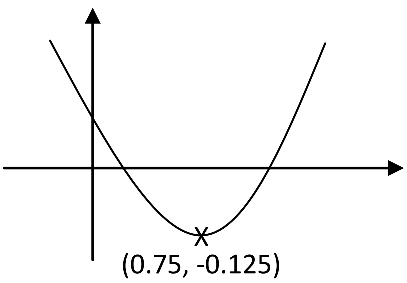
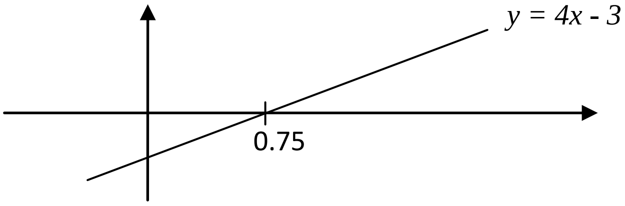
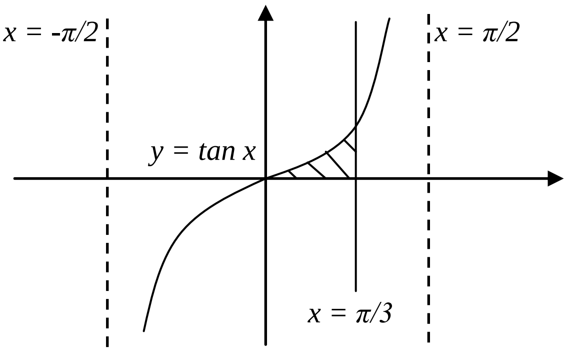
PU3 H2 Mathematics Paper 1 2012 Prelim II

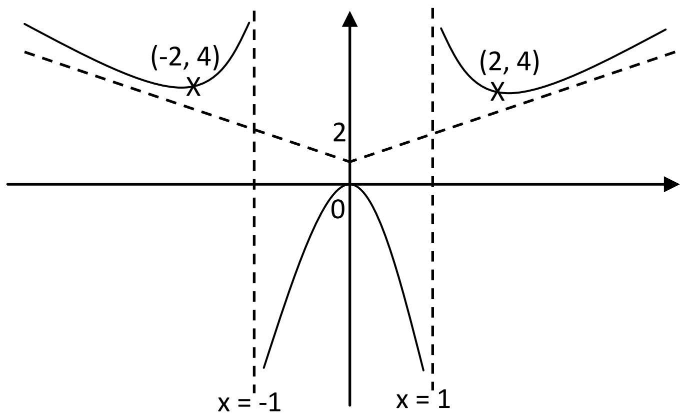
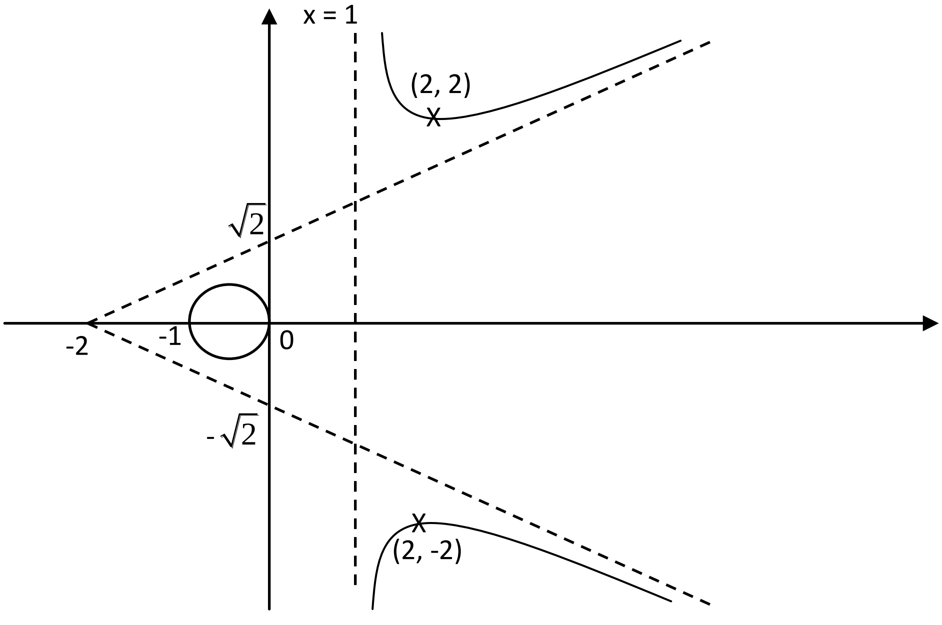
1(i)	$\sqrt{(4p)^2+(7p)^2+(-4p)^2}=1$ $p^2=\frac{1}{81}$ $p=\frac{1}{9}\quad (reject-ve)$
1(ii)	Length of projection of a on b.
1(iii)	$a\times b=\begin{pmatrix}3\\-2\\6\end{pmatrix}\times\begin{pmatrix}\frac{4}{9}\\\frac{7}{9}\\-\frac{4}{9}\end{pmatrix}=\begin{pmatrix}-\frac{34}{9}\\4\\\frac{29}{9}\end{pmatrix}$
2	$\frac{2x^2+2x+1}{x^2-2x+1}\geq 1$ $\frac{2x^2+2x+1}{(x-1)^2}\geq 1$ $x^2+4x\geq 0$ $x\leq -4\ \ and\ \ x\geq 0$ $x\neq 1$ $\frac{2(\ln x)^2+2(\ln x)+1}{(\ln x)^2-2(\ln x)+1}\geq 1$ $\ln x\leq -4,\ \ln x\geq 0$ $x\leq e^{-4},\ \ x\geq 1$ $x\neq 1\Rightarrow x\neq e$
3(i)	$a-b+c=6$ $1.21a+1.1b+c=0.12$ $2.25a+1.5b+c=1$ $a=2,\quad b=-3,\quad c=1$
3(ii)	 $f(x)$ is increasing for $x\geq 0.75$.

