RIVER VALLEY HIGH SCHOOL
2023 JC1 Promotional Examination
Higher 2

Name	

CLASS	2	3	J		

INDEX NUMBER		

FURTHER MATHEMATICS 9649/01

Paper 1 **27 September 2023**

3 hours

Additional Materials: List of Formulae (MF26)
Answer Papers
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

Write your name and class on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
Up to **2 marks may be deducted** for poor presentation in your answers.

At the end of the examination, place the cover page on top of your answer paper and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

1. ~~**(a)** Prove by mathematical induction that~~ $\cancel{5^n - 2^n}$ ~~is divisible by 3 for all positive integers~~ ~~n.~~ ~~[4]~~

~~**(b)** Hence, find the greatest common divisor of the numbers in the set~~

$$\cancel{\left\{2\left(5^{n+1}\right) - 5\left(2^{n+1}\right),\ n \in \mathbb{Z}^{+}\right\}}.$$

~~(The greatest common divisor of a set of numbers is the largest integer that can divide all the numbers in the set.)~~ ~~[3]~~

2. **(i)** Use Simpson's rule with five ordinates to find an approximation to $I = \int_4^5 \frac{2}{\sqrt{8x - x^2 - 7}}\,\mathrm{d}x$, giving your answer to 4 decimal places. [2]

(ii) Find the exact value of I and deduce from part **(i)** an approximate value of $\sin^{-1}\left(\frac{1}{3}\right)$, giving your answer to 4 decimal places. [3]

3. ~~Solve the differential equation~~ $\cancel{\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} - 4z = 8\mathrm{e}^{2x} - \cos 2x + 4\sin 2x}$, ~~given that~~ $\cancel{z = 0}$ ~~and~~ $\cancel{\frac{\mathrm{d}z}{\mathrm{d}x} = 1}$ ~~when~~ $\cancel{x = 0}$. ~~[7]~~

4. ~~A deck of 25 playing cards consists of 3 different colours – blue, yellow and red. Each card has a picture of a fruit on it. The number of cards of each type are given in the following table:~~

	~~Apple~~	~~Kiwi~~	~~Watermelon~~	~~Grapes~~
~~Blue~~	~~1~~	~~3~~	~~2~~	4
~~Yellow~~	~~1~~	~~2~~	~~1~~	~~2~~
~~Red~~	~~2~~	~~2~~	~~1~~	4

~~A card is drawn at random from the deck. Events~~ ~~B~~ ~~and~~ ~~G~~ ~~are defined as follows:~~

~~B: The card is Blue.~~

~~G: The card shows Grapes.~~

~~**(i)** Determine if events~~ ~~G~~ ~~and~~ ~~B'~~ ~~are independent.~~ ~~[2]~~

~~The card is placed back into the deck and now two cards are drawn randomly from the deck.~~

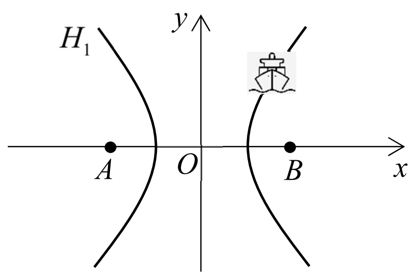
~~**(ii)** Find the probability that the two cards contain exactly one Apple card given that at least one card is Red.~~ ~~[5]~~

5. The sequence $\{x_n\}$ where $n \geq 2$ is defined by the recurrence relation $x_n - 4x_{n-1} + 4x_{n-2} = \dfrac{9}{2^n}$. It is given that $x_0 = 1$ and $x_1 = 3$.

By considering the sequence $\{y_n\}$, where $y_n = x_n - 2^{-n}$ for $n \geq 0$, show that $y_n - 4y_{n-1} + 4y_{n-2} = 0$. Hence, find an expression for x_n as a function of *n*. [7]

6. ~~Loran, short for Long Range Navigation, is a radio-based navigation system that was widely used before the advent of Global Positioning System (GPS). It was developed during World War II and continued to be used for navigation purposes until it was largely phased out in the late 20th century. Loran operates by using the time difference of arrival (TDOA) of radio signals from multiple ground-based transmitters to determine the position of a receiver, typically a ship or aircraft.~~

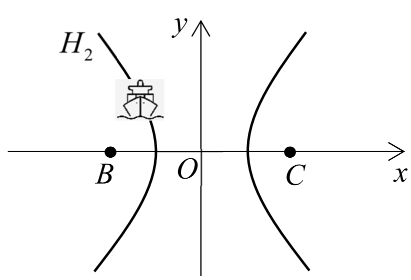
~~Two transmitter stations *A* and *B* are 50