Chapter 10

COMPLEX NUMBERS

The set of real numbers can be extended to complex numbers. This set of numbers arose, historically, from the question of whether a negative number can have a square root.
For example, what are the roots of $x^2+1=0$?
From this problem, a new number was discovered: the square root of negative one. This number is denoted by i, a symbol assigned by Leonhard Euler (1707-1783, Swiss mathematician).
i.e. $\mathrm{i}=\sqrt{-1}$

Complex numbers consist of all numbers of the form $a+b\,\mathrm{i}$, where a and b are real numbers.

Symbolically, $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$

Set of natural numbers **Set of integers** **Set of rational numbers** **Set of real numbers** **Set of complex numbers**

Complex numbers have applications in a variety of sciences and related areas such as signal processing, control theory, electromagnetism, quantum mechanics, cartography, and many others.

10.1 Introduction

Let us start by solving the equation $x^2-8x+20=0$. Applying the formula, we have

$$\begin{aligned} x &= \frac{8 \pm \sqrt{64-80}}{2} \\ &= 4 \pm 2\sqrt{-1} \end{aligned}$$

If we let $\mathrm{i}=\sqrt{-1}$, it follows that $x=4+2\mathrm{i}$ or $x=4-2\mathrm{i}$.

The numbers $4+2\mathrm{i}$ and $4-2\mathrm{i}$ are called complex numbers.

A **complex number**, usually denoted by the letter z, is defined as any number of the form $z=a+b\mathrm{i}$ where $\mathrm{i}=\sqrt{-1}$ and $a, b \in \mathbb{R}$, and this is known as the **Cartesian form** of complex numbers.

The real number a is called the **real part of z**, written as a = **Re(z)**.

The real number b is called the **imaginary part of z**, written as b = **Im(z)**.

The set of complex numbers is denoted by $\mathbb{C}$.

Note:	THINKZONE:
(i) $i^2 = -1$ If $z = a + bi$ (ii) If $b = 0$, then the complex number $z = a$ is real. (iii) If $a = 0$, then the complex number $z = bi$ is said to be **purely imaginary**.	Simplify $i^3,\ i^4,\ i^5,\ i^6,\ i^7,\ i^8,\ i^9,\ i^{10},\ \ldots$ $-i \quad 1 \quad i \quad -1 \quad -i \quad i$ Do you see any pattern?

10.2 Operation on Complex Numbers

We shall now see how we can add, subtract, multiply and divide two complex numbers.
Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, where $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Equality

$z_1 = z_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$

For example, $2 + ai = b - 3i$ then $a = -3$ and $b = 2$.
In general, $a + bi = 0 \Leftrightarrow a = b = 0$

Addition and Subtraction

$$\begin{aligned} z_1 + z_2 &= x_1 + y_1 i + x_2 + y_2 i \\ &= (x_1 + x_2) + (y_1 + y_2)i. \end{aligned}$$

For example, $(2 + 3i) + (4 - 2i) = (2 + 4) + i(3 - 2) = 6 + i$.

$$\begin{aligned} z_1 - z_2 &= x_1 + y_1 i - (x_2 + y_2 i) \\ &= (x_1 - x_2) + (y_1 - y_2)i. \end{aligned}$$

For example, $(1 - 2i) - (3 + 5i)$
$= (1 - 3) + i(-2 - 5) = -2 - 7i$

Multiplication

$$\begin{aligned} z_1 z_2 &= (x_1 + y_1 i)(x_2 + y_2 i) \\ &= x_1 x_2 + x_1 y_2 i + x_2 y_1 i + y_1 y_2 i^2 \\ &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i \quad \text{since } i^2 = -1. \end{aligned}$$

Therefore $(9 + i)(-2 + 3i) = -18 + 27i - 2i + 3i^2$
$= -18 + 25i - 3$
$= -21 + 25i$

Multiplication by a Real Number

For any $k \in \mathbb{R}$, $kz = k(x + iy)$
$= kx + kyi$.

For example, $-2(4 - 9i) = -8 + 18i$

Conjugate of a Complex Number

If $z = x + iy$, we define the **conjugate of z**, denoted by z^* to be $z^* = x - iy$.

For example, $(-3 + 4i)^* = -3 - 4i$, $i^* = -i$ and $5^* = 5$.

The conjugate pair z and z^* have the following properties:

(a)	$z + z^* = 2\,\mathrm{Re}(\,z\,)$	
(b)	$z - z^* = 2\,\mathrm{Im}(\,z\,)\,\mathrm{i}$	
(c)	$zz^* = x^2 + y^2 = \lvert z \rvert^2$	where $\lvert z \rvert = \sqrt{x^2 + y^2}$ (to be discussed in 10.5.1)
(d)	$(z^*)^* = z$	

Proof:

(a) $z + z^* = (x + \mathrm{i}y) + (x - \mathrm{i}y) = 2x$

$= 2\,\mathrm{Re}(z)$

(b) $z - z^* = (x + \mathrm{i}y) - (x - \mathrm{i}y) = 2\mathrm{i}y$

$= 2\,\mathrm{Im}(z)\,\mathrm{i}$

(c) $zz^* = (x + \mathrm{i}y)(x + \mathrm{i}y)^*$

$= (x + \mathrm{i}y)(x - \mathrm{i}y)$

$= x^2 - (\mathrm{i}y)^2$

$= x^2 + y^2 = \lvert z \rvert^2$ where $\lvert z \rvert = \sqrt{x^2 + y^2}$

(d) $(z^*)^* = (\,(x + \mathrm{i}y)^*)^*$

$= (x - \mathrm{i}y)^*$

$= x + \mathrm{i}y$

$= z$

$z_1 = a + ib$

$z_2 = c + id$

(a) $(z_1 + z_2)^* = (a + ib + c + id)^*$

$= ((a+c) + i(b+d))^*$

$= ((a+c) - i(b+d))$

$= a - ib + c - id$

$= (a+ib)^* + (c+id)^*$

$= z_1^* + z_2^*$

(b) $(z_1 z_2)^* = ((a+ib)(c+id))^*$

$= (ac + ibc + ida - bd)^*$

$= (ac - bd + i(bc+da))^*$

$= ac - bd - i(bc+da)$

$= ac - bd - ibc - ida$

$= c(a - ib) - d(b + ia)$

$= c(a - ib) - di(\frac{b}{i} + a)$

Note:	**THINKZONE:**
(a) $(z_1 + z_2)^* = z_1^* + z_2^*$,	Can you derive these results?
(b) $(z_1 z_2)^* = z_1^* z_2^* \quad \Rightarrow \quad (z^2)^* = (z^*)^2$	
(c) $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1{}^*}{z_2{}^*}$	

$= c(a - ib) - di(\frac{b}{i} + a)(-i^2)$

$= c(a - ib) - di(-bi + a)$

$= (a - ib)(c - di)$

$= z_1^* z_2^*$

Division

$$\frac{z_1}{z_2} = \frac{x_1 + y_1\mathrm{i}}{x_2 + y_2\mathrm{i}}$$

$$= \frac{(x_1 + y_1\mathrm{i})\ (x_2 - y_2\mathrm{i})}{(x_2 + y_2\mathrm{i})\ (x_2 - y_2\mathrm{i})}$$

$$= \frac{x_1x_2 - x_1y_2\mathrm{i} + x_2y_1\mathrm{i} - y_1y_2\mathrm{i}^2}{x_2{}^2 - y_2{}^2\mathrm{i}^2}$$

$$= \frac{(x_1x_2 + y_1y_2) + (x_2y_1 - x_1y_2)\,\mathrm{i}}{x_2{}^2 + y_2{}^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2{}^2 + y_2{}^2} + \left(\frac{x_2y_1 - x_1y_2}{x_2{}^2 + y_2{}^2}\right)\mathrm{i}$$

Example 1

Express $\dfrac{1}{2\ +\ 3\text{i}}$ and $\dfrac{2+\text{i}}{1+\text{i}}$ in the form of a + ib were a, $b \in \mathbb{R}$.

Solution:	THINKZONE:
$\dfrac{1}{2+3i} = \dfrac{1(2-3i)}{(2+3i)(2-3i)} = \dfrac{2-3i}{4+9}$ $= \dfrac{2}{13} - \dfrac{3}{13}i$ $\dfrac{2+\text{i}}{1+\text{i}}$ $=\dfrac{2+\text{i}}{1+\text{i}}\times\dfrac{1-\text{i}}{1-\text{i}}$ $=\dfrac{3-\text{i}}{1+1}$ $=\dfrac{3}{2}-\dfrac{1}{2}\text{i}$	$(a+b\text{i})(a-b\text{i})$ $=a^2-(b\text{i})^2$ $=a^2-b^2\text{i}^2$ $=a^2-b^2(-1)$ since $\text{i}^2=-1$ $=a^2+b^2$

Example 2

If $z=\dfrac{1+\text{i}}{2-\text{i}}$, find the real and imaginary parts of (i) z^2 (ii) $z-\dfrac{1}{z}$.

Solution:	THINKZONE:
$z = \dfrac{1+i}{2-i}$ $= \dfrac{(1+i)}{2-i} \times \dfrac{2+i}{2+i}$ $= \dfrac{1+3i}{2^2+1^2}$ $= \dfrac{1}{5}(1+3i)$ $z^2 = \dfrac{1+3i}{5} \times \dfrac{1+3i}{5} = \dfrac{1}{25}(-8+6i)$ $= -\dfrac{8}{25} + \dfrac{6}{25}i$ $\text{Re}(z^2) = -\dfrac{8}{25}$, $\text{Im}(z^2) = \dfrac{6}{25}$	Use GC to check answer: To enter the complex number i, press [2nd] [.]

(ii) $\dfrac{1}{z}=\dfrac{2-\mathrm{i}}{1+\mathrm{i}}$

$=\dfrac{(2-\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}$

$=\dfrac{1}{2}-\dfrac{3}{2}\mathrm{i}$

$z-\dfrac{1}{z}$

$=\dfrac{1+3\mathrm{i}}{5}-\left(\dfrac{1}{2}-\dfrac{3}{2}\mathrm{i}\right) = -\dfrac{3}{10}+\dfrac{21}{10}\mathrm{i}$

$\mathrm{Re}\,(z-\dfrac{1}{z})=-\dfrac{3}{10}$, $\quad \mathrm{Im}(z-\dfrac{1}{z})=\dfrac{21}{10}$

Example 3

(a) Find the square roots of 3 – 4i.

(b) Find z if $\dfrac{z}{1+z} = \dfrac{1}{1-3\mathrm{i}}$.

Solution:

(a) Let $z=x+iy$, where $x, y \in \mathbb{R}$ such that $z^2=3-4i$

$\Rightarrow (x+yi)^2=3-4i$

$x^2-y^2+2xyi=3-4i$

Comparing the real and imaginary parts, we have

$x^2-y^2=3$ — (1)

$2xy=-4 \Rightarrow y=-\frac{2}{x}$ — (2)

$x^2-(-\frac{2}{x})^2=3$

$x^4-3x^2-4=0$

$(x^2-4)(x^2+1)=0$

$x^2=4$ or $x^2=-1$ (reject) $\because x \in \mathbb{R}$

$x=\pm 2$ thus $y=\mp 1$

$\therefore$ the square roots of $3-4i$ is $(2-i)$ and $(-2+i)$

(b) $\dfrac{z}{1+z}=\dfrac{1}{1-3i}$

$z(1-3i)=1+z$

$z-3iz=1+z$

$z=\dfrac{1}{-3i}=\dfrac{i}{-3i^2}=\dfrac{1}{3}i$

THINKZONE:

To find the square roots of 3 – 4i means to solve $z^2=3-4\mathrm{i}$

Cross multiply

Make z the subject of the equation

Example 4

Express $(2-\mathrm{i})^3$ in the form $a+\mathrm{i}b$. Hence, find a root of the equation $(z-\mathrm{i})^3 = -11-2\mathrm{i}$.

Solution:

$$(2-\mathrm{i})^3 = 2^3 + \binom{3}{1}2^2(-\mathrm{i}) + \binom{3}{2}2(-\mathrm{i})^2 + (-\mathrm{i})^3$$

$$= 8+3\times4\times(-\mathrm{i})+3\times2\times\mathrm{i}^2-\mathrm{i}^3$$

$$= 8-12\mathrm{i}-6+\mathrm{i} = 2-11\mathrm{i}$$

$(z-i)^3 = -11-2i = -i(2-11i)$

$= -i(2-i)^3 = i^3(2-i)^3$

$\Rightarrow z-i = i(2-i) = 1+2i$

$\Rightarrow z = 1+3i$ is a root of equation.

THINKZONE:

In general, $a-b\mathrm{i} = -\mathrm{i}(b+a\mathrm{i})$.
So $(-11)-2\mathrm{i} = -\mathrm{i}(2-11\mathrm{i})$.

Or, you can use GC to evaluate $\dfrac{-11-2\mathrm{i}}{2-11\mathrm{i}}$.

Why do we write $-\mathrm{i} = \mathrm{i}^3$?

$1+3\mathrm{i}$ is just one of the roots for $(z-\mathrm{i})^3 = -11-2\mathrm{i}$, there are 2 more roots. Can you find them?