3(iii)	
4	$(4)^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}=2\left[1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2+\ldots\right]$ $=2+\frac{x}{4}-\frac{x^2}{64}+\ldots$ $\left\|\frac{x}{4}\right\|<1\Rightarrow\|x\|<4$ $x=\frac{1}{2}$ $\sqrt{\frac{1}{2}+4}=\sqrt{\frac{9}{2}}=\frac{3}{\sqrt{2}}$ $\frac{3}{\sqrt{2}}=2+\frac{\frac{1}{2}}{4}-\frac{\left(\frac{1}{2}\right)^2}{64}+\ldots$ $\frac{3}{\sqrt{2}}\approx\frac{543}{256}$ $2\sqrt{2}=\frac{512}{181}$
5(a)	$a=2000$ $r=1.02$ $d=-50$ $T_{n+1}=2000(1.02)^n-50\left[1.02^{n-1}+1.02^{n-2}+\ldots+1\right]$ $=2000(1.02)^n-50\left(\frac{1-1.02^n}{1-1.02}\right)$ $=2000(1.02)^n+2500\left(1-1.02^n\right)$ $=2000(1.02)^n-500\left(1.02^n\right)$ $=500\left(5-1.02^n\right)$ $500\left(5-1.02^n\right)\le 0$ $\ln 5\le n\ln 1.02$ $n\ge 81.27$ It can last 81 years. $500\left(5-1.02^{30}\right)=\$1594.32$

