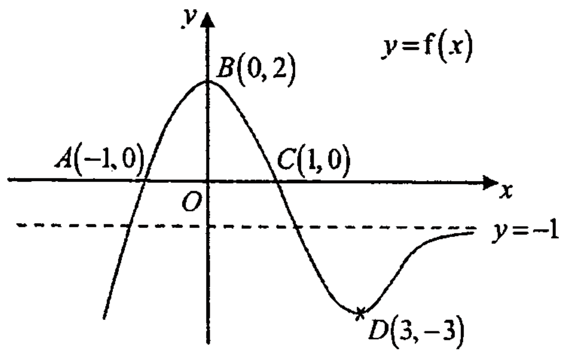
VJC JC1 2023 Promotional Examination

1 **(a)** Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x if $y=\tan^{-1}\sqrt{x^2-1}$. [3]

(b) It is given that $y=x^{\cos x}$ for $x>0$. By taking logarithm first, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. [3]

2 The diagram shows the curve $y=\mathrm{f}(x)$. The curve passes through the points $A(-1,0)$, $B(0,2)$, $C(1,0)$ and $D(3,-3)$.



On separate clearly labelled diagrams, sketch the graphs of

(a) $y=-\frac{1}{2}\mathrm{f}(x)$, [2]

(b) $y=\mathrm{f}(1-x)$. [2]

3 Given that $a\in$ and $a>1$, use an algebraic method to solve the inequality

$$\frac{ax-a+2}{(x-a)(1-x)}\leq 1 .$$ [5]

Hence find the set of values of x such that $\frac{a|x|-a+2}{(|x|-a)(1-|x|)}\leq 1$. [2]

4 **(a)** Differentiate $\mathrm{e}^{\tan x}$ with respect to x. [1]

(b) Hence find $\int \mathrm{e}^{\tan x}\sec^4 x\,\mathrm{d}x$. [4]

5 The curve C has equation $x^2-4y^2=4$.

(a) Sketch C, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

(b) Describe a pair of transformations which transforms the graph of C onto the graph of $(x-2)^2-y^2=4$. [2]

6 **(a)** Find $\int \frac{3-2x}{\sqrt{5+4x-x^2}}\,dx$. [4]

(b) Show that

$$\int_{\sqrt{2}}^{4} |x-3|\,dx = a - b\sqrt{2},$$

where a and b are constants to be determined. [3]

(c) Using the substitution $x=\sin u$, find the exact value of $\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1+x^2}{\sqrt{1-x^2}}\,dx$. [4]

7 **Do not use a calculator in answering this question.**

(a) One of the roots of the equation $3x^3+px^2+qx+3=0$, where p and q are real, is $1+\sqrt{2}\,\mathrm{i}$. Find the other roots and the values of p and q. [5]

(b) The complex numbers w and z, with $|w|<|z|$, satisfy the simultaneous equations

$$wz^*=3-\mathrm{i} \text{ and } w+z^*=2-\mathrm{i}.$$

Find w and z. [5]

8 It is given that $\mathrm{f}(x)=\ln(1+2x^2)$.

(a) A polynomial $p(x)=ax^2+bx+c$ is used to approximate $\mathrm{f}(x)$ for $1\le x\le 2$. Given that $p(1)=\mathrm{f}(1)$, $p(1.5)=\mathrm{f}(1.5)$ and $p(2)=\mathrm{f}(2)$, find the values of the coefficients a, b and c. [3]

(b) Using standard series from the List of Formulae (MF26), find the Maclaurin series for $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^6. [2]

(c) Find the set of values of x for the expansion in part **(b)** to be valid. [2]

(d) Use $x^2=\frac{1}{6}$ in your series from part **(b)** to show that $\ln\left(\frac{4}{3}\right)\approx\frac{m}{162}$, where m is an integer to be determined. [2]

(e) Using your series from part **(b)**, find the series expansion for $\ln\left(\frac{1+2x^2}{1+2x+x^2}\right)$, up to and including the term in x^4. Give the coefficients in exact form. [2]

9 **Do not use a calculator in answering this question.**

(a) It is given that $\alpha = -\sqrt{3} + \mathrm{i}$ and $\beta = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}$.

Find $\frac{\alpha^5}{\beta^*}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

(b) The complex number z is given by $z = \cos\theta + \mathrm{i}\sin\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Show that $\frac{1}{1+z^2} = \frac{z^*}{2\cos\theta}$. Hence find the modulus and argument of $\frac{1}{1+z^2}$ in terms of θ.