Self review 1: Given that $(1+5\mathrm{i})p-2q = 3+7\mathrm{i}$, find p and q when:

a) p and q are real

b) p and q are complex conjugates. [a: $p=7/5, q=-4/5$; b: $p=2-\mathrm{i}, q=2+\mathrm{i}$]

Solution:

(a)

$$(1+5\mathrm{i})p-2q = 3+7\mathrm{i}$$

$$\Rightarrow \quad p-2q+5p\mathrm{i} = 3+7\mathrm{i}$$

Comparing real and imaginary parts:
$p-2q=3;\quad 5p=7$

$$\Rightarrow p=\frac{7}{5},\quad q=-\frac{4}{5}$$

(b) Let $p=a+b\mathrm{i}$ where $a,b\in\mathbb{R}$.
Thus $q=a-b\mathrm{i}$.

$$(1+5\mathrm{i})p-2q = 3+7\mathrm{i}$$

$$\Rightarrow (1+5\mathrm{i})(a+b\mathrm{i})-2(a-b\mathrm{i}) = 3+7\mathrm{i}$$

$$\Rightarrow a-5b+(5a+b)\mathrm{i}-2a+2b\mathrm{i} = 3+70\mathrm{i}$$

Comparing real and imaginary parts:
$-a-5b=3;\quad 5a+3b=7$

$$\Rightarrow a=2,\quad b=-1$$

Thus $p=2-\mathrm{i},\quad q=2+\mathrm{i}$

Complex Numbers in Graphic Calculator

Press MODE to display mode setting.
Use arrow keys to select

(i) a+bi to display the complex number in Cartesian form, or

(ii) re^θi to display it in exponential form (to be discussed in 10.6).

Note:

- The **radian mode** is to be used for calculations involving complex numbers
- To enter the complex number i, press 2nd .

You can find operations or functions for complex numbers in the **MATH CPLX menu** (press MATH ❯ ❯):

1: conj(	gives the complex conjugate
2: real(	gives the real part of a complex number
3: imag(	gives the imaginary part of a complex number
4: angle(	gives the principal argument of a complex number (to be discussed in 10.5)
5: abs(	gives the modulus of a complex number
6: ▶Rect	displays the result in Cartesian form
7: ▶Polar	displays the result in **exponential** form

Example 5:

If $z=\dfrac{1-\mathrm{i}}{2+\mathrm{i}}$, find $z-\dfrac{1}{z}$. Hence, or otherwise, find a complex number w in Cartesian form ($a+b\mathrm{i}$ form) such that $w-\dfrac{1}{w}=-\dfrac{3}{10}+\dfrac{21}{10}\mathrm{i}$

Solution:	**THINKZONE:**
$z-\dfrac{1}{z} = \dfrac{1-\mathrm{i}}{2+\mathrm{i}}-\dfrac{2+\mathrm{i}}{1-\mathrm{i}} = -\dfrac{3}{10}-\dfrac{21}{10}\mathrm{i}$ (using GC) $w-\dfrac{1}{w}=-\dfrac{3}{10}+\dfrac{21}{10}\mathrm{i}=\left(-\dfrac{3}{10}-\dfrac{21}{10}\mathrm{i}\right)^*$ $=\left(z-\dfrac{1}{z}\right)^*$ $\Rightarrow w-\dfrac{1}{w}=z^*-\dfrac{1}{z^*}$ By observation, $w=z^*$ or $w=-\dfrac{1}{z^*}$ Thus, $w=z^* = \left(\dfrac{1-\mathrm{i}}{2+\mathrm{i}}\right)^* = \dfrac{1}{5}+\dfrac{3}{5}\mathrm{i}$ $w=-\dfrac{1}{z^*} = -\left(\dfrac{2+\mathrm{i}}{1-\mathrm{i}}\right)^* = \dfrac{2-\mathrm{i}}{1+\mathrm{i}} = -\dfrac{1}{2}+\dfrac{3}{2}\mathrm{i}$	Use the property of conjugate: $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$ Upon getting the answer for w, key in the answer into GC to check answer!

Self-Review 2 (a): If $z = 4 - 3\mathrm{i}$, express $z+\dfrac{1}{z}$ in Cartesian form. Hence, find a complex number w in Cartesian form such that $-w-\dfrac{1}{w}=\dfrac{104}{25}+\dfrac{72}{25}\mathrm{i}$. [$w=-4-3\mathrm{i}$ or $\dfrac{1}{25}(-4+3\mathrm{i})$]

Solution:

Using GC, $z+\dfrac{1}{z}=\dfrac{104}{25}-\dfrac{72}{25}\mathrm{i}$

Given that $-w-\dfrac{1}{w}=\dfrac{104}{25}+\dfrac{72}{25}\mathrm{i} \Rightarrow \left(-w-\dfrac{1}{w}\right)^*=\left(\dfrac{104}{25}+\dfrac{72}{25}\mathrm{i}\right)^* = \dfrac{104}{25}-\dfrac{72}{25}\mathrm{i}$

$$\Rightarrow (-w^*)-\frac{1}{w^*}=\frac{104}{25}-\frac{72}{25}\mathrm{i} \Rightarrow z+\frac{1}{z}=(-w^*)+\frac{1}{-w^*}$$

By comparison, $-w^*=z$ or $\dfrac{1}{z}$. Thus $w=-z^*=-4-3\mathrm{i}$ or $w=-\dfrac{1}{z^*}=-\dfrac{1}{4+3\mathrm{i}}=\dfrac{1}{25}(-4+3\mathrm{i})$.

Self-Review 2 (b): If $z = 4 - 3i$, express $z - \frac{1}{z}$ in Cartesian form. Hence, find a complex number w in Cartesian form such that $w + \frac{1}{w} = \frac{78}{25} + \frac{96}{25}i$. [$w = 3 + 4i$ or $\frac{1}{25}(3 - 4i)$]

Solution:

Using GC, $z - \frac{1}{z} = \frac{96}{25} - \frac{78}{25}i$

$$w + \frac{1}{w} = \frac{78}{25} + \frac{96}{25}i = i\left(\frac{96}{25} - \frac{78}{25}i\right) = i\left(z - \frac{1}{z}\right) = iz - \frac{i}{z} = iz + \frac{1}{iz}$$

By comparison, $w = iz$ or $\frac{1}{iz}$.

Thus, $w = i(4-3i) = 3+4i$ or $w = \frac{1}{iz} = \frac{1}{3+4i} = \frac{1}{25}(3-4i)$.

10.3 Solving equations involving Complex numbers

without conjugate: remove anyone using substitution

with conjugate: get rid of one without conjugate then replace with $(x+iy)$

Example 6:
Find the two roots of the equation $ww^* = 4 + 2i + 2iw^*$, giving your answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

Solution:

Let $w = a + bi$, where $a, b \in \mathbb{R}$

Given: $ww^* = 4 + 2i + 2iw^*$

$\Rightarrow (a+bi)(a-bi) = 4 + 2i + 2i(a - bi)$

$\Rightarrow a^2 + b^2 = 4 + 2i + 2ai + 2b$

$a^2 + b^2 = 4 + 2b + i(2 + 2a)$

Comparing real and imaginary parts,

$a^2+b^2 = 4+2b$ — (1)

$0 = 2+2a$ — (2)

From 2, $a = -1$

Sub $a = -1$ in (1):

$(-1)^2 + b^2 - 4 - 2b = 0$

$\Rightarrow b^2 - 2b - 3 = 0$

$\Rightarrow (b-3)(b+1) = 0$

$b = -1$ or $b = 3$

Hence the two roots are $-1 - i$ and $-1 + 3i$.

THINKZONE:

Notice the complexity when we substitute $w = a + ib$ into the equation.

The problem develops into a system of 2 unknowns and subsequently 2 equations.

If there are 2 or more unknown complex numbers involved, this method would be too tedious as an approach and inadvisable in general.

$2iz^* - w = 1+7i \Rightarrow w = 2iz^* - 1 - 7i$

$z + 2w = 16 - 2i$

$z + 2(2iz^* - 1 - 7i) = 16 - 2i$

$z + 4iz^* - 2 - 14i = 16 - 2i$

$(x+iy) + 4i(x-iy) - 2 - 14i - 16 + 2i = 0$

$x + i(y+4x) + 4y = 18 + 12i$

$x + 4y = 18$

$y + 4x = 12$

solve simultaneously

Example 7:

Two complex numbers w and z are such that $z - \mathrm{i}w = 2$ and $2w + (1+2\mathrm{i})z = \mathrm{i}$. Find w and z, giving each answer in the form $x + y\mathrm{i}$.

Solution:

$z - \mathrm{i}w = 2 \quad \Rightarrow \quad z = 2 + \mathrm{i}w$ ---(1)

$2w + (1+2\mathrm{i})z = \mathrm{i}$ --- (2)

Substitute (1) into (2):

$2w + (1+2\mathrm{i})(2+\mathrm{i}w) = \mathrm{i}$

$\Rightarrow \quad 2w + (2+\mathrm{i}w) + (4\mathrm{i} - 2w) = \mathrm{i}$

$\Rightarrow \quad \mathrm{i}w = -2 - 3\mathrm{i}$

$$\Rightarrow \quad w = \frac{-2-3\mathrm{i}}{\mathrm{i}} = \frac{(-2-3\mathrm{i})(-\mathrm{i})}{\mathrm{i}(-\mathrm{i})} = -3 + 2\mathrm{i}$$

Substitute $w = -3 + 2\mathrm{i}$ into (1):

$z = 2 + \mathrm{i}(-3+2\mathrm{i}) = 2 - 2 - 3\mathrm{i} = -3\mathrm{i}$

THINKZONE:

If we adopt the same approach just as in Example 6, substituting $z = a + b\mathrm{i}$ and $w = c + d\mathrm{i}$, we would eventually end up with solving for 4 unknowns.

Do you think this is wise?

Example 8:

The complex numbers p and q satisfy $p = q\mathrm{i} + 2$ and $p^2 - q + 6 + 2\mathrm{i} = 0$. By eliminating q or otherwise, solve the simultaneous equations.

Solution:

Given:

$p = q\mathrm{i} + 2$ ---(1) and $p^2 - q + 6 + 2\mathrm{i} = 0$ --(2)

From (2), $q = p^2 + 6 + 2\mathrm{i}$ ----(3)

Substituting (3) into (1):

$p = (p^2 + 6 + 2\mathrm{i})\mathrm{i} + 2$

$p = p^2\mathrm{i} + 6\mathrm{i} - 2 + 2$

$\mathrm{i}p^2 - p + 6\mathrm{i} = 0$

$$p = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(i)(6i)}}{2i} = \frac{1 \pm \sqrt{25}}{2i}$$

$$p = \frac{1-5}{2i} = \frac{-4}{2i} = \frac{-2 \cdot i}{i \cdot i} = 2i$$

$$\text{or } p = \frac{1+5}{2i} = \frac{6}{2i} = \frac{3i}{i \cdot i} = -3i$$

Substituting the values of p into (3),

$q = (2i)^2 + 6 + 2i = 2 + 2i$.

Or $q = (-3i)^2 + 6 + 2i = -3 + 2i$

THINKZONE:

What are some keywords in this question?

$az^2 + bz + c = 0$

$$\Rightarrow \quad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is still true if $a, b, c \in \mathbb{C}$

Would this method work if the discriminant is a complex number?

Self-Review 3: Solve the simultaneous equations $iz+2w=2$, $z-(1+i)w=4$, giving your answers in the form $x+iy$, where $x,\ y\in\mathbb{R}$. [$z=6-4i$, $w=-1-3i$]

Given: $iz+2w=2$ ----- (1)

$z-(1+i)w=4$ ----- (2)

$(1)-i\times(2)$; $2w+i(1+i)w=2-4i$

Thus $[2+i(1+i)]w=2-4i$

$$\Rightarrow w=\frac{2-4i}{2+i+i^2}=\frac{2-4i}{1+i}=-1-3i$$

Hence, $z=4+(1+i)w=4+(1+i)(-1-3i)=6-4i$.

Conjugate Root Theorem

Let f(z) be a polynomial in z with **real coefficients**. If $\boldsymbol{\alpha}$ is a complex root of f(z) = 0, then $\boldsymbol{\alpha^*}$ is also a root.

E.g. Suppose the equation $az^4+bz^3+cz^2+dz+e=0$, where a, b, c, d and e are real numbers, has a complex root $\boldsymbol{\alpha}$.

Then $a\alpha^4+b\alpha^3+c\alpha^2+d\alpha+e=0$

$\Rightarrow (a\alpha^4+b\alpha^3+c\alpha^2+d\alpha+e)^*=0^*=0$

$\Rightarrow (a\alpha^4)^*+(b\alpha^3)^*+(c\alpha^2)^*+(d\alpha)^*+e^*=0$

$\Rightarrow a^*(\alpha^*)^4+b^*(\alpha^*)^3+c^*(\alpha^*)^2+d^*(\alpha^*)+e=0$

$\Rightarrow a(\alpha^*)^4+b(\alpha^*)^3+c(\alpha^*)^2+d(\alpha^*)+e=0$

Then α^* is also a root.

In general, whenever a polynomial equation with **real coefficients** has complex roots, by Conjugate Root Theorem, the complex roots will occur in conjugate pairs.

Example 9:
Without the use of GC, solve the equation $z^4+z^3-8z^2+14z-8=0$ given that $1+i$ is one of the roots.

Solution:
Since $z^4+z^3-8z^2+14z-8=0$ has **real coefficients**, by Conjugate Root Theorem, $1-i$ is another root.

Then $[z-(1+i)]$ and $[z-(1-i)]$ are factors of the expression on the RHS of the given equation.