5(b)	$a+8d=50$ $\frac{15}{2}\left(2a+14d\right)=570$ $a=-46,\quad d=12$ $S_n>500$ $\frac{n}{2}\left[-92+\left(n-1\right)12\right]>500$ $12n^2-104n-1000>0$ $n<-9.59$ (rej) or $n>9.59$ $\therefore$ least n is 10.
6(i)	$x^2-y^2=2xy-1$ $2x-2y\left(\frac{dy}{dx}\right)=2\left[y+x\left(\frac{dy}{dx}\right)\right]$ $2x-2y=2x\left(\frac{dy}{dx}\right)+2y\left(\frac{dy}{dx}\right)$ $\frac{dy}{dx}=\frac{2x-2y}{2x+2y}$ $\frac{dy}{dx}=\frac{x-y}{x+y}$
6(ii)	$\frac{x-y}{x+y}=0$ $x=y$ $x^2-x^2=2x^2-1$ $x^2=\frac{1}{2}$ $x=\pm\sqrt{\frac{1}{2}}$ $x=\frac{1}{\sqrt{2}},\quad y=\frac{1}{\sqrt{2}}$ $x=-\frac{1}{\sqrt{2}},\quad y=-\frac{1}{\sqrt{2}}$
7	$\frac{dy}{dx}=\frac{1}{2}\left(x+y\right)^2$ $\frac{d^2y}{dx^2}=\left(x+y\right)\left(\frac{dy}{dx}+1\right)$ $\frac{d^3y}{dx^3}=\left(1+\frac{dy}{dx}\right)^2+\left(x+y\right)\left(\frac{d^2y}{dx^2}\right)$ $\frac{d^3y}{dx^3}-\left(1+\frac{dy}{dx}\right)^2-\left(x+y\right)\left(\frac{d^2y}{dx^2}\right)=0$

	When $x=0,\ y=1$ $\frac{dy}{dx}=\frac{1}{2}(0+1)^2=\frac{1}{2}$ $\frac{d^2y}{dx^2}=(0+1)\left(\frac{1}{2}+1\right)=\frac{3}{2}$ $\frac{d^3y}{dx^3}=\left(1+\frac{1}{2}\right)^2+(0+1)\left(\frac{3}{2}\right)=\frac{15}{4}$ $y=1+\frac{1}{2}x+\frac{\frac{3}{2}x^2}{2!}+\frac{\frac{15}{4}x^3}{3!}+\ldots$ $y=1+\frac{1}{2}x+\frac{3}{4}x^2+\frac{5}{8}x^3+\ldots$
8	 $V=\pi\int_0^{\pi/3}(\tan x)^2\,dx$ $=\pi\int_0^{\pi/3}\left(\sec^2 x-1\right)dx$ $=\pi\left[\tan x-x\right]_0^{\pi/3}$ $=\pi\left[\sqrt{3}-\frac{\pi}{3}\right]$ $=\sqrt{3}\pi-\frac{\pi^2}{3}$ units^3
9(a)	$\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^5\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^9$ $=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^5\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)^9$ $=\left(\cos\frac{5\pi}{2}+i\sin\frac{5\pi}{2}\right)\left(\cos\left(-\frac{9\pi}{6}\right)+i\sin\left(-\frac{9\pi}{6}\right)\right)$ $=\left(e^{i\frac{5\pi}{2}}\right)\left(e^{-i\frac{9\pi}{6}}\right)$ $=e^{i\pi}$ $=-1$

9(b)	$z^3 = 27e^{i\frac{\pi}{2}}$ $z^3 = 27e^{i\left(\frac{\pi}{2}+2k\pi\right)}$ $z = 3e^{i\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)}, \quad k = -1,0,1$ $z_1 = 3\left(0-i\right) = -3i$ $z_2 = 3\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = \frac{3\sqrt{3}}{2}+\frac{3}{2}i$ $z_3 = 3\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = -\frac{3\sqrt{3}}{2}+\frac{3}{2}i$
10(i)	GC: $\alpha = 1.052$ $\beta = 5.505$
10(ii)	$x_{n+1} = \ln\left(x_n - 1\right) + 4$ If a sequence converges, as $n \to \infty$, $x_n \to L$ and $x_{n+1} \to L$. $L = \ln\left(L-1\right)+4$ $\ln\left(L-1\right) - L + 4 = 0$ Hence, $L = \alpha$ or $L = \beta$.
10(iii)	Maximum point: $\left(2,2\right)$ $x_{n+1} = \ln\left(x_n - 1\right) + 4$ $x_{n+1} - x_n = \ln\left(x_n - 1\right) - x_n + 4$ Since $\ln\left(x-1\right) - x + 4 \le 2$, $x_{n+1} - x_n \le 2$ $x_{n+1} \le x_n + 2$
11	$\frac{dx}{dt} = m - px$ When $x = 0.5, \frac{dx}{dt} = 0$ $0 = m - p\left(0.5\right)$ $m = 0.5p$ $\frac{dx}{dt} = \left(0.5p\right) - px$ $= -\frac{1}{2}p\left(2x-1\right)$ $= -k\left(2x-1\right)$ where $k > 0$.