[4]

10 The curve C has equation $y = \frac{3x^2 + ax + 3}{x+1}$, where a is a constant, $a \neq 6$.

(a) Find the set of values of a such that C does not cut the x-axis. [2]

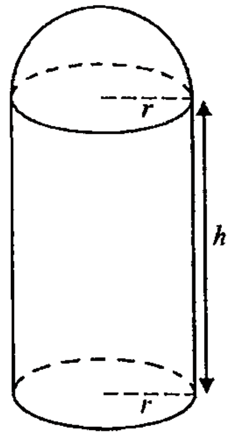
(b) It is given that the equation of C is $y = \frac{3x^2 - 6x + 3}{x+1}$. Sketch C and give the equations of any asymptotes. Also, state the coordinates of any points where C crosses the axes and of any turning points. [4]

(c) Let k be a positive constant. By sketching a suitable graph in the same diagram in part **(b)**, find an inequality satisfied by k such that the equation

$$k^2(x-1)^2 + \left(\frac{3x^2 - 6x + 3}{x+1} + 12\right)^2 = k^2$$

has 2 real and distinct roots. [2]

11 A company manufactures a closed container made of glass as shown in the figure below.



The closed container, of negligible thickness is made up of two components. The bottom component is a cylinder of base radius r cm and a height of h cm. The top component is a hemisphere of radius r cm. The company requires the volume of the container to be fixed at 108π cm^3.

The external surface area of the container is denoted by A cm^2. The company wants the value of A to be as small as possible to reduce the cost of production.

[The volume of a sphere of radius r is given by $\frac{4}{3}\pi r^3$ and its surface area is given by $4\pi r^2$.]

(a) Show that $A=\left(\frac{5r^2}{3}+\frac{216}{r}\right)\pi$. [3]

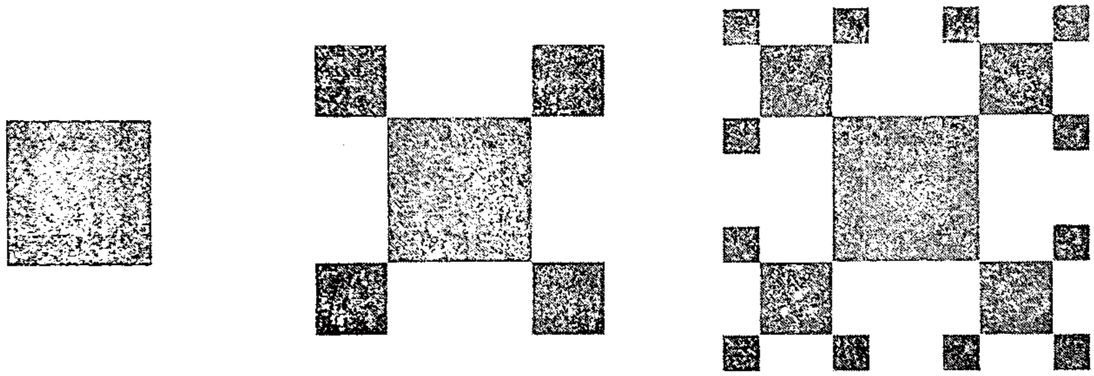
(b) Using differentiation, find the exact value of r that gives the minimum value of A, proving that A is a minimum. [4]

(c) Sketch the graph showing the external surface area of the container as the radius of the hemisphere varies, stating the coordinates of the end point of the graph. [3]

The company decides to produce the glass container with $r=3$. To use the container as a decorative piece, the glass container is filled completely with a viscous liquid.

(d) A small crack at the bottom of the container causes the viscous liquid to leak out of the container at a constant rate of 5 cm^3 per second. Find the rate of decrease of the height of the viscous liquid in the container 14 seconds after the container cracked. [3]

12 The diagrams below show a sequence of patterns formed by squares. Stage 0 is represented by a square of length 1. At each successive stage, squares with half the length of the smallest square in the previous stage are added to the unoccupied vertices of the squares in the previous stage.



Stage 0 Stage 1 Stage 2

Let A_0 be the area of the square in Stage 0 and A_n be the area of each new square added in Stage n. For example, $A_0=1$ and $A_1=\frac{1}{4}$.

(a) Find an expression for A_n, giving your answer in terms of n. [2]

(b) Show that the total area of new squares added in Stage n is given by $\left(\frac{3}{4}\right)^{n-1}$. [2]

(c) Show that the total area of all the squares in Stage n is given by $5-4\left(\frac{3}{4}\right)^n$. [3]

A square fractal is formed when this process of adding new squares at each stage continues indefinitely.