Hence, $[z-(1+i)][z-(1-i)]$ is also a factor of it.

$[z-(1+i)][z-(1-i)]$

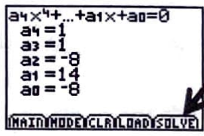
$=z^2-[(1+i)+(1-i)]z+(1+i)(1-i)$

$=z^2-2z+2$

THINKZONE:

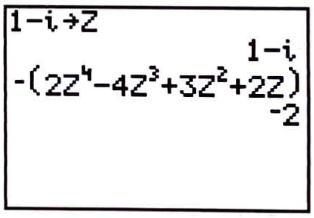
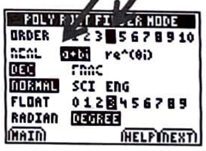
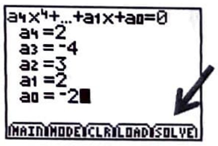
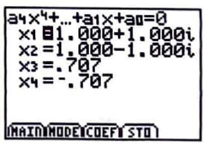
If α and β are roots of a quadratic equation, then the equation can be written as
$(x-\alpha)(x-\beta)=0$
i.e. $x^2-(\alpha+\beta)x+\alpha\beta=0$

That is,
x^2 – (sum of roots)x + (product of roots) = 0

Alternatively, we recall the relationships between a quadratic equation and its two roots : Sum of roots = $(1+i)+(1-i)=2$ Product of roots = $(1+i)(1-i)=2$ So, the quadratic factor is z^2-2z+2. By long division (or comparing coefficients) $z^4+z^3-8z^2+14z-8$ $=(z^2-2z+2)(z^2+3z-4)$ $=(z^2-2z+2)(z+4)(z-1)$ The roots of equation are $1+i,\ 1-i,\ -4,\ 1$	Use of GC to check answer: press **APPS > 5:PlySmlt2 >1: POLY ROOT FINDER** select **ORDER 4** and **a+bi** and press $\boxed{\textbf{graph}}$  

Example 10:

Find the real value of k such that $2x^4-4x^3+3x^2+2x+k=0$ has a complex root $1-i$. Hence factorise $2x^4-4x^3+3x^2+2x+k$ into a product of one quadratic and two linear factors with real coefficients.

Solution: $k=-(2(1-i)^4-4(1-i)^3+3(1-i)^2+2(1-i))$ $=-2$ (using GC) Since $2x^4-4x^3+3x^2+2x-2=0$ has real coefficients, by Conjugate Root Theorem, $1+i$ is also a complex root. These two complex roots are the same roots in Example 9, so we use the results from Example 9. Hence, the quadratic factor is x^2-2x+2. Thus, we can let, $2x^4-4x^3+3x^2+2x-2=\left(x^2-2x+2\right)\left(2x^2+ax+b\right)$ Comparing constants: $-2=2b \Rightarrow b=-1$ Comparing coeff of x^3: $-4=a-4 \Rightarrow a=0$ Hence, $2x^4-4x^3+3x^2+2x-2$ $=\left(x^2-2x+2\right)\left(2x^2-1\right)$ $=\left(x^2-2x+2\right)\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right)$	**THINKZONE:** Use of GC to get answer.  We can use the GC to check the accuracy of our answers too:  

$az^3+bz^2+cz+d=a(z-(2+i))(z-(2-i))(z+3)$
$az^3+bz^2+cz+d=a(z-2-i)(z-2+i)(z+3)$

Example 11: (AJC Prelim 2018/I/7)

Given that $z=-2+3\text{i}$ is a root of the equation $2z^2+(-1+4\text{i})z+c=0$, find the complex number c and the other root.

Solution:	**THINKZONE:**
Let $\alpha=-2+3\text{i}$ and the other root be β $\alpha+\beta=\frac{-1+4i}{2}$ $\Rightarrow \beta=\frac{1}{2}-2i-(-2+3i)=\frac{5}{2}-5i$ $\alpha\beta=\frac{c}{2} \Rightarrow c=2\alpha\beta$ $\Rightarrow c=2(-2+3i)(\frac{5}{2}-5i)=20+35i$	Recall that, if α,β are the roots of the quadratic equation $ax^2+bx+c=0$, then, $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$ Why can't we use Conjugate Root Theorem?

Self-Review 4: (NJC Prelim 2007/I/1) Without the use of GC, verify that $-2+3\text{i}$ is a root of the equation $z^3+5z^2+17z+13=0$ and hence determine the other roots of the equation.

[– 2 –3i, – 1]

Solution:

$(-2+3\text{i})^2=(-2)^2+2(-2)(3\text{i})+(3\text{i})^2=4-12\text{i}+9\text{i}^2=-5-12\text{i}$

$(-2+3\text{i})^3=(-2+3\text{i})^2(-2+3\text{i})=(-5-12\text{i})(-2+3\text{i})=10-15\text{i}+24\text{i}+36=46+9\text{i}$

Substitute $z=-2+3\text{i}$ into the equation $z^3+5z^2+17z+13=0$,

$$(-2+3\text{i})^3+5(-2+3\text{i})^2+17(-2+3\text{i})+13=46+9\text{i}+5(-5-12\text{i})-21+51\text{i}=0$$

Hence $-2+3\text{i}$ is a root of the equation.

By conjugate root theorem, since the coefficients are real numbers, the conjugate is also a root.

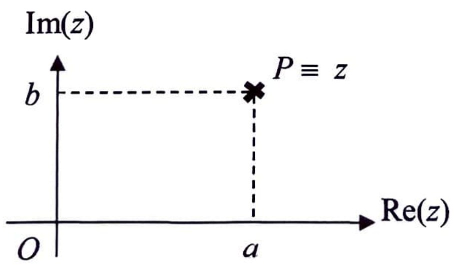
Thus $(z-(-2+3i))(z-(-2-3i))$ is a factor, i.e., $z^2+4z+13$ is a factor.

Since $z^3+5z^2+17z+13=(z^2+4z+13)(z+1)$. Thus the roots are $z=-2\pm3\text{i},-1$

10.4 The Argand Diagram

The x-axis of the Cartesian plane represents the real part of a complex number, while the y-axis represents the imaginary part. Hence, the x-axis is called real axis while the y-axis is the imaginary axis.

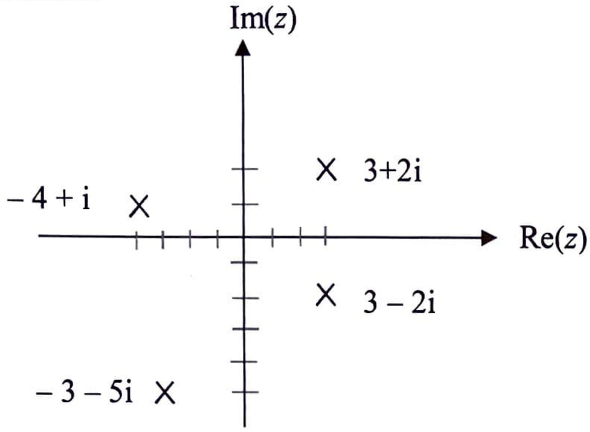
We use the point P with coordinates (a, b) to represent the complex number $z = a + b\text{i}$.



A diagram in which complex numbers are represented this way is called an **Argand Diagram.** (This idea was introduced by the French Mathematician Argand, hence this diagram is known as the **Argand diagram.**) We say that the complex number z is represented by the point P. Sometimes this is written as $P \equiv z$.

Example 12:
Show on the Cartesian plane the following complex numbers $3 + 2i$, $-4 + i$, $-3 - 5i$, $3 - 2i$.

Solution:



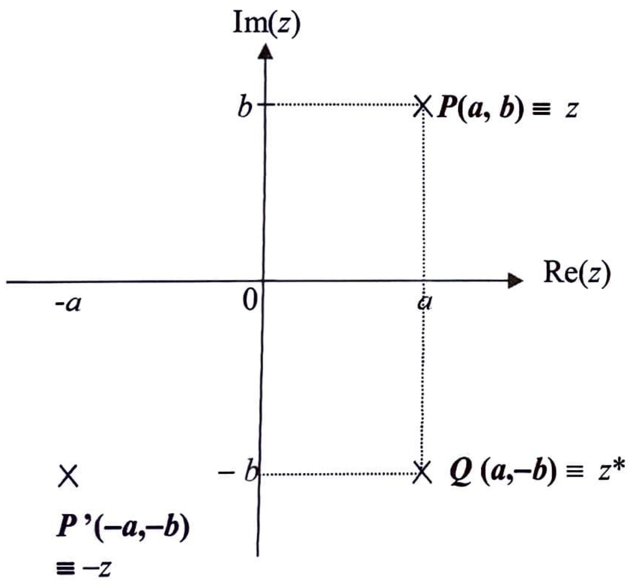
THINKZONE:

It is always good to draw to scale for all Argand diagram.

A distorted diagram will distort your view.

Notice, $3 + 2i$ and $3 - 2i$ are reflections of each other in the real axis.

Conjugation
Again for simplicity, let $z = a + bi$ where $a > 0$ and $b > 0$.
The conjugate of z, $z^* = a - bi$. Let $P \equiv z$ and $Q \equiv z^*$.

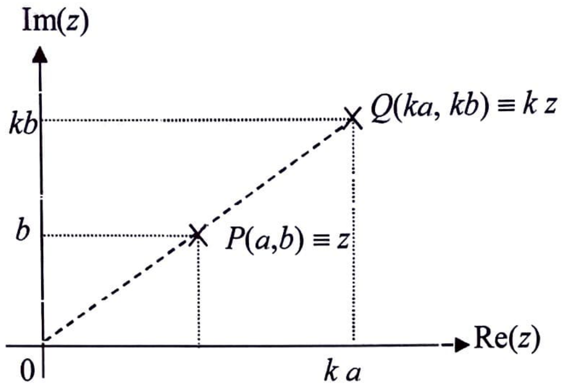


Note:
(a) z^* will be represented by Q is the reflection of $P\ (a, b)$ in the Real axis, i.e. $Q\ (a, -b)$.
(b) $-z$ will be represented by $P'\ (-a, -b)$. It is the reflection in the origin of the point $P\ (a, b)$ representing z.

Multiplication of a complex number z by a real number *k*

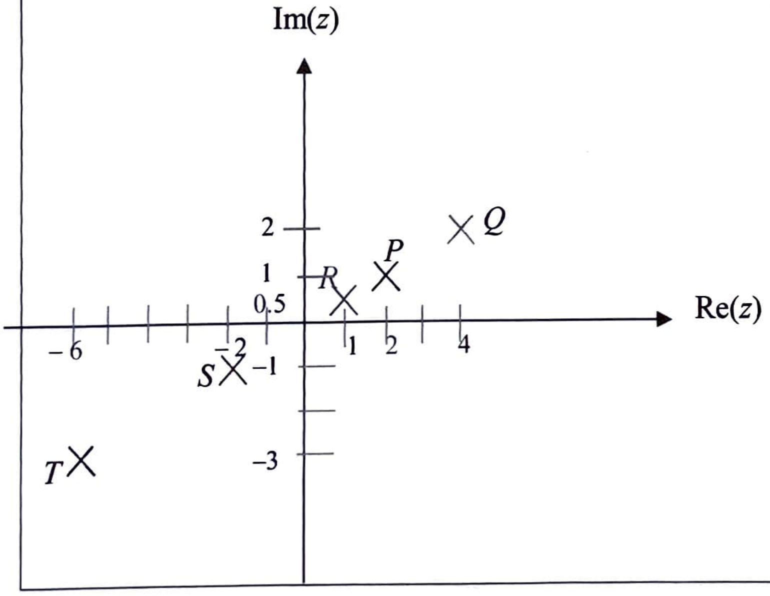
Geometrical Interpretation of *kz* where $k \in \mathbb{R}$ and $z \in \mathbb{C}$

If k is a positive real number, then the complex number $kz = k(a + bi)$ is represented by the line segment OQ where the points $O, P(\equiv z)$ and $Q(\equiv kz)$ are collinear and is such that $OQ = k\,OP$.



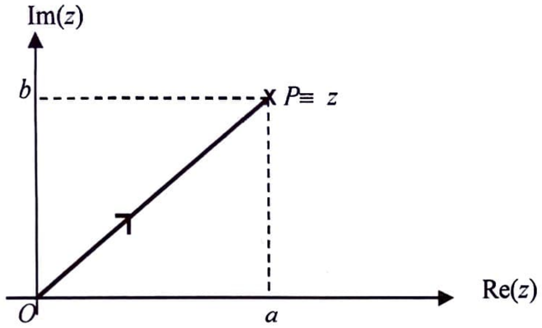
In general, for any point $z \in \mathbb{C}, k \in \mathbb{R}$, the point Q representing kz lies on the straight line passing through the origin O and the point P representing z.
In particular, if $k > 0$ then P and Q lies on the same side of origin. If $k < 0$ then P and Q would lie on opposite sides of the origin. Furthermore, $OQ = |k|\,OP$.

Example 13:
Let the point P represent the complex number $z = 2 + i$ on the Argand diagram. Locate the points representing $2z$, $\frac{1}{2}z$, $-z$, $-3z$.

Solution:	**THINKZONE:**
Let Q, R, S, and T be the points representing $2z$, $\frac{1}{2}z$, $-z$ and $-3z$ respectively. 	Do you notice the points lie on a straight line?

