_{-\ln 3}^{\ln 3}\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx$$

$$=2\int_{0}^{\ln 3}\sqrt{1+\left(\frac{e^{x}-e^{-x}}{2}\right)^{2}}\,dx$$

$$=2\int_{0}^{\ln 3}\sqrt{1+\frac{e^{2x}-2+e^{-2x}}{4}}\,dx$$

$$=2\int_{0}^{\ln 3}\sqrt{\frac{e^{2x}+2+e^{-2x}}{4}}\,dx$$

$$=2\int_{0}^{\ln 3}\sqrt{\frac{\left(e^{x}+e^{-x}\right)^{2}}{4}}\,dx$$

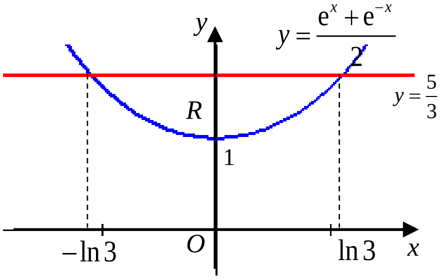
$$=\int_{0}^{\ln 3}e^{x}+e^{-x}\,dx$$

$$=\left[e^{x}-e^{-x}\right]_{0}^{\ln 3}$$

$$=\left(3-\frac{1}{3}\right)-(1-1)$$

$$=2\frac{2}{3}\text{ units}$$

(b)



By shell method,

$$V=\text{volume of cylinder}-\ 2\pi\int_{0}^{\ln 3}xy\,dx$$

$$=\pi(\ln 3)^{2}\left(\frac{5}{3}\right)-2\pi\int_{0}^{\ln 3}x\left(\frac{e^{x}+e^{-x}}{2}\right)dx$$

$$=\frac{5}{3}\pi(\ln 3)^{2}-\pi\int_{0}^{\ln 3}x\left(e^{x}+e^{-x}\right)dx$$

	$u = x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{x} + \mathrm{e}^{-x}$ $\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \qquad v = \mathrm{e}^{x} - \mathrm{e}^{-x}$ $\int_0^{\ln 3} x\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)\mathrm{d}x = \left[x(\mathrm{e}^{x} - \mathrm{e}^{-x})\right]_0^{\ln 3} - \int_0^{\ln 3} (\mathrm{e}^{x} - \mathrm{e}^{-x})\,\mathrm{d}x$ $= \ln 3\left[\left(3 - \frac{1}{3}\right) - 0\right] - \left[\mathrm{e}^{x} + \mathrm{e}^{-x}\right]_0^{\ln 3}$ $= \frac{8}{3}\ln 3 - \left[\left(3 + \frac{1}{3}\right) - (1+1)\right]$ $= \frac{8}{3}\ln 3 - \frac{4}{3}$ $V = \frac{5}{3}\pi(\ln 3)^2 - \pi\left(\frac{8}{3}\ln 3 - \frac{4}{3}\right)$ $= \frac{\pi}{3}\left[5(\ln 3)^2 - 8\ln 3 + 4\right]$ $\therefore\ p = 5,\ q = -8,\ r = 4$
(c)	Since $y = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}$, $y = 1, \quad x = 0$ $y = \frac{5}{3}, \quad x = \ln 3$ $\bar{y} = \frac{\pi}{V}\int_1^{\frac{5}{3}} x^2 y\ \mathrm{d}y$ $= \frac{\pi}{V}\int_0^{\ln 3} x^2\left(\frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}\right)\mathrm{d}x$ $= \frac{\pi}{4V}\int_0^{\ln 3} x^2\left(\mathrm{e}^{2x} - \mathrm{e}^{-2x}\right)\mathrm{d}x$ (Shown)
(d)	Using GC, $\bar{y} = \frac{\pi}{4\left(\frac{\pi}{3}\left[5(\ln 3)^2 - 8\ln 3 + 4\right]\right)}\int_0^{\ln 3} x^2\left(\mathrm{e}^{2x} - \mathrm{e}^{-2x}\right)\mathrm{d}x$ $= 1.4408$ Hence, height of centroid = 1.4408 – 1 = 0.4408 ≈ 0.441 units

10

(a)

$$C_{n+1} = 2C_n\left(1-\frac{\sqrt{C_n}}{4}\right)$$

As $n \to \infty$, $C_{n+1} \to L$ and $C_n \to L$,

$$L = 2L\left(1-\frac{\sqrt{L}}{4}\right)$$

$$L\left(2\left(1-\frac{\sqrt{L}}{4}\right)-1\right)=0$$

$$L = 0_{\text{(rejected)}} \quad \text{or} \quad 1-\frac{\sqrt{L}}{4}=\frac{1}{2}$$

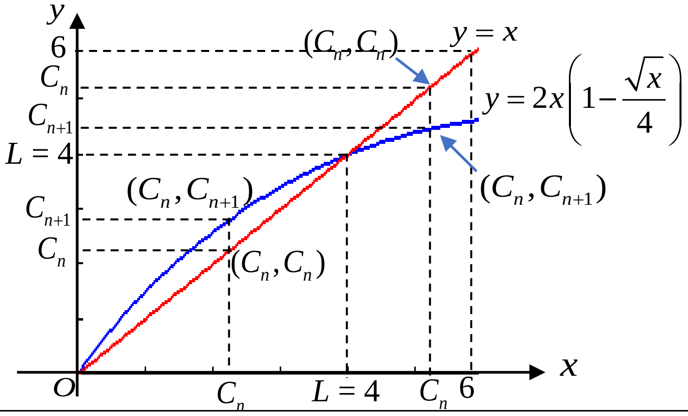
$$\frac{\sqrt{L}}{4}=\frac{1}{2}$$

$$\sqrt{L}=2$$

$$L=4$$

(b)

Sketch the graphs of $y = 2x\left(1-\frac{\sqrt{x}}{4}\right)$ and $y = x$



(i)

From the graph, when $0 < C_n < L = 4$, the graph of $y = 2x\left(1-\frac{\sqrt{x}}{4}\right)$ lies above the graph of $y = x$.

Comparing the *y*-coordinates, $C_n < C_{n+1} < L = 4$. (Shown)

(ii)

Similarly, when $C_n > L = 4$, the graph of $y = 2x\left(1-\frac{\sqrt{x}}{4}\right)$ lies below the graph of $y = x$.

Comparing the *y*-coordinates, $L = 4 < C_{n+1} < C_n$. (Shown)

(c)

Initial population $C_0 = 5 > 4$,

From **(b)(ii)**, $L = 4 < C_{n+1} < C_n$.

Which means, $C_0 > C_1 > C_2 > C_3 > ... > 4$.

Hence, the population **decreases** and **converges to 400**.

(d)	$\dfrac{\mathrm{d}P}{\mathrm{d}t}=2P\left(1-\dfrac{P}{k}\right)$ $2P\left(1-\dfrac{P}{k}\right)=0$ $P=0$ or $P=k$ Since the equilibrium population value is L (= 4) hundred, $k=4$
(e)	k is the **carrying capacity** of the environment. It is the **maximum population size of the birds** that the environment can sustain in the long term given the resources available.
(f)