(d) Give a reason why the area of the square fractal converges and write down its value. [2]

(e) The total area of the squares in Stage m first exceeds 90% of the area of the square fractal. Find the value of m and find the total number of squares at Stage m. [3]

VJC JC1 2023 Promotional Examination Solutions

1 (a)

$$\frac{d}{dx}\tan^{-1}\sqrt{x^2-1} = \frac{1}{1+\left(\sqrt{x^2-1}\right)^2}\left(\frac{1}{2}\right)\left(x^2-1\right)^{-\frac{1}{2}}(2x)$$

$$= \frac{1}{1+x^2-1}\frac{x}{\sqrt{x^2-1}}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

(b)

$$y = x^{\cos x}$$

$$\ln y = \cos x \ln x$$

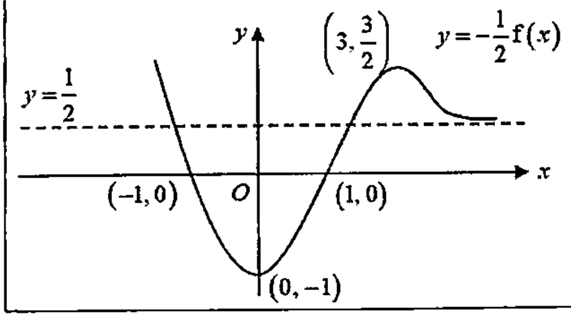
$$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{1}{x} + \ln x(-\sin x)$$

$$\frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

2

(1) $y=\mathrm{f}(x)\xrightarrow{y\text{ replaced by }-y}y=-\mathrm{f}(x)$

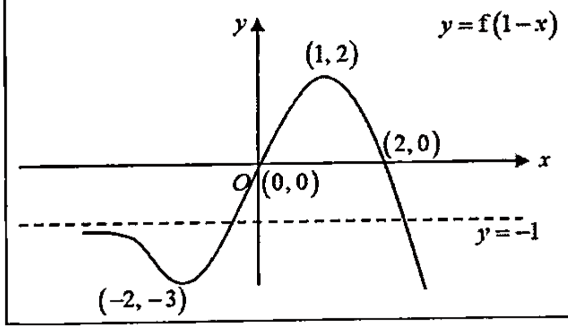
(2) $y=-\mathrm{f}(x)\xrightarrow{y\text{ replaced by }\frac{1}{2}y}y=-\frac{1}{2}\mathrm{f}(x)$



(b) $y=\mathrm{f}(1-x)$.

(1) $y=\mathrm{f}(x)\xrightarrow{x\text{ replaced by }x+1}y=\mathrm{f}(x+1)$

(2) $y=\mathrm{f}(x+1)\xrightarrow{x\text{ replaced by }-x}y=\mathrm{f}(1-x)$



3

$$\frac{ax-a+2}{(x-a)(1-x)} \le 1$$

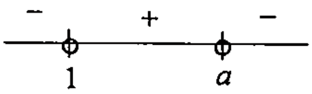
$$\frac{ax-a+2-(x-a)(1-x)}{(x-a)(1-x)} \le 0$$

$$\frac{ax-a+2-x+a+x^2-ax}{(x-a)(1-x)} \le 0$$

$$\frac{x^2-x+2}{(x-a)(1-x)} \le 0$$

$$\frac{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}{(x-a)(1-x)} \le 0$$

Since $\left(x-\frac{1}{2}\right)^2+\frac{7}{4}>0$ for all $x \in \mathbb{R}$, $(x-a)(1-x)<0$.



$x<1$ or $x>a$

For $\frac{a|x|-a+2}{(|x|-a)(1-|x|)} \le 1$, replace x with $|x|$ in the result above to get the following.

$|x|<1$ or $|x|>a$

$-1<x<1$ or $x<-a$ or $x>a$

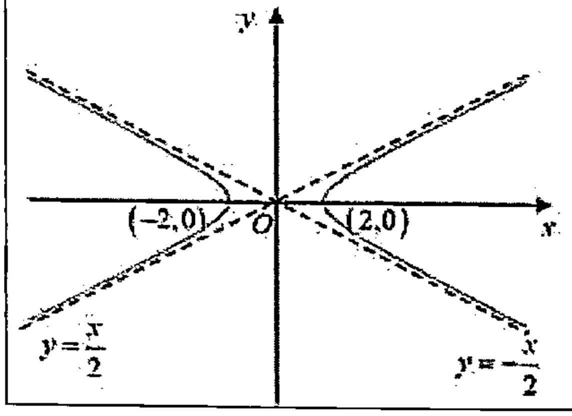
$\{x \in \mathbb{R} : x<-a \text{ or } -1<x<1 \text{ or } x>a\}$