Vector Representation

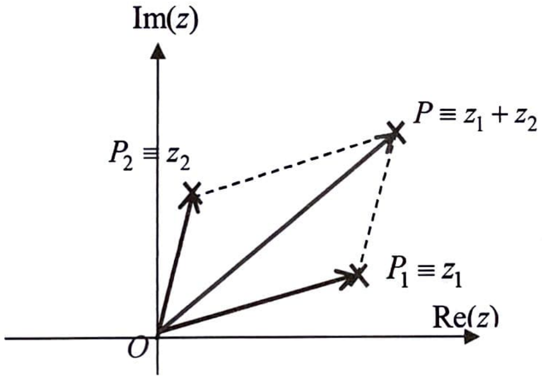
In vector form, the complex number $a + bi$ may be alternatively represented in the Argand diagram by the position vector $\overrightarrow{OP}$ where P is (a, b).



Note: Do **not** write $\overrightarrow{OP} = z$ (A vector **cannot** be equal to a complex number). Instead, we will write $P \equiv z$

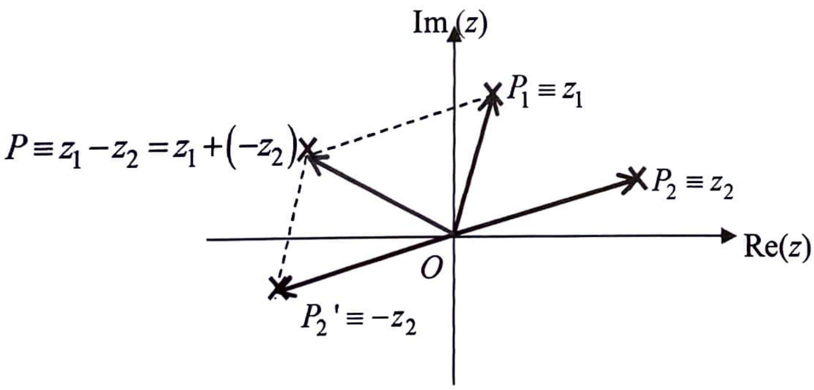
Addition of Two Complex Numbers in an Argand Diagram

Geometrically, if P_1, P_2 represent the complex numbers z_1, z_2, then $P \equiv z_1 + z_2$ where P is the 4th vertex of the parallelogram determined by the points, O, P_1, P_2 .



Subtraction of Two Complex Numbers in an Argand Diagram

Writing $z_1 - z_2$ as $z_1 +(- z_2)$, the point P, the 4th vertex of the parallelogram determined by the points O, P_1, P_2' represents $z_1 - z_2$.



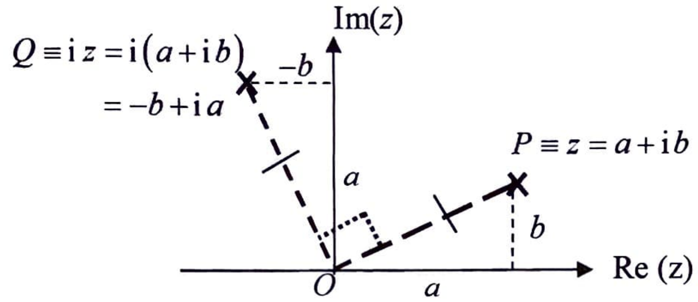
Note: Addition and subtraction of two complex numbers are 'the same' as vector addition and subtraction.

Multiplication of a complex number by i

For the sake of simplicity, let $z = a + bi$ where $a, b \in \mathbb{R}$ and $a > 0$, $b > 0$.

$\mathrm{i}\, z = \mathrm{i}\,(a + b\mathrm{i}) = a\,\mathrm{i} + b\,\mathrm{i}^2 = -b + a\mathrm{i}$.

Let $P \equiv z$ and $Q \equiv \mathrm{i}z$.



From above, we see that $\angle POQ = 90°$ and

length $OP = |\mathrm{i}z| = |z| = \sqrt{a^2+b^2}$ = length OQ

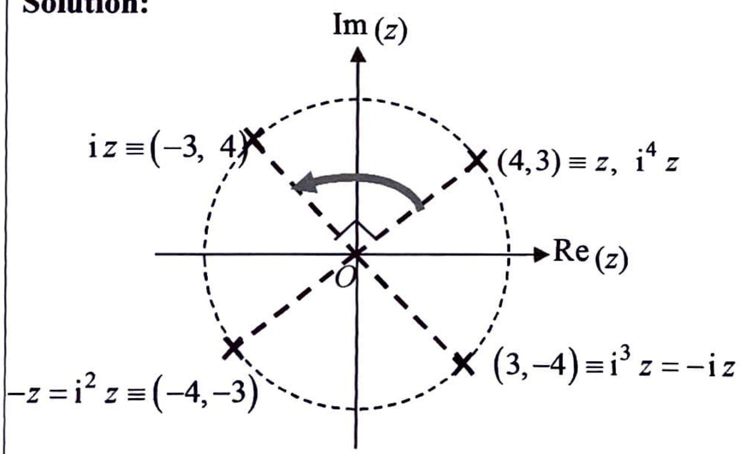
Therefore the vector representing the complex number $\mathrm{i}z$ **is obtained by rotating the vector representing** z **about** O **through 90° in an anti-clockwise sense.**

Question: What is the corresponding effect of multiplying a complex number z **by** $-\mathrm{i}$**?**
Ans: The vector representing the complex number $-\mathrm{i}z$ is obtained by rotating the vector representing z about O through 90° in a clockwise sense.

Example 14:

If $z = 4 + 3\mathrm{i}$, locate the points representing the complex numbers z, $\mathrm{i}z$, i^2z, i^3z and i^4z on the Argand diagram.

Solution:



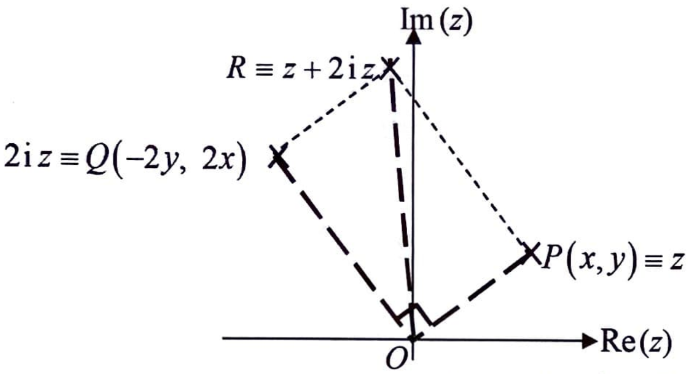
THINKZONE:
This is done by an anti-clockwise rotation of the corresponding vector through an angle of 90° about the origin.

For example, the point $(-3,\ 4)$ is obtained by doing an anti-clockwise rotation of the line segment joining (4, 3) and (0, 0) through an angle of 90° about the origin.

Example 15 (N92/2/12):

The complex number z is given by $z = x + y\mathrm{i}$, where $x > 0$ and $y > 0$. Sketch an Argand diagram, with origin O, showing the points P, Q and R representing z, $2\mathrm{i}z$ and $(z + 2\mathrm{i}z)$ respectively. State the size of angle POQ, and describe briefly the geometrical relationship between O, P, Q and R.

Solution:



multiply z by 2i means double the length of OP, and then rotating it by an angle of 90° anti-clockwise

hence ∡POQ = 90°

Also OQ = 2OP

The points O, P, Q, R form a rectangle OPRQ (describe in an anticlockwise sense) where the length OQ is twice the length of OP.

THINKZONE:

For *P*: If $x > 0$ and $y > 0$, which quadrant will the complex number lie in?

For *Q*: What is the corresponding effect of multiplying the complex number *z* with i and followed by 2?

For *R*: What is the effect of addition of two complex numbers in an Argand diagram?

Could you find the area of the quadrilateral *OPQR* in terms of *x* and *y*?

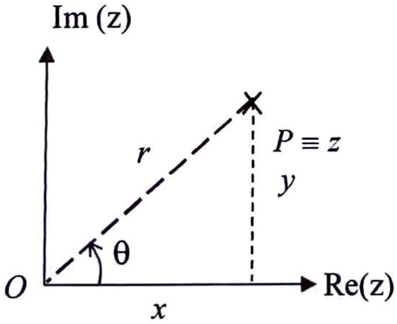
How do you think you will be graded if you do not draw the diagram accurately? Is accuracy paramount?

You have to ensure that the diagram is drawn to scale.

10.5 Trigonometric form (or Polar form) of a Complex Number

10.5.1 Modulus and argument:

Let point *P* represents the complex number $z = x + iy$.
Let the length of the line segment *OP* be *r* and the angle subtended by *OP* and the real axis be θ.



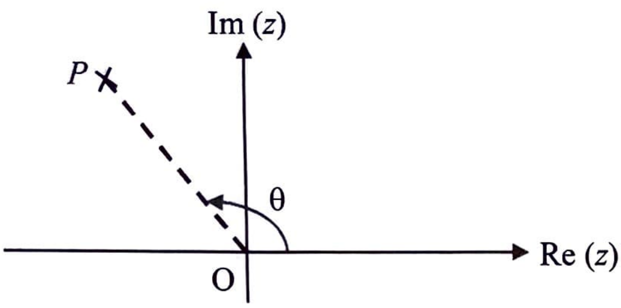
Then $x = r\cos\theta, y = r\sin\theta$ and $r = \sqrt{x^2 + y^2}$.

r is called the **modulus of** *z* and is denoted by $|z|$, i.e. $r = |z| = |x + iy| = \sqrt{x^2 + y^2}$.

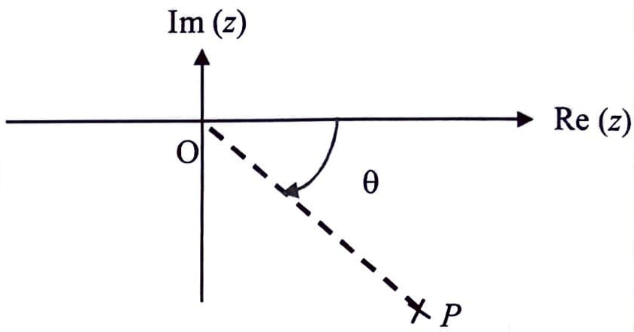
$\boldsymbol{\theta}$ is called the **argument of** *z* and is denoted by $\arg z$, i.e. $\theta = \arg z = \arg(x + iy)$.

Note:

If P is **above** the real axis, θ is measured **anti-clockwise sense** and θ is **positive**.



If P is **below** the real axis, θ is measured **clockwise sense** and θ is **negative**.



Conventionally, we restrict θ to the range $-\pi < \theta \leq \pi$, called the **principal range**. The **unique** value of θ lying in the principal range is called the **principal argument**.

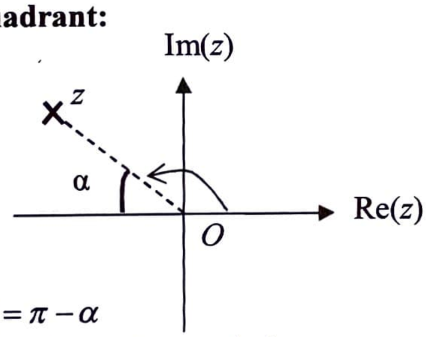
From now onwards, the argument of a complex number is taken to be the principal argument unless otherwise stated. Follow the **ABS** approach to get the principal argument correctly.

Step 1. Argand diagram: To find the argument of a complex number, always **draw an Argand diagram** and indicate on the diagram which quadrant the complex number lies in.

Step 2. Basic angle: Find basic angle $= \alpha = \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

Step 3. Sign: Find the sign and magnitude of the argument of a complex number. This depends on which quadrant the complex number is in.

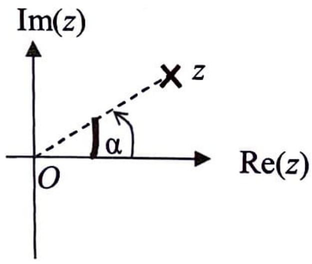
2nd Quadrant:



$\arg(z) = \pi - \alpha$

Argument is positive and obtuse.

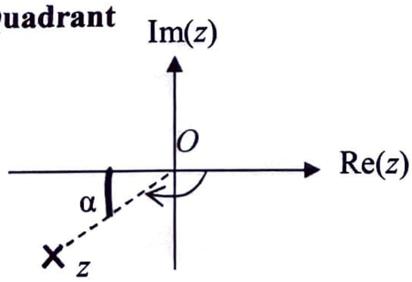
1st Quadrant:



$\arg(z) = \alpha$

Argument is positive and acute.

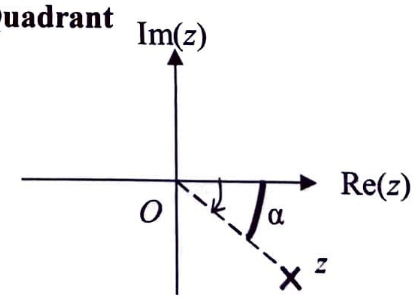
3rd Quadrant



$\arg(z) = -(\pi - \alpha)$

Argument is negative and obtuse.

4th Quadrant

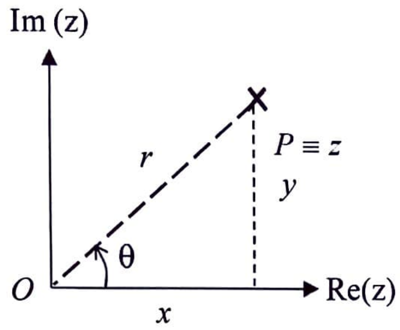


$\arg(z) = -\alpha$

Argument is negative and acute.