	When $t=0,\ x=1.5,\ \frac{dx}{dt}=-0.02,$ $\frac{dx}{dt}=-k\left(2x-1\right)$ $\frac{1}{2x-1}\left(\frac{dx}{dt}\right)=-k$ $\int \frac{1}{2x-1}\,dx=-k\int 1\,dt$ $\frac{1}{2}\ln\left\|2x-1\right\|=-kt+c$ $c=\frac{1}{2}\ln 2$ $-0.02=-k\left(3-1\right)$ $k=0.01$ $\frac{1}{2}\ln\left\|2x-1\right\|=-0.01t+\frac{1}{2}\ln 2$ When $x=1.01,$ $\frac{1}{2}\ln\left\|2\left(1.01\right)-1\right\|=-0.01t+\frac{1}{2}\ln 2$ $t=50\ln\left(\frac{100}{51}\right)\approx 33.7\,\text{sec}$
12(i)	
12(ii)	

13(i)	$\overrightarrow{BA}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$ $\overrightarrow{CB}=\begin{pmatrix}2\\-8\\4\end{pmatrix}$ $\overrightarrow{BA}\times\overrightarrow{CB}=\begin{pmatrix}2\\1\\-1\end{pmatrix}\times\begin{pmatrix}2\\-8\\4\end{pmatrix}=2\begin{pmatrix}2\\5\\9\end{pmatrix}$ $\begin{pmatrix}0\\-2\\1\end{pmatrix}\bullet\begin{pmatrix}2\\5\\9\end{pmatrix}=0-10+9=-1$ $\therefore\ p:r\cdot\begin{pmatrix}2\\5\\9\end{pmatrix}=-1$
13(ii)	$l_1:r=\begin{pmatrix}1\\-4\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\4\end{pmatrix}$ $l_2:r=\begin{pmatrix}0\\1\\-1\end{pmatrix}+\mu\begin{pmatrix}k\\2\\3\end{pmatrix}$ $-4+\lambda=1+2\mu$ $-1+4\lambda=-1+3\mu$ $\lambda=-3,\quad \mu=-4$ $1+2(-3)=0+k(-4)$ $k=\frac{5}{4}$
13(iii)	$\begin{pmatrix}2\\1\\4\end{pmatrix}\cdot\begin{pmatrix}2\\5\\9\end{pmatrix}=45\neq 0$ Since l_1 not perpendicular to plane's normal, l_1 does not lie in the plane.

13(iv)	$\left[\begin{pmatrix}1\\-4\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\4\end{pmatrix}\right]\square\begin{pmatrix}2\\5\\9\end{pmatrix}=-1$ $45\lambda=26$ $\lambda=\frac{26}{45}$ $r=\begin{pmatrix}1\\-4\\-1\end{pmatrix}+\left(\frac{26}{45}\right)\begin{pmatrix}2\\1\\4\end{pmatrix}=\begin{pmatrix}\frac{97}{45}\\-\frac{154}{45}\\\frac{59}{45}\end{pmatrix}$
13(v)	$\cos\theta=\dfrac{\frac{5}{2}+10+27}{\left(\sqrt{\frac{233}{16}}\right)\left(\sqrt{110}\right)}$ $\theta=0.16191rad$ $\square\ =\frac{\pi}{2}-0.16191=1.41\,rad\ (80.7^{\circ})$