4

$$\frac{d}{dx}e^{\tan x} = \sec^2 x\, e^{\tan x}$$

$$\int \sec^4 x\, e^{\tan x}\, dx$$
$$= \int \left(\sec^2 x\right)\sec^2 x\, e^{\tan x}\, dx$$
$$= \sec^2 x \cdot e^{\tan x} - \int e^{\tan x} \cdot 2\sec x \sec x \tan x\, dx$$
$$= \sec^2 x \cdot e^{\tan x} - 2\int e^{\tan x} \cdot \sec^2 x \cdot \tan x\, dx$$
$$= \sec^2 x \cdot e^{\tan x} - 2\left[\left(e^{\tan x}\right) \cdot \tan x - \int e^{\tan x} \cdot \sec^2 x\, dx\right]$$
$$= e^{\tan x}\sec^2 x - 2e^{\tan x}\tan x + 2e^{\tan x} + C$$
$$= e^{\tan x}\left(\sec^2 x - 2\tan x + 2\right) + C$$

5



$x^2 - 4y^2 = 4 \xrightarrow{\text{Replace } y \text{ with } \frac{y}{2}} x^2 - y^2 = 4$

$x^2 - y^2 = 4 \xrightarrow{\text{Replace } x \text{ with } x-2} (x-2)^2 - y^2 = 4$

(1) Stretch the graph of C by a factor of 2, parallel to the y-axis with x-axis invariant, followed by
(2) translating the resultant graph by 2 units in the positive x-direction.

6

$$\int \frac{3-2x}{\sqrt{5+4x-x^2}}\,dx = \int \frac{4-2x-1}{\sqrt{5+4x-x^2}}\,dx$$

$$= \int \frac{4-2x}{\sqrt{5+4x-x^2}} - \frac{1}{\sqrt{9-(x-2)^2}}\;dx$$

$$= 2\sqrt{5+4x-x^2} - \sin^{-1}\left(\frac{x-2}{3}\right) + C$$

$$\int_{\sqrt{2}}^{4} |x-3|\,dx = \int_{\sqrt{2}}^{3} -(x-3)\;dx + \int_{3}^{4} x-3\;dx$$

$$= -\left[\frac{x^2}{2} - 3x\right]_{\sqrt{2}}^{3} + \left[\frac{x^2}{2} - 3x\right]_{3}^{4}$$

$$= -\left(\frac{9}{2} - 9 - 1 + 3\sqrt{2}\right) + \left(8 - 12 - \frac{9}{2} + 9\right)$$

$$= 6 - 3\sqrt{2}$$

$$\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1+x^2}{\sqrt{1-x^2}}\,\mathrm{d}x \qquad\qquad x=\sin u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u}=\cos u$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\sin^2 u}{\sqrt{1-\sin^2 u}}\times\cos u\ \mathrm{d}u$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\sin^2 u}{|\cos u|}\times\cos u\ \mathrm{d}u$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\sin^2 u}{\cos u}\times\cos u\ \mathrm{d}u \qquad \left(\because \frac{\pi}{4}<u<\frac{\pi}{3}\Rightarrow \cos u>0\right)$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1+\frac{1-\cos 2u}{2}\ \mathrm{d}u$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3-\cos 2u}{2}\ \mathrm{d}u$$

$$=\left[\frac{3}{2}u-\frac{\sin 2u}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$=\frac{3}{2}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)-\frac{1}{4}\left(\sin\frac{2\pi}{3}-\sin\frac{\pi}{2}\right)$$

$$=\frac{\pi}{8}-\frac{\sqrt{3}}{8}+\frac{1}{4}$$

7

Since all the coefficients are real, $1-\sqrt{2}\,\mathrm{i}$ is also a root.

Quadratic factor $=\left[x-\left(1-\sqrt{2}\,\mathrm{i}\right)\right]\left[x-\left(1+\sqrt{2}\,\mathrm{i}\right)\right]$

$=(x-1)^2-2(-1)$

$=x^2-2x+3$

Let $3x^3+px^2+qx+3=\left(x^2-2x+3\right)(ax+b)$.

By inspection or by comparing coefficient of x^3 and constant term, $a=3$ and $b=1$.

$3x^3+px^2+qx+3=\left(x^2-2x+3\right)(3x+1)$

By comparing coefficients,

$p=1-6=-5$

$q=-2+9=7$

$3x^3+px^2+qx+3=\left(x^2-2x+3\right)(3x+1)=0$

Other than $1+\sqrt{2}\,\mathrm{i}$, the other roots are $1-\sqrt{2}\,\mathrm{i}$ and $-\frac{1}{3}$.