10.5.2 Complex Number in Trigonometric form (or Polar form)

Consider the complex number $z = x + iy$ in the Argand diagram shown below:



If the *principal argument* of $z = x + yi$ is defined to be θ, where $-\pi < \theta \le \pi$,

$$z = x + yi$$
$$= r\cos\theta + r\sin\theta\,\mathrm{i} \qquad \sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$
$$= r(\cos\theta + \mathrm{i}\sin\theta) \qquad r\sin\theta = y \qquad r\cos\theta = x$$
$$= re^{i\theta}$$

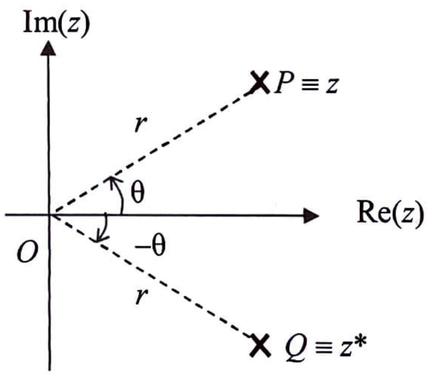
$z = r$ **(cos θ + i sin θ)** where $-\pi < \theta \le \pi$ is called the **trigonometric form, modulus argument form or polar form** of the complex number.

Note:

(i) If $z = r(\cos\theta + \mathrm{i}\sin\theta)$, then

$$z^* = r(\cos\theta - \mathrm{i}\sin\theta)$$
$$= r[\cos(-\theta) + \mathrm{i}\sin(-\theta)]$$

Both z and z^* are represented on an Argand diagram as shown.



It is easy to notice that: $|z| = |z^*| = r$, $\arg z^* = -\arg z = -\theta \quad (-\pi < \theta \le \pi)$

(ii) z is purely real $\Leftrightarrow$ principal argument of z is 0 or π.

(iii) z is purely imaginary $\Leftrightarrow$ principal argument of z is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Remark:

It is important to remember the trigonometric ratios of special angles: i.e. $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$. The following examples will illustrate the relationship between "special" complex numbers with these special angles.

Example 16:

Express the following in the trigonometric form:

(a) $z=1+\sqrt{3}\,\mathrm{i}$ (b) $z=2\mathrm{i}$ (c) $z=-\sqrt{3}+\mathrm{i}$ (d) $z=-3$

(e) $z=-1-\mathrm{i}$ (f) $z=-2\mathrm{i}$ (g) $z=1-\sqrt{3}\,\mathrm{i}$ (h) $z=5$

Solution:

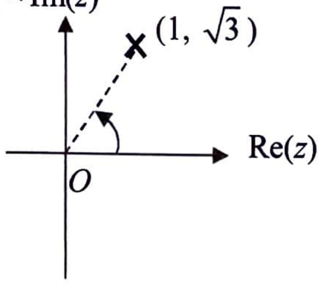
(a) $z=1+\sqrt{3}\,\mathrm{i}$

$r=|z|=\sqrt{1+3}=2$

basic $\measuredangle = \left|\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)\right| = \frac{\pi}{3}$ rad

$\arg z = \frac{\pi}{3}$ rad

$z=2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$



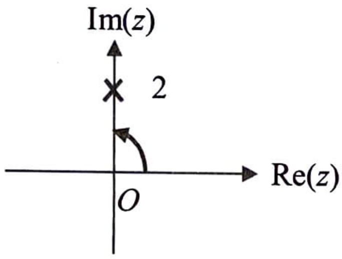
(b) $z=2\mathrm{i}$

$|z|=2$

basic $\measuredangle=\dfrac{\pi}{2}$

$\arg z=\dfrac{\pi}{2}$

$z=2\left(\cos\dfrac{\pi}{2}+\mathrm{i}\sin\dfrac{\pi}{2}\right)$



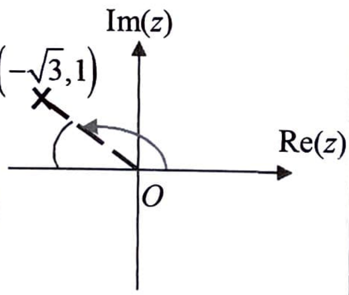
(c) $z=-\sqrt{3}+\mathrm{i}$

$r=|z|=\sqrt{3+1}=2$

basic $\measuredangle=\left|\tan^{-1}\frac{1}{\sqrt{3}}\right|=\frac{\pi}{6}$

$\arg z=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

$z=2\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)$



(d) $z=-3$

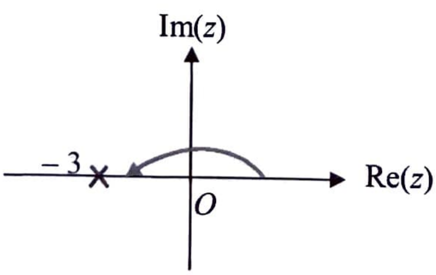
$|z|=3$

basic $\measuredangle$

$=\tan^{-1}\left(\left|\dfrac{0}{-3}\right|\right)=0$

$\arg z=\pi$

$z=3\left(\cos\pi+\mathrm{i}\sin\pi\right)$



THINKZONE:

Always represent the complex number on an Argand diagram to visualize.

Followed the **ABS** approach.

Use of GC to check/find the modulus and argument of a complex number

θ is in the 1st quadrant. It is positive and acute.

θ is between 1st and 2nd quadrant.

It is positive and perpendicular to the real axis.

θ is in 2nd quadrant.

It is positive and obtuse.

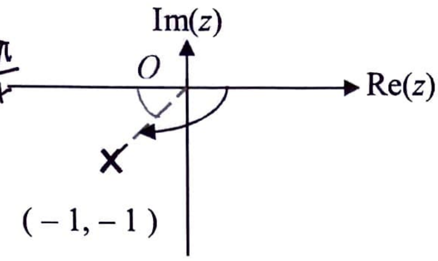
(e) $z=-1-\mathrm{i}$

$|z|=\sqrt{1+1}=\sqrt{2}$

basic $\measuredangle = \left|\tan^{-1}\left(\frac{-1}{-1}\right)\right| = \frac{\pi}{4}$

$\arg z = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

$z = \sqrt{2}\left(\cos -\frac{3\pi}{4} + i \sin -\frac{3\pi}{4}\right)$



θ is between 2nd and 3rd quadrant.

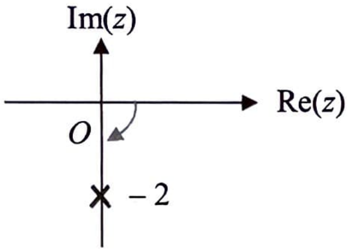
Note that : $\arg z \neq -\pi$ as $-\pi < \arg z \leq \pi$

(f) $z=-2\mathrm{i}$

$|z|=2$

basic $\measuredangle = \frac{\pi}{2}$

$\arg z = -\frac{\pi}{2}$



$$z=2\left(\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$$

θ is in the 3rd quadrant. It is negative and obtuse.

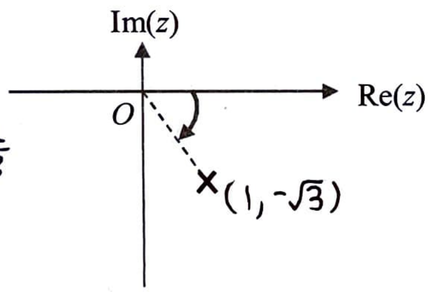
(g) $z=1-\sqrt{3}\,\mathrm{i}$

$|z|=\sqrt{1+3}=2$

basic $\Delta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$\arg z = -\frac{\pi}{3}$

$z = 2\left(\cos -\frac{\pi}{3} + i \sin -\frac{\pi}{3}\right)$



θ is between 3rd and 4th quadrant. It is negative.

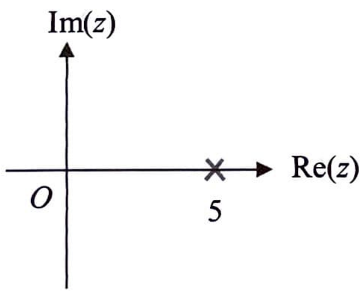
(h) $z=5$

$|z|=5$

basic $\measuredangle = 0$

$\arg z = 0$

$z=5(\cos 0+\mathrm{i}\sin 0)$

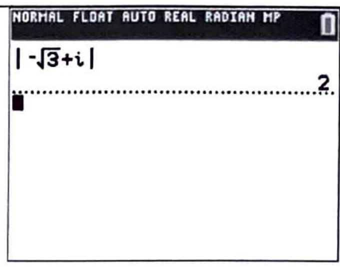
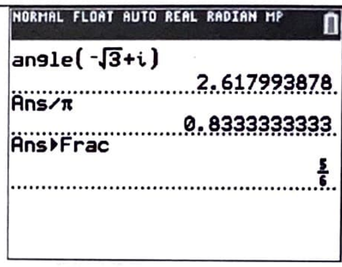


θ is in the 4th quadrant. It is negative and acute.

GC basics to calculate $|z|$ and $\arg(z)$:

You may use the GC to calculate the modulus and the argument of a complex number.

Take for e.g. in Example 16(c), $z=-\sqrt{3}+\mathrm{i}$.

To find the modulus, we can key alpha window (A-lock tblset f2) to access the modulus function, and type in our complex number accordingly. (note: if it is a irrational number, you can square the number, then square root it back to get the nice surd form)	
To find the angle, we can key math (test A) > 4 (L4 T) to get the argument of the complex number (note: the number given is usually irrational, we can divide our answer by π to see if $\arg(z)$ is a nice number in terms of π.	

10.5.3 Multiplication in Trigonometric Form

If $z_1=r_1(\cos\theta_1+\mathrm{i}\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+\mathrm{i}\sin\theta_2)$, then

$$\begin{aligned}z_1z_2&=r_1r_2(\cos\theta_1+\mathrm{i}\sin\theta_1)(\cos\theta_2+\mathrm{i}\sin\theta_2)\\&=r_1r_2\left[(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+\mathrm{i}(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)\right]\\&=r_1r_2\left[\cos(\theta_1+\theta_2)+\mathrm{i}\sin(\theta_1+\theta_2)\right]\end{aligned}$$

$$|z_1z_2|=r_1r_2=|z_1||z_2|$$

$$\arg(z_1z_2)=\theta_1+\theta_2=\arg(z_1)+\arg(z_2) \quad \text{where} \quad -\pi<\arg(z_1z_2),\ \arg(z_1),\ \arg(z_2)\le\pi$$

In general,

$$z_1z_2....z_n=r_1r_2......r_n\left[\cos(\theta_1+\theta_2+.....+\theta_n)+\mathrm{i}\sin(\theta_1+\theta_2+.....+\theta_n)\right]$$

Taking $z_1=z_2=....=z_n=z$

$$z^n=r^n(\cos n\theta+\mathrm{i}\sin n\theta)$$

$$|z^n|=|z|^n, \quad \arg(z^n)=n\theta=n\arg(z) \quad \text{where} \quad -\pi<\arg(z^n),\ \arg(z)\le\pi$$

10.5.4 Division in Trigonometric Form

If $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$, then

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)} = \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 - \mathrm{i}\sin\theta_2)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)(\cos\theta_2 - \mathrm{i}\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right]}{(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2)\right]$$

$$\therefore \quad \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2) \text{ where } -\pi < \arg\left(\frac{z_1}{z_2}\right),\ \arg(z_1),\ \arg(z_2) \le \pi$$

Properties of modulus and argument of complex numbers:

- $|z_1z_2| = |z_1||z_2|$ $\quad\quad$ $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$
- $|z^n| = |z|^n$ $\quad\quad$ $\arg(z^n) = n\arg(z)$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $\quad\quad$ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
- $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ $\quad\quad$ $\arg\left(\frac{1}{z}\right) = \arg(1) - \arg(z) = 0 - \arg(z) = -\arg(z)$
- $zz^* = |z|^2$
- $|z^*| = |z|$ $\quad\quad$ $\arg z^* = -\arg z$

Note:

i) The values of $\arg(z_1 z_2)$, $\arg\left(\frac{z_1}{z_2}\right)$ and $\arg(z^n)$ obtained may not be the principal argument. If this occurs, then we have to add or subtract **multiples of** 2π to obtain the desired principal argument (i.e. $-\pi < \theta \le \pi$).

ii) It is worth noting that the complex function '**arg**' behaves just like the **logarithmic functions** 'ln' in view of the properties above.

Example 17:

The complex numbers p and q are given by $p = \sqrt{3} - \mathrm{i}$ and $q = -4 + 4\mathrm{i}$. Without the use of GC, find the modulus and argument of

(a) $\frac{1}{p}$ $\quad$ (b) q^* $\quad$ (c) p^3 $\quad$ (d) pq $\quad$ (e) $\frac{p}{q^2}$.