Alternative

$3\left(1+\sqrt{2}\mathrm{i}\right)^3+p\left(1+\sqrt{2}\mathrm{i}\right)^2+q\left(1+\sqrt{2}\mathrm{i}\right)+3=0$

$3\left(1+3\sqrt{2}\mathrm{i}-6-2\sqrt{2}\mathrm{i}\right)+p\left(1+2\sqrt{2}\mathrm{i}-2\right)+q\left(1+\sqrt{2}\mathrm{i}\right)+3=0$

$(-12-p+q)+(3+2p+q)\sqrt{2}\mathrm{i}=0$

Comparing real and imaginary parts,

$\begin{cases}-12-p+q=0\\3+2p+q=0\end{cases}\Rightarrow\begin{cases}-p+q=12\\2p+q=-3\end{cases}\Rightarrow p=-5,\ q=7.$

Since all the coefficients are real, $1-\sqrt{2}\,\mathrm{i}$ is also a root.

Quadratic factor $=\left[x-\left(1-\sqrt{2}\,\mathrm{i}\right)\right]\left[x-\left(1+\sqrt{2}\,\mathrm{i}\right)\right]$

$=(x-1)^2-2(-1)$

$=x^2-2x+3$

$3x^3-5x^2+7x+3=0$

$\left(x^2-2x+3\right)(3x+1)=0$

$x=1\pm2\mathrm{i}$ or $x=-\frac{1}{3}$

Other than $1+\sqrt{2}\,\mathrm{i}$, the other roots are $1-\sqrt{2}\,\mathrm{i}$ and $-\frac{1}{3}$.

$w+z^*=2-\mathrm{i}\quad\Rightarrow\quad z^*=2-\mathrm{i}-w$

$w(2-\text{i}-w)=3-\text{i}$

$w^2-(2-\text{i})w+(3-\text{i})=0$

$$w=\frac{2-\text{i}\pm\sqrt{(2-\text{i})^2-4(3-\text{i})}}{2}$$

$$=\frac{2-\text{i}\pm\sqrt{4-4\text{i}-1-12+4\text{i}}}{2}$$

$$=\frac{2-\text{i}\pm\sqrt{-9}}{2}$$

$$=\frac{2-\text{i}\pm 3\text{i}}{2}$$

$w=1-2\text{i}$ or $w=1+\text{i}$

When $w=1-2\text{i}$:

$z^*=2-\text{i}-(1-2\text{i})=1+\text{i}$

$z=1-\text{i}$

(reject since it's given that $|w|<|z|$)

When $w=1+\text{i}$:

$z^*=2-\text{i}-(1+\text{i})=1-2\text{i}$

$z=1+2\text{i}$

Hence, $w=1+\text{i}$ and $z=1+2\text{i}$.

8

$$a + b + c = \ln 3$$
$$2.25a + 1.5b + c = \ln 5.5$$
$$4a + 2b + c = \ln 9$$

By GC, $a = -0.227$, $b = 1.78$ and $c = -0.455$ (to 3 s.f.).

$$\mathrm{f}(x) = \ln\left(1 + 2x^2\right)$$
$$= 2x^2 - \frac{1}{2}\left(2x^2\right)^2 + \frac{1}{3}\left(2x^2\right)^3 + \ldots$$
$$= 2x^2 - 2x^4 + \frac{8}{3}x^6 + \ldots$$

The expansion is valid when $-1 < 2x^2 \le 1$.

$2x^2 \le 1 \qquad \left(\because 2x^2 \ge 0 \text{ for all } x \in \mathbb{R}\right)$

$2x^2 - 1 \le 0$

$\left(\sqrt{2}x - 1\right)\left(\sqrt{2}x + 1\right) \le 0$

$\left\{x \in \mathbb{R} : -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}\right\}$

$$\ln\left(1+2x^2\right)\approx 2x^2-2x^4+\frac{8}{3}x^6$$

Substitute $x^2=\frac{1}{6}$:

$$\ln\left[1+2\left(\frac{1}{6}\right)\right]\approx 2\left(\frac{1}{6}\right)-2\left(\frac{1}{6}\right)^2+\frac{8}{3}\left(\frac{1}{6}\right)^3$$

$$\ln\left(\frac{4}{3}\right)\approx\frac{47}{162}$$

$\therefore m=47$

$$\begin{aligned}\ln\left(\frac{1+2x^2}{1+2x+x^2}\right)&=\ln\left(1+2x^2\right)-\ln\left(1+x\right)^2\\&=\ln\left(1+2x^2\right)-2\ln\left(1+x\right)\\&=\left(2x^2-2x^4+\ldots\right)-2\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right)\\&=-2x+3x^2-\frac{2}{3}x^3-\frac{3}{2}x^4+\ldots\end{aligned}$$

Alternative

$$\ln\left(\frac{1+2x^2}{1+2x+x^2}\right)=\ln\left(1+2x^2\right)-\ln\left(1+2x+x^2\right)$$

$$=\left(2x^2-2x^4+\ldots\right)-\left(\left(2x+x^2\right)-\frac{\left(2x+x^2\right)^2}{2}+\frac{\left(2x+x^2\right)^3}{3}-\frac{(2x)^4}{4}+\ldots\right)$$

$$=2x^2-2x^4-2x-x^2+\frac{4x^2+4x^3+x^4}{2}-\frac{8x^3+3\left(4x^4\right)}{3}+4x^4+\ldots$$

$$=-2x+3x^2-\frac{2}{3}x^3-\frac{3}{2}x^4+\ldots$$

9

$\alpha=-\sqrt{3}+\mathrm{i}=2\mathrm{e}^{\frac{5\pi}{6}\mathrm{i}}$ and $\beta=\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}=\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}$

$$\left|\frac{\alpha^5}{\beta^*}\right| = \frac{|\alpha|^5}{|\beta^*|} = \frac{2^5}{1} = 32$$

$$5\arg(\alpha) - \arg(\beta^*) = 5\left(\frac{5\pi}{6}\right) - \left(-\frac{\pi}{3}\right) = \frac{27\pi}{6} = \frac{9\pi}{2}$$

$$\arg\left(\frac{\alpha^5}{\beta^*}\right) = \frac{\pi}{2}$$

$$\frac{\alpha^5}{\beta^*} = 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Alternative

$$\alpha = -\sqrt{3} + i = 2e^{\frac{5\pi}{6}i} \text{ and } \beta = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = e^{\frac{\pi}{3}i}$$

$$\frac{\alpha^5}{\beta^*} = \frac{\left(2e^{\frac{5\pi}{6}i}\right)^5}{e^{-\frac{\pi}{3}i}}$$

$$= \frac{32e^{\frac{25\pi}{6}i}}{e^{-\frac{\pi}{3}i}}$$

$$= \frac{32e^{\frac{\pi}{6}i}}{e^{-\frac{\pi}{3}i}} \qquad \left(\because e^{\frac{25\pi}{6}i} = e^{\frac{25\pi}{6}i - 4\pi i}\right)$$

$$= 32e^{\frac{\pi}{2}i}$$

$$= 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

(b)

9b	$z=\cos\theta+\mathrm{i}\sin\theta=\mathrm{e}^{\mathrm{i}\theta} \quad \Rightarrow \quad z^*=\mathrm{e}^{-\mathrm{i}\theta}$ $\frac{1}{1+z^2}=\frac{1}{1+\mathrm{e}^{2\theta\mathrm{i}}}$ $=\frac{1}{\mathrm{e}^{\mathrm{i}\theta}\left(\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\theta}\right)}$ $=\frac{\mathrm{e}^{-\mathrm{i}\theta}}{\left(\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\theta}\right)}$ $=\frac{z^*}{2\cos\theta} \qquad \left(\because z^*+z=2\mathrm{Re}(z)=2\cos\theta\right)$ $\left\lvert\frac{1}{1+z^2}\right\rvert=\frac{\lvert z^*\rvert}{\lvert 2\cos\theta\rvert}=\frac{1}{2\cos\theta} \qquad \left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\Rightarrow 2\cos\theta>0\right)$ $\arg\left(\frac{1}{1+z^2}\right)=\arg\left(\frac{z^*}{2\cos\theta}\right)=\arg(z^*)=-\arg(z)=-\theta$ ***Alternative*** $\frac{1}{1+z^2}=\frac{1}{1+(\cos\theta+\mathrm{i}\sin\theta)^2}$ $=\frac{1}{1+\cos^2\theta+2\mathrm{i}\sin\theta\cos\theta-\sin^2\theta}$ $=\frac{1}{2\cos^2\theta+2\mathrm{i}\sin\theta\cos\theta} \qquad \left(\because 1-\sin^2\theta=\cos^2\theta\right)$ $=\frac{1}{2\cos\theta(\cos\theta+\mathrm{i}\sin\theta)}$ $=\frac{\cos\theta-\mathrm{i}\sin\theta}{2\cos\theta(\cos\theta+\mathrm{i}\sin\theta)(\cos\theta-\mathrm{i}\sin\theta)}$ $=\frac{\cos\theta-\mathrm{i}\sin\theta}{2\cos\theta(\cos^2\theta+\sin^2\theta)}$ $=\frac{z^*}{2\cos\theta} \qquad \left(\because \cos^2\theta+\sin^2\theta=1\right)$ $\left\lvert\frac{1}{1+z^2}\right\rvert=\frac{\lvert z^*\rvert}{\lvert 2\cos\theta\rvert}=\frac{1}{2\cos\theta} \qquad \left(-\frac{\pi}{2}<\theta<\frac{\pi}{2}\Rightarrow 2\cos\theta>0\right)$ $\arg\left(\frac{1}{1+z^2}\right)=\arg\left(\frac{z^*}{2\cos\theta}\right)=\arg(z^*)=-\arg(z)=-\theta$