$\left|\frac{1}{p}\right| = \frac{1}{|p|} = \frac{1}{\sqrt{3+1}} = \frac{1}{2}$

$\arg\left(\frac{1}{p}\right) = -\arg(p) = \tan^{-1}\left(\left|\frac{1}{\sqrt{3}}\right|\right) = \frac{\pi}{6}$

$|q^*| = |q| = \sqrt{4^2+4^2} = \sqrt{32}$

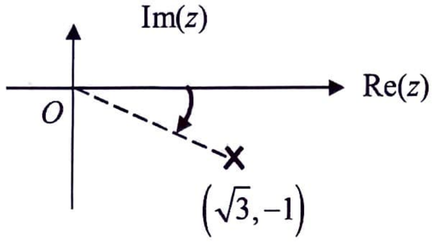
basic arg $|q^*| = -\arg|q| = \tan^{-1}\left(\left|\frac{4}{-4}\right|\right) = \frac{\pi}{4}$

Solution:

$|p| = \sqrt{3+1} = 2$

basic $\measuredangle = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

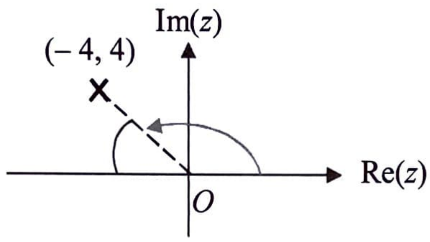
$\arg p = -\frac{\pi}{6}$



$|q| = \sqrt{16+16} = 4\sqrt{2}$

basic $\measuredangle = \tan^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{4}$

$\arg q = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$



(a) $\left|\frac{1}{p}\right| = \frac{1}{|p|} = \frac{1}{2}$

$\arg\left(\frac{1}{p}\right) = \arg(1) - \arg(p) = 0 - \left(-\frac{\pi}{6}\right) = \frac{\pi}{6}$

(b) $|q^*| = |q| = 4\sqrt{2}$

$\arg(q^*) = -\arg(q) = -\frac{3\pi}{4}$

(c) $|p^3| = |p|^3 = 8$

$\arg(p^3) = 3\arg(p) = -\frac{\pi}{2}$

(d) $|pq| = |p||q| = 8\sqrt{2}$

$\arg(pq) = \arg p + \arg q = -\frac{\pi}{6} + \frac{3\pi}{4} = \frac{7\pi}{12}$

(e) $\left|\frac{p}{q^2}\right| = \frac{|p|}{|q|^2} = \frac{2}{(4\sqrt{2})^2} = \frac{1}{16}$

$\arg\left(\frac{p}{q^2}\right) = \arg p - \arg q^2$

$= \arg p - 2\arg p$

$= -\frac{\pi}{6} - 2\left(\frac{3\pi}{4}\right)$

Since $-\pi < \arg z \le \pi$, we have

$\arg\left(\frac{p}{q^2}\right) = -1\frac{2}{3}\pi + 2\pi = \frac{\pi}{3}$

THINKZONE:

Complex function '**arg**' behaves just like the **logarithmic functions** 'ln'.

We have to add or subtract **multiples of** 2π to obtain the desired principal argument (i.e. $-\pi < \theta \le \pi$).

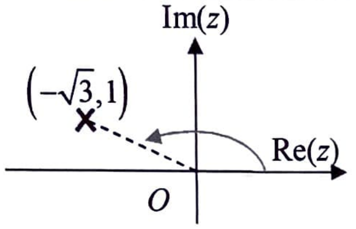
Example 18:

Write down the moduli and argument of $-\sqrt{3}+\mathrm{i}$ and $4+4\mathrm{i}$. Hence express the complex number $z=\dfrac{\left(-\sqrt{3}+\mathrm{i}\right)^3}{\left(4+4\mathrm{i}\right)^4}$ in the trigonometric form.

Solution:

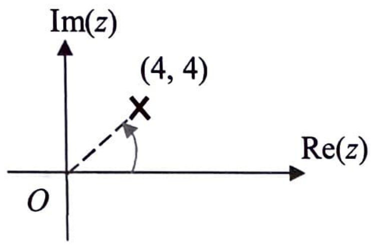
$\left|-\sqrt{3}+\mathrm{i}\right|=\sqrt{3+1}=2$

$\arg(-\sqrt{3}+\mathrm{i})=\dfrac{5}{6}\pi$ (use GC)



$|4+4\mathrm{i}|=\sqrt{16+16}=4\sqrt{2}$

$\arg(4+4\mathrm{i})=\dfrac{\pi}{4}$ (use GC)



$|z|=\left|\dfrac{(-\sqrt{3}+i)^3}{(4+4i)^4}\right|=\dfrac{|-\sqrt{3}+i|^3}{|4+4i|^4}=\dfrac{2^3}{(4\sqrt{2})^4}$

$=\dfrac{1}{128}$

$\arg z=\arg\left[\dfrac{(-\sqrt{3}+i)^3}{(4+4i)^4}\right]$

$=3\arg(-\sqrt{3}+i)-4\arg(4+4i)$

$=3\left(\frac{5}{6}\pi\right)-4\left(\frac{\pi}{4}\right)=\frac{3}{2}\pi$

But $-\pi<\arg z\le\pi$

then $\arg z=\frac{3}{2}\pi-2\pi=-\frac{\pi}{2}$

Hence, $z=\dfrac{(-\sqrt{3}+i)^3}{(4+4i)^4}=\dfrac{1}{128}\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

THINKZONE:

You are allowed to use GC to find the argument of a complex number (unless otherwise stated).

Do check the argument by drawing a simple Argand diagram just to confirm.

Put $|z|$ and $\arg(z)$ together using $z=r\left(\cos\theta+\mathrm{i}\sin\theta\right)$

Example 19: (N08/II/3(a))

The complex number w has modulus r and argument θ, where $0<\theta<\frac{1}{2}\pi$, and w^* denotes the conjugate of w. State the modulus and argument of p, where $p=\dfrac{w}{w^*}$.

Given that p^5 is real and positive, find the possible values of θ.

Solution:

$|p|=\left|\dfrac{w}{w^*}\right|=\dfrac{|w|}{|w^*|}=\dfrac{|w|}{|w|}=1$

$\arg(p)=\arg\left(\dfrac{w}{w^*}\right)=\arg(w)-\arg(w^*)$

$=\arg(w)+\arg(w)=2\arg(w)=2\theta$

Note that $-\pi<2\theta<\pi$ since $0<\theta<\frac{1}{2}\pi$, thus θ

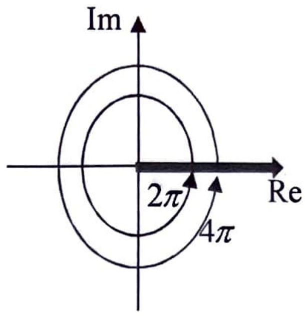
THINKZONE:

Use properties of modulus and arguments.

$0<\theta<\dfrac{1}{2}\pi\Rightarrow\ 0<2\theta<\pi$.

$\arg(p^5) = 5\arg(p) = 10\theta$, which is obviously out of principal argument range
since p^5 is real and positive, $\arg(p^5) = 0$
Thus, $10\theta = 0, 2\pi, 4\pi, 6\pi \ldots$
$\theta = 0, \frac{2\pi}{10}, \frac{4\pi}{10}, \frac{6\pi}{10} \ldots$
Since $0 < \theta < \frac{1}{2}\pi$, thus $\theta = \frac{\pi}{5}, \frac{2\pi}{5}$

So 2θ is in the principal argument range.



Self-Review 5: Without the use of calculator, find the moduli and arguments of

(a) $\dfrac{1+\mathrm{i}}{1-\mathrm{i}}$ and (b) $\dfrac{\sqrt{2}}{1-\mathrm{i}}$. $[1, \frac{\pi}{2}, 1, \frac{\pi}{4}]$

Solution:

(a)

$$\left|\frac{1+\mathrm{i}}{1-\mathrm{i}}\right| = \frac{|1+\mathrm{i}|}{|1-\mathrm{i}|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$$\arg\left(\frac{1+\mathrm{i}}{1-\mathrm{i}}\right) = \arg(1+\mathrm{i}) - \arg(1-\mathrm{i}) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(b)

$$\left|\frac{\sqrt{2}}{1-\mathrm{i}}\right| = \frac{\sqrt{2}}{|1-\mathrm{i}|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$$\arg\left(\frac{\sqrt{2}}{1-\mathrm{i}}\right) = \arg\left(\sqrt{2}\right) - \arg(1-\mathrm{i}) = 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

10.6 Exponential form of a Complex number

Using Maclaurin's series expansion,

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \ldots, \quad \theta \in \mathbb{R}$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \ldots, \quad \theta \in \mathbb{R}$$

From MF26, $e^{\theta} = 1 + \theta + \frac{1}{2!}\theta^2 + \frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}\theta^5 + \ldots, \quad \theta \in \mathbb{R}$

We have

$$e^{i\theta} = 1+(i\theta)+\frac{1}{2!}(i\theta)^2+\frac{1}{3!}(i\theta)^3+\frac{1}{4!}(i\theta)^4+\frac{1}{5!}(i\theta)^5+\ldots$$
$$= 1+i\theta-\frac{1}{2!}\theta^2-\frac{1}{3!}i\theta^3+\frac{1}{4!}\theta^4+\frac{1}{5!}i\theta^5+\ldots$$
$$= \left(1-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4-\ldots\right)+i\left(\theta-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5-\ldots\ldots\right)$$
$$= \cos\theta + i\sin\theta$$

($e^{i\theta} = \cos\theta + i\sin\theta$ is called **Euler's formula**)

$$\boxed{\therefore \quad z = r(\cos\theta + i\sin\theta) = re^{i\theta}, \text{ where } -\pi < \theta \le \pi}$$

This is called the exponential form of z and θ must be in radians.

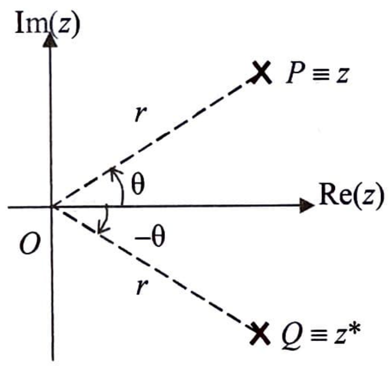
Note:

(1) $(re^{i\theta})^* = re^{i(-\theta)} = re^{-i\theta}$

(2) $e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$

(3) $re^{i(\theta+2n\pi)} = re^{i\theta}$, $n \in \mathbb{Z}$

(4) $r_1 e^{i\theta} . r_2 e^{i\alpha} = r_1 r_2 e^{i(\theta+\alpha)}$



$\therefore$ $[r_1(\cos\theta + i\sin\theta)][r_2(\cos\alpha + i\sin\alpha)] = r_1r_2[\cos(\theta+\alpha) + i\sin(\theta+\alpha)]$

Note: The exponential form is especially useful for multiplication and division of complex numbers as the index laws are applicable.

Example 20:
Express $1 + \sqrt{3}\,i$ and $1 - i$ in exponential form. Hence, find, in exponential form, the following complex numbers:

(i) $(1 + \sqrt{3}\,i)^3(1 - i)$, (ii) $\dfrac{(1+\sqrt{3}i)^3}{1-i}$.

Solution:

$|1 + \sqrt{3}\,i| = \sqrt{1+3} = 2$, $\arg(1 + \sqrt{3}\,i) = \frac{\pi}{3}$

Hence, $1+\sqrt{3}i = 2e^{\frac{\pi}{3}i}$

$|1 - i| = \sqrt{1+1} = \sqrt{2}$, $\arg(1 - i) = -\frac{\pi}{4}$

Hence, $1-i = \sqrt{2}e^{-\frac{\pi}{4}i}$

THINKZONE:

You can use GC to find argument

(i) $(1+\sqrt{3}\,i)^3(1-i)$

$$= \left(2e^{\frac{\pi}{3}i}\right)^3\left(\sqrt{2}e^{-\frac{\pi}{4}i}\right) = 8\sqrt{2}e^{\frac{3\pi}{4}i}$$

(ii) $$\frac{(1+\sqrt{3}i)^3}{1-i} = \frac{(2e^{\frac{\pi}{3}i})^3}{\sqrt{2}e^{-\frac{\pi}{4}i}} = \frac{8e^{\pi i}}{\sqrt{2}e^{-\frac{\pi}{4}i}}$$

$$= \frac{8}{\sqrt{2}}e^{(\pi+\frac{\pi}{4})i} = \frac{8\sqrt{2}}{2}e^{\frac{5}{4}\pi i}$$

Reminder: Always **ensure** that argument is in the principal range by adding or subtracting multiples of 2π

Example 21: (Method of "taking out half power")

Show that the modulus of the complex number z_1, where $z_1 = 1+e^{i\pi/3}$, is $\sqrt{3}$, and find its argument.

Solution:

$$z_1 = 1+e^{i\pi/3} = e^{i\pi/3}+1$$
$$= (e^{i\pi/6})^2 + e^{i\pi/6}\times e^{-i\pi/6}$$
$$= e^{i\pi/6}(e^{i\pi/6}+e^{-i\pi/6})$$
$$= e^{i\pi/6}\left(2\cos\left(\frac{\pi}{6}\right)\right)$$
$$= \cancel{e^{i\pi/6}} = \sqrt{3}e^{i\pi/6}$$

Thus $|z_1| = \sqrt{3}$, $\arg(z_1) = \frac{\pi}{6}$

THINKZONE:
Observe the expression in the question is just nice for us to apply the strategy of "Taking out half powers".

Recall, we have the useful results:

$$e^{i\theta}+e^{-i\theta} = 2\cos\theta$$
$$e^{i\theta}-e^{-i\theta} = 2i\sin\theta$$

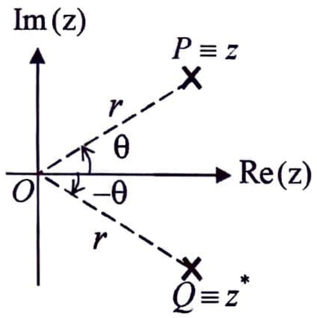
"Taking out half power" is a fast way to simplify a complex number. However, it works only when we have expressions of the form $1+e^{ik\theta}$ or $1-e^{ik\theta}$ or $e^{ik\theta}-1$.

So in general, $$e^{i\theta}+1 = e^{\frac{i\theta}{2}}\left(e^{\frac{i\theta}{2}}+e^{\frac{-i\theta}{2}}\right) = e^{\frac{i\theta}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}+\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right) = \left(2\cos\frac{\theta}{2}\right)e^{\frac{i\theta}{2}}$$

Example 22:

Show that $e^{i2\theta}+e^{-i2\theta}$ is a real number for all θ. The complex number w is given by $w = \dfrac{2}{1+e^{i4\theta}}$ where $0<\theta<\pi$. Hence show that $\text{Re}(w)=1$.

Solution:



THINKZONE:
What are some keywords in this question?

$$e^{i2\theta}+e^{-i2\theta}$$
$$=(\cos 2\theta+i\sin 2\theta)+(\cos(-2\theta)+i\sin(-2\theta))$$
$$=\cos 2\theta+i\sin 2\theta+\cos 2\theta-i\sin 2\theta$$
$$=2\cos 2\theta$$
$$=2\cos 2\theta+0i$$

since $\cos(-\alpha)=\cos\alpha$ and $\sin(-\alpha)=-\sin\alpha$

This is a real number since $\mathrm{Im}\left(e^{i2\theta}+e^{-i2\theta}\right)=0$.

$$w=\frac{2}{1+e^{i4\theta}}=\frac{2}{e^{i4\theta}+1}=\frac{2}{(e^{i2\theta}+e^{-i2\theta})e^{i2\theta}}$$

Why do we factorise $e^{i2\theta}$?

$$=\frac{2e^{-i2\theta}}{e^{i2\theta}+e^{-i2\theta}}=\frac{2e^{-i2\theta}}{2\cos 2\theta}$$

$$=\frac{\cos 2\theta-i\sin 2\theta}{\cos 2\theta}=\frac{\cos 2\theta}{\cos 2\theta}-\frac{i\sin 2\theta}{\cos 2\theta}=1-i\tan 2\theta$$

Hence, $\mathrm{Re}(w)=1$.

Self-Review 6: Given that $w=\dfrac{1}{e^{i4\theta}-1}$, show that $\mathrm{Re}(w)=-\dfrac{1}{2}$ and find $\mathrm{Im}(w)$ in terms of θ.

$[-\frac{1}{2}\cot 2\theta]$

Solution:

$$w=\frac{1}{e^{i4\theta}-1}=\frac{1}{e^{i2\theta}\left(e^{i2\theta}-e^{-i2\theta}\right)}=\frac{e^{-i2\theta}}{2i\sin 2\theta}=\frac{\cos 2\theta-i\sin 2\theta}{2i\sin 2\theta}$$

$$=\frac{\cos 2\theta}{2i\sin 2\theta}-\frac{i\sin 2\theta}{2i\sin 2\theta}$$

$$=\frac{1}{2i}\times\cot 2\theta-\frac{1}{2}=-\frac{1}{2}-\frac{i}{2}\cot 2\theta$$

Thus $\mathrm{Re}(w)=-\dfrac{1}{2}$ and $\mathrm{Im}(w)=-\dfrac{1}{2}\cot(2\theta)$

Learning Experiences

You may visit the following websites for interesting articles related to complex numbers.

1. Quadratic Equations: https://plus.maths.org/content/os/issue30/features/quadratic/index
2. About Euler: https://plus.maths.org/content/os/issue42/features/wilson/index
3. Complex Numbers in Movies: https://plus.maths.org/content/os/issue42/features/lasenby/index
4. Fractals: http://www.intmath.com/complex-numbers/fractals.php
5. More on Fractals: https://plus.maths.org/content/os/issue40/features/devaney/index

NANYANG JUNIOR COLLEGE

DEPARTMENT OF MATHEMATICS

YEAR 1 (2019) **H2 MATHEMATICS TUTORIAL 10A** **Complex Numbers**

OPERATION WITH COMPLEX NUMBERS

1. Simplify the following complex numbers without the use of GC.

(a) $(3-7\mathrm{i})(-4+\mathrm{i})$ (b) $\dfrac{7+5\mathrm{i}}{4-3\mathrm{i}}$

(c) $\left(\dfrac{1+\sqrt{3}\,\mathrm{i}}{1-\sqrt{3}\,\mathrm{i}}\right)^2$ (d) $\dfrac{3-4\mathrm{i}}{-\mathrm{i}+2}-\dfrac{7-4\mathrm{i}}{3+2\mathrm{i}}$ (e) $\dfrac{(7+8\mathrm{i})^{10}}{(-8+7\mathrm{i})^{11}}$

(Check your answers using GC.)

2. **Without the use of a graphing calculator**, find the square roots of $21-20\mathrm{i}$.

3. **N2002/I/5**

(i) The complex number $x+\mathrm{i}y$ is such that $(x+\mathrm{i}y)^2=\mathrm{i}$. Find the possible values of the real numbers x and y, giving your answers in exact form.

(ii) Hence find the possible values of the complex number w such that $w^2=-\mathrm{i}$

4. (a) Two complex numbers w and z are such that

$$2w+z=12\mathrm{i} \quad , \quad w+2z=\frac{-13+4\mathrm{i}}{2-\mathrm{i}}.$$

Find w and z, giving each answer in the form $x+\mathrm{i}y$.

(b) Two complex numbers w and z are such that

$$2w^*+z=12\mathrm{i} \quad , \quad w+2z=-6-\mathrm{i}.$$

Find w and z, giving each answer in the form $x+\mathrm{i}y$.

(c) **2016/MJC/Prelim/I/Q10(a)**

Solve the simultaneous equations

$$z=w+2\mathrm{i}-1 \text{ and } z^2-\mathrm{i}w+\frac{5}{2}=0,$$

giving z and w in the form $x+y\mathrm{i}$ where x and y are real.

5. **N2008/I/8**

A graphic calculator is not to be used in answering this question.

(i) It is given that $z_1=1+\sqrt{3}\mathrm{i}$. Find the value of $z_1{}^3$, showing clearly how you obtain your answer.

(ii) Given that $1+\sqrt{3}\mathrm{i}$ is a root of the equation $2z^3+az^2+bz+4=0$, find the values of the real numbers a and b.

(iii) For these values of a and b, solve the equation in part (ii), and show all the roots on an Argand diagram.

6. **2011/RI/Prelim/II/1**

Given that one of the roots of the equation $z^4 - az^3 + 10z - 25 = 0$ is $1+2\text{i}$ where a is real, show that $a = 2$. Without using the graphic calculator, find the other roots of the equation in exact form. Hence find the roots of the equation $(w-1)^4 + 2(w-1)^3 - 10(w-1) - 25 = 0$, giving your answers in exact form.

7. **9233/N2007/I/9 (Modified)**

The equation $az^4 + bz^3 + cz^2 + dz + e = 0$ has a root $z = k\text{i}$, where k is real and non-zero. Given that the coefficients a, b, c, d and e are real, show that

$$ad^2 + b^2e = bcd.$$

Verify that this condition is satisfied for the equation

$$z^4 + 3z^3 + 13z^2 + 27z + 36 = 0$$

And hence find two roots of this equation which are of the form $z = k\text{i}$, where k is real.

Hence express $z^4 + 3z^3 + 13z^2 + 27z + 36$ as a product of 2 quadratic factors.

8. **2016/DHS/Prelim/I/Q2**

The complex number w is such that $kw^2 + kww^* + \text{i}w - \text{i}w^* - 1 = 0$, where w^* is the complex conjugate of w and k is a real and non-zero constant.

(i) For $w = a + b\text{i}$ where a and b are real numbers, obtain an expression for b in terms of a and k. Explain why w is either purely real or purely imaginary.

(ii) Using your result in part (i), or otherwise, find the real roots of the equation $2w^2 + 2ww^* + \text{i}w - \text{i}w^* - 1 = 0$.

9. **2016/HCI/Prelim/II/Q2(a)**

Given that $z_1 = -\dfrac{\text{i}}{2}$ is a root of the equation $2z^3 + (\text{i}-8)z^2 + az + 13\text{i} = 0$, find the complex number a and solve the equation, giving your answer in Cartesian form $x + \text{i}y$.

Hence, find in Cartesian form the roots of the equation

$$2w^3 + (1+8\text{i})w^2 - aw - 13 = 0.$$

<u>**Answers**</u>

1. (a) $-5+31\text{i}$ (b) $\dfrac{13+41\text{i}}{25}$ (c) $-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\text{i}$ (d) $1+\text{i}$ (e) $\dfrac{8+7\text{i}}{113}$

2. $\pm(5-2\text{i})$

3. (i) $x=\pm\frac{\sqrt{2}}{2}$ $y=\pm\frac{\sqrt{2}}{2}$, (ii) $\pm\frac{\sqrt{2}}{2}(-1+\mathrm{i})$

4. (a) $w=2+\frac{25}{3}\mathrm{i}$, $z=-4-\frac{14}{3}\mathrm{i}$,(b) $w=2-5\mathrm{i}, z=-4+2\mathrm{i}$

   (c) $w=\frac{3}{2}-\frac{1}{2}\mathrm{i}$, $z=\frac{1}{2}+\frac{3}{2}\mathrm{i}$ or $w=\frac{1}{2}-\frac{5}{2}\mathrm{i}$, $z=-\frac{1}{2}-\frac{1}{2}\mathrm{i}$

5. (i) -8, (ii) $-3, 6$ (iii) $1\pm\sqrt{3}\mathrm{i}$, $-\frac{1}{2}$

6. $z=1\pm 2\mathrm{i}, \pm\sqrt{5}$, $w=\pm 2\mathrm{i}, 1\pm\sqrt{5}$

7. $\pm 3\mathrm{i}$

8. (i) $b=\frac{2ka^2-1}{2}$ (ii) $w=-\frac{1}{2}$ or $w=\frac{1}{2}$

9. $a=26-4\mathrm{i}$, $z=-\frac{\mathrm{i}}{2}$ or $z=2\pm 3\mathrm{i}$, $w=-\frac{1}{2}$ or $w=\pm 3-2\mathrm{i}$

Assignment

1. **[CJC/Prelim/2017/II/Q4 part]**

(a) The complex numbers z and w satisfy the simultaneous equations

$$z+w^*+5\mathrm{i}=10 \quad \text{and} \quad |w|^2=z+18+\mathrm{i}.$$

Find z and w. [4]

(b) (i) It is given that $2+\mathrm{i}$ is a root of the equation $z^2-5z+7+\mathrm{i}=0$. Find the second root of the equation in cartesian form, showing your working clearly. [2]

(ii) Hence find the roots of the equation $-\mathrm{i}w^2+5w+7\mathrm{i}-1=0$. [2]

2. **[YJC Prelim 9758/2017/02/Q3] Do not use a calculator in answering this question.**

Given that $z=1+\mathrm{i}$ is a root of the equation $2z^4+az^3+7z^2+bz+2=0$, find the values of the real numbers a and b and the other roots. [5]

Deduce the roots of the equation $2z^4+bz^3+7z^2+az+2=0$. [2]

Geometry of Complex Numbers and Polar Representation

1. Express the following complex numbers in the polar form, stating the modulus and argument, as well as in exponential form. Represent each on an Argand diagram:
   (a) $1+\sqrt{3}\,\mathrm{i}$ (b) $-6-6\,\mathrm{i}$ (c) $-2\,\mathrm{i}$

2. Express the complex number $z=\dfrac{7+5\mathrm{i}}{6-\mathrm{i}}$ in polar form. Find the modulus and argument of $2z$, $1/z$, z^* and iz.

3. **MI/2015/1/11(c)**

   Given that $\left|z^2\right|=2$, $\left|wz\right|=2\sqrt{2}$, $\arg(-\mathrm{i}z)=\dfrac{\pi}{4}$ and $\arg\left(\dfrac{z^2}{w}\right)=-\dfrac{5\pi}{6}$, find w in the form $a+bi$, where a and b are real coefficients to be determined.

4. Let z be a complex number with $0<\arg z<\dfrac{\pi}{2}$. Let A represents z on the Argand diagram. Sketch on **a single Argand diagram**, A and the points representing the following complex numbers:
   (a) $2z$ (b) $-z$ (c) z^* (d) $-z^*$ (e) iz (f) -iz

   You should illustrate the geometrical relationship for each point with respect to A clearly on your diagram.

5. In an Argand diagram the points A, B and C represent the complex numbers a, $6+8\mathrm{i}$ and c respectively. $OABC$ is a square described in an anti-clockwise sense. Find c in terms of a. Find by calculation, a and c.

6. **N2002/11/2b**

   Given that $\arg(a+ib)=\theta$, where $a>0$, $b>0$, find, in terms of θ and π, the value of

   (i) $\arg(-a+ib)$ (ii) $\arg(-a-ib)$ (iii) $\arg(b+ia)$

7. **N1996/II/12(a)**

   The complex numbers z and w are given by $z=-3+2i$ and $w=5+4i$.

   a) Find $|z|$ and $\arg z$, expressing the modulus in surd form and the argument in radians correct to 3 significant figures.
   b) Express $\dfrac{z}{w}$ in the form $a+bi$, where a and b are exact fractions.
   c) In an Argand diagram the points Z and W represent the complex numbers z and w respectively, and Z' represents z^*, the complex conjugate of z. The point P is such that $ZWZ'P$ (in that order) is a parallelogram. Find the complex number p represented by P.

8. **ACJC/2010/1/4**

The complex number w has modulus 3 and argument $\frac{2\pi}{3}$. Find the modulus and argument of $\frac{-i}{w^*}$, where w^* is the complex conjugate of w. Hence express $\frac{-i}{w^*}$ in the form $a+ib$, where a and b are real, giving the exact values of a and b in non-trigonometrical form. Find the possible values of n such that $\left(\frac{-i}{w^*}\right)^n$ is purely imaginary.

9. **N2000/II/14(a)**

The complex number w has modulus $\sqrt{2}$ and argument $-\frac{3}{4}\pi$, and the complex number z has modulus 2 and argument $-\frac{1}{3}\pi$. Find the modulus and argument of wz, giving each answer exactly.
By expressing w and z in the form $x + iy$, find the real and imaginary parts of wz.
Hence show that $\sin\left(\frac{1}{12}\pi\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

10. **J90/II/12(a)**

Find the modulus of the complex number $\frac{2\mathrm{i}}{3-4\mathrm{i}}$, and show that the argument, in radians, is 2.5, correct to one decimal place. Hence, find correct to one decimal place, the value of x and a value of y such that $\mathrm{e}^{x+\mathrm{i}y} = \frac{2\mathrm{i}}{3-4\mathrm{i}}$.

11. **N1998/2/12b**

The complex number q is given by $q = \frac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}}$, where $0 < \theta < 2\pi$. In either order,

(i) find the real part of q, (ii) show that the imaginary part of q is $\frac{1}{2}\cot\left(\frac{1}{2}\theta\right)$.

12. **N99/II/12**

(a) The complex numbers $2\mathrm{e}^{\frac{1}{12}\pi\mathrm{i}}$ and $2\mathrm{e}^{\frac{5}{12}\pi\mathrm{i}}$ are represented by the points A and B respectively in an Argand diagram with origin O. Show that triangle OAB is equilateral.

(b) The complex numbers z and w each have modulus R, and have argument α and β respectively, where $0 < \alpha < \beta < \frac{1}{2}\pi$. In either order,

(i) show that $z + w = 2R\cos\frac{1}{2}(\beta-\alpha)\mathrm{e}^{\frac{1}{2}(\alpha+\beta)\mathrm{i}}$, express $|z+w|$ and $\arg(z+w)$ in terms of R, α and β, as appropriate. Show also that $|z-w| = 2R\sin\frac{1}{2}(\beta-\alpha)$.

(ii) The complex number z and w are represented by the points Z and W respectively in an Argand diagram with origin O. Triangle OZW has area Δ. Show that $|z^2 - w^2| = 4\Delta$.

Answers:

1. (a) $2\left(\cos\frac{\pi}{3}+\text{i}\sin\frac{\pi}{3}\right)$ (b) $6\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right)+\text{i}\sin\left(-\frac{3\pi}{4}\right)\right]$ (c) $2\left[\cos\left(-\frac{\pi}{2}\right)+\text{i}\sin\left(-\frac{\pi}{2}\right)\right]$

2. $\sqrt{2}\left(\cos\frac{\pi}{4}+\text{i}\sin\frac{\pi}{4}\right)$; $2\sqrt{2},\frac{\pi}{4}$; $\frac{1}{\sqrt{2}},-\frac{\pi}{4}$; $\sqrt{2},-\frac{\pi}{4}$; $\sqrt{2},\ \frac{3\pi}{4}$

3. $1+\sqrt{3}\text{i}$

5. $a=7+\text{i},\quad c=-1+7\text{i}$

6. (i) $\pi-\theta$ (ii) $-\pi+\theta$ (iii) $\frac{\pi}{2}-\theta$

7. (a) $\sqrt{13}$, 2.55 (b) $-\frac{7}{41}+\frac{22}{41}\text{i}$ (c) $-11-4\text{i}$

8. $\frac{1}{3}$, $\frac{\pi}{6}$, $\frac{\sqrt{3}}{6}+\frac{i}{6}$, $3(2k+1), k\in\mathbb{Z}$

9. $2\sqrt{2}$, $\frac{11}{12}\pi$; $\text{Re}(wz)=-1-\sqrt{3}$, $\text{Im}(wz)=\sqrt{3}-1$

10. $\frac{2}{5}$, $x=\ln\frac{2}{5}$, $y=2.5$

11. $-\frac{1}{2}$

<u>Assignment</u>

1. **ACJC/Prelim/2007/1/5**

Show that complex number $z=1+\text{e}^{-\text{i}\frac{\pi}{3}}$ can be expressed as $\sqrt{3}\text{e}^{-\text{i}\frac{\pi}{6}}$. The complex number zw has modulus 5 and argument $\frac{3}{4}\pi$. Find the complex number w in exact polar form.

2. **SAJC Prelim 9758/2017/01/Q7**

(i) Show that for any complex number $z=re^{\text{i}\theta}$, where $r>0$, and $-\pi<\theta\le\pi$,

$$\frac{z}{z-r}=\frac{1}{2}-\frac{1}{2}\left(\cot\frac{\theta}{2}\right)\text{i}.$$ [3]

(ii) Given that $z=2\text{e}^{\text{i}\left(\frac{\pi}{3}\right)}$ is a root of the equation $z^2-2z+4=0$. State, in similar form the other root of the equation. [1]

(iii) Using parts **(i)** and **(ii)**, solve the equation $\frac{4w^2}{(w-1)^2}-\frac{4w}{w-1}+4=0$. [4]

3. **CJC Prelim 9758/2017/02/Q4c**

The complex number z is given by $z=-a+a\text{i}$, where a is a positive real number.

(i) It is given that $w=-\frac{\sqrt{2}z^*}{z^4}$. Express w in the form $re^{\text{i}\theta}$, in terms of a, where $r>0$ and $-\pi<\theta\le\pi$. [4]

(ii) Find the two smallest positive whole number values of n such that $\text{Re}\,(w^n)=0$. [3]

Extra Practice Questions

(*The questions will not be discussed during tutorials. Full solutions will be uploaded to SLS*)

1 **RVHS/2012 Prelim/I/Q1**

One root of the equation $z^4-2z^3+14z^2+az+b=0$, where a and b are real, is $z=1+2\text{i}$. Find the values of a and b and the other roots. [5]

Deduce the roots of the equation $z^4+2iz^3-14z^2-18iz+45=0$. [2]

2 **TJC/2012 Prelim/I/Q2**

Consider the equation $2z^3+(1-2\mathrm{i})z^2-(a+b\mathrm{i})z+2+2\mathrm{i}=0$, where a and b are real. Given that -2 is a root of the equation, find the values of a and b.

Given also that $1+\mathrm{i}$ is another root, find the third root of the equation. [3]

3 **JJC/2014 Prelim/I/Q10a**

Given that the complex number $z=(1+\mathrm{i})t+\dfrac{1-\mathrm{i}}{t}$ is represented by the point P on an Argand diagram where t is a non-zero real constant. Find the Cartesian equation of the locus of the point P. [3]

4 **ACJC/2016 Prelim/I/Q8**

The complex number z is given by $z=k+\mathrm{i}$ where k is a non-zero real number.

(i) Find the possible values of k if $z=k+\mathrm{i}$ satisfies the equation $z^3-\mathrm{i}z^2-2z-4\mathrm{i}=0$. [3]

(ii) For the complex number z found in part **(i)** for which $k>0$, find the smallest integer value of n such that $|z^n|>100$ and z^n is real. [3]

5 **CJC/2016 Prelim/I/Q8**

Do not use a calculator in answering this question.

(i) It is given that complex numbers z_1 and z_2 are the roots of the equation $z^2-6z+36=0$ such that $\arg(z_1)>\arg(z_2)$. Find exact expressions of z_1 and z_2 in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [4]

(ii) Find the complex number $\dfrac{z_1^{\,4}}{\mathrm{i}z_2}$ in exact polar form. [3]

Find the smallest positive integer n such that $z_2^{\,n}$ is a positive real number. [2]

6 **HCI/2016 Prelim/II/Q2 (part)**

Given that $z_1=-\dfrac{\mathrm{i}}{2}$ is a root of the equation $2z^3+(\mathrm{i}-8)z^2+az+13\mathrm{i}=0$, find the complex number a and solve the equation, giving your answer in Cartesian form $x+\mathrm{i}y$. [4]

Hence, find in Cartesian form the roots of the equation $2w^3+(1+8\mathrm{i})w^2-aw-13=0$. [2]

7 **MJC/2016 Prelim/I/Q10**

(a) Solve the simultaneous equations

$$z = w + 2\mathrm{i} - 1 \text{ and } z^2 - \mathrm{i}w + \frac{5}{2} = 0,$$

giving z and w in the form $x + y\mathrm{i}$ where x and y are real. [5]

(b) (i) Given that $z = w - \frac{1}{w}$ where $w = 2(\cos\theta + \mathrm{i}\sin\theta)$, $-\pi < \theta \le \pi$, express the real and imaginary parts of z in terms of θ. [3]

(ii) Hence show that locus of z on an Argand diagram lies on the curve with cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are constants. [3]

8 **NYJC/2016 Prelim/II/Q3**

(a) Solve $z^3 - 2(2-\mathrm{i})z^2 + (8-3\mathrm{i})z - 5 + \mathrm{i} = 0$, given that one of the three roots is real. **[5]**

(b) The complex number u is given by $u = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \frac{\pi}{2}$.

(i) Show that $1 - u^2 = -2\mathrm{i}u\sin\theta$ and hence find the modulus and argument of $1 - u^2$ in terms of θ.

[4]

Given that $(1-u^2)^{10}$ is real and negative, find the possible values of θ in terms of π. **[3]**

9 **SAJC/2017 Prelim/I/Q7**

(i) Show that for any complex number $z = re^{\mathrm{i}\theta}$, where $r > 0$, and $-\pi < \theta \le \pi$,

$$\frac{z}{z-r} = \frac{1}{2} - \frac{1}{2}\left(\cot\frac{\theta}{2}\right)\mathrm{i}.$$ **[3]**

(ii) Given that $z = 2e^{\mathrm{i}\left(\frac{\pi}{3}\right)}$ is a root of the equation $z^2 - 2z + 4 = 0$. State, in similar form, the other root of the equation. **[1]**

(iii) Using parts **(i)** and **(ii)**, solve the equation $\frac{4w^2}{(w-1)^2} - \frac{4w}{w-1} + 4 = 0$. **[4]**

10 **ACJC/2017 Prelim/I/Q7**

(a) Given that $2z+1=|w|$ and $2w-z=4+8\text{i}$, solve for w and z. [5]

(b) Find the exact values of x and y, where $x, y \in \mathbb{R}$, such that $2\text{e}^{-\left(\frac{3+x+\text{i}y}{\text{i}}\right)}=1-\text{i}$ [4]

Answers:

1. $a=-18$ and $b=45$; $z=1+2\text{i}$, $z=1-2\text{i}$, $z=3\text{i}$, $z=-3\text{i}$; $z=-\text{i}+2$, $z=-\text{i}-2$, $z=3$, $z=-3$

2. $a=5$, $b=3$; $\dfrac{1}{2}$

3 $x^2-y^2=4$

4. (i) $k=\pm\sqrt{3}$; (ii) 12

5 (i) $z_1=6e^{\text{i}\frac{\pi}{3}}$ and $z_2=6e^{-\text{i}\frac{\pi}{3}}$ **(ii)** $6^3\left[\cos\left(\dfrac{-5\pi}{6}\right)+i\sin\left(\dfrac{-5\pi}{6}\right)\right]$ **(iii)** 6

6. $a=26-4\text{i}$; $\therefore z=-\dfrac{\text{i}}{2}$ or $z=2\pm3\text{i}$.; $\therefore w=-\dfrac{1}{2}$ or $w=\pm3-2\text{i}$

7. (a) $w=\dfrac{3}{2}-\dfrac{1}{2}\text{i}$, $z=\dfrac{1}{2}+\dfrac{3}{2}\text{i}$ or $w=\dfrac{1}{2}-\dfrac{5}{2}\text{i}$, $z=-\dfrac{1}{2}-\dfrac{1}{2}\text{i}$

(b)(i) $\text{Re}(z)=\dfrac{3}{2}\cos\theta$, $\text{Im}(z)=\dfrac{5}{2}\sin\theta$

8. Roots: 1, $2-3\text{i}$, $1+\text{i}$ **b(i)** $|1-u^2|=2\sin\theta$ $\arg(1-u^2)=-\dfrac{\pi}{2}+\theta$ **(ii)** $\theta=\dfrac{1}{5}\pi, \dfrac{2}{5}\pi$

9. ii) $z=2\text{e}^{\text{i}\left(-\frac{\pi}{3}\right)}$ (iii) $w=\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\text{i}$ or $w=\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\text{i}$

10. (a) $z=2$, $w=3+4\text{i}$; (b) $x=-\dfrac{\pi}{4}-3$, $y=\dfrac{1}{2}\ln 2$