10

Since C does not cut the x-axis, $\Rightarrow y\neq 0$

$$\Rightarrow \frac{3x^2+ax+3}{x+1} \neq 0$$

$\Rightarrow 3x^2+ax+3 \neq 0$

i.e. $3x^2+ax+3=0$ has no real solutions.

Discriminant < 0

$a^2-4(3)(3)<0$

$(a-6)(a+6)<0$

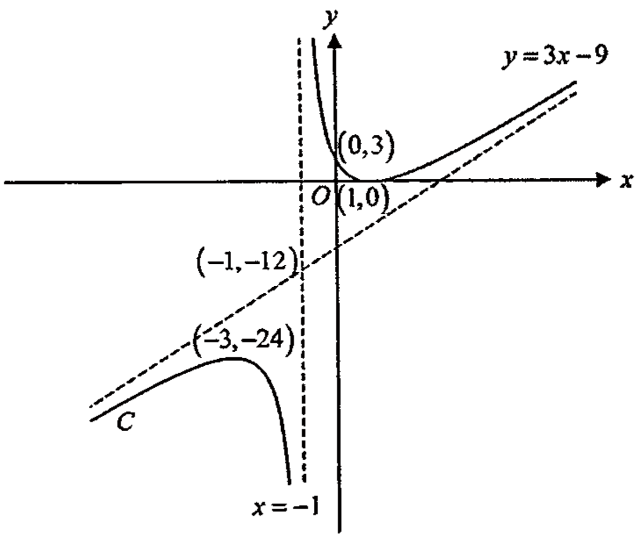
$-6<a<6$

$\{a \in \mathbb{R} : -6<a<6\}$

$$y=\frac{3x^2-6x+3}{x+1}=3x-9+\frac{12}{x+1}$$

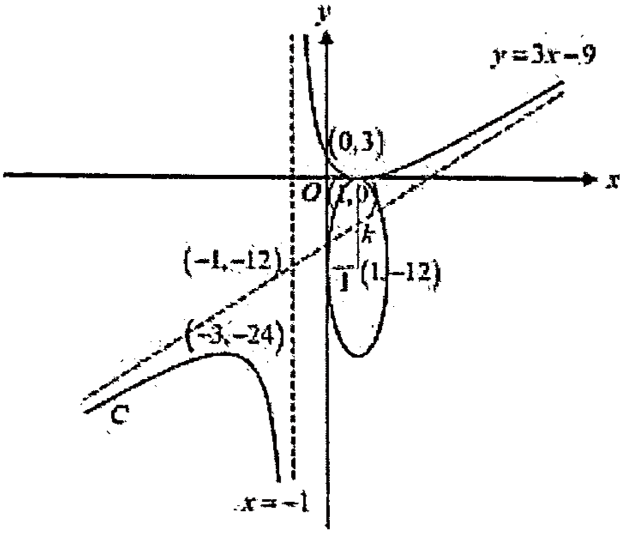
Equations of asymptotes: $y=3x-9, \quad x=-1$

Intercepts: $(0,3), \quad (1,0)$



$$k^2(x-1)^2+\left(\frac{3x^2-6x+3}{x+1}+12\right)^2=k^2$$

Sketch $(x-1)^2+\frac{(y+12)^2}{k^2}=1$ in the same diagram as C.



From the sketch, $k > 12$.

11

$$\left(\frac{1}{2}\right)\left(\frac{4}{3}\pi r^3\right)+\pi r^2 h=108\pi$$

$$\frac{2}{3}r^3+r^2h=108$$

$$h=\frac{108-\frac{2}{3}r^3}{r^2}=\frac{108}{r^2}-\frac{2}{3}r$$

$$A=\frac{1}{2}\left(4\pi r^2\right)+2\pi rh+\pi r^2$$

$$=3\pi r^2+2\pi r\left(\frac{108}{r^2}-\frac{2}{3}r\right)$$

$$=3\pi r^2+\frac{216\pi}{r}-\frac{4\pi}{3}r^2$$

$$=\frac{5\pi}{3}r^2+\frac{216\pi}{r}$$

$$=\left(\frac{5r^2}{3}+\frac{216}{r}\right)\pi$$

$$A=\left(\frac{5r^2}{3}+\frac{216}{r}\right)\pi$$

$$\frac{dA}{dr}=\left(\frac{10r}{3}-\frac{216}{r^2}\right)\pi$$

For stationary value of A, $\frac{dA}{dr}=0$.

$$\frac{10r}{3}-\frac{216}{r^2}=0$$

$$10r^3=648$$

$$r^3=\frac{324}{5}$$

$$r=\sqrt[3]{\frac{324}{5}}$$

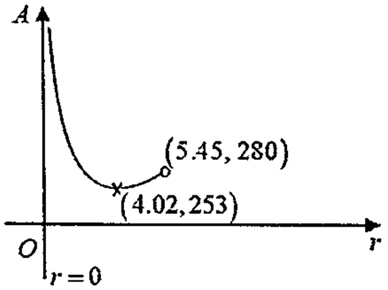
$$\frac{dA}{dr}=\left(\frac{10r}{3}-\frac{216}{r^2}\right)\pi$$

$$\frac{d^2A}{dr^2}=\left(\frac{10}{3}+\frac{432}{r^3}\right)\pi$$

When $r=\sqrt[3]{\frac{324}{5}}$, $\frac{d^2A}{dr^2}=10\pi=31.416>0$. [*OR evaluate by GC*]

Hence, A is minimum when $r=\sqrt[3]{\frac{324}{5}}$.

When $h=\frac{108-\frac{2}{3}r^3}{r^2}=0$, $r=\sqrt[3]{\frac{324}{2}}\approx 5.4514$.



After 14 seconds, amount of liquid leaked is 70 cm^3.
Total volume of hemisphere

$=\frac{2}{3}\pi(3)^3=18\pi=56.549<70$.

After 14 seconds, the remaining liquid lies within the cylinder.

Since the cylinder has a uniform cross-section, rate of decrease of height is $\frac{5}{\pi(3)^2}=\frac{5}{9\pi}\approx 0.177$ cm per second.

Alternative

After 14 seconds, amount of liquid leaked is 70 cm^3.
Total volume of hemisphere

$=\frac{2}{3}\pi(3)^3=18\pi=56.549<70$.

Let V cm^3 be the volume of the liquid in the cylinder, and l cm be the height of the liquid.

$V=\pi r^2 l=9\pi l \Rightarrow \frac{dV}{dl}=9\pi$

$\frac{dl}{dt}=\frac{dl}{dV}\times\frac{dV}{dt}=\frac{1}{9\pi}\times(-5)=-\frac{5}{9\pi}\approx -0.177$

Height of liquid decreases at 0.177 cm per second.

12

$$A_n = \left(\frac{1}{2^n}\right)^2 = \frac{1}{2^{2n}} = \frac{1}{4^n}$$

Total area of new squares added in stage $n = \frac{1}{4^n} \times 4 \times 3^{n-1}$

$$= \left(\frac{3}{4}\right)^{n-1}$$

$$\text{Total area in stage } n = 1+\left(\frac{3}{4}\right)^0+\left(\frac{3}{4}\right)^1+\left(\frac{3}{4}\right)^2+\ldots+\left(\frac{3}{4}\right)^{n-1}$$

$$=1+\frac{\left(\frac{3}{4}\right)^0\left[1-\left(\frac{3}{4}\right)^n\right]}{1-\frac{3}{4}}$$

$$=1+4\left[1-\left(\frac{3}{4}\right)^n\right]$$

$$=5-4\left(\frac{3}{4}\right)^n$$

As $n \to \infty, \left(\frac{3}{4}\right)^n \to 0$

$5-4\left(\frac{3}{4}\right)^n \to 5$

Hence, area of the square fractal converges.

Area of square fractal is 5.

$5-4\left(\frac{3}{4}\right)^n > 0.9(5)$

$\left(\frac{3}{4}\right)^n < \frac{1}{8}$

$n > \frac{\ln\frac{1}{8}}{\ln\frac{3}{4}} = 7.23$

Hence, $m = 8$.

Total number of squares $= 1+\sum_{n=1}^{8}\left(4\times 3^{n-1}\right) = 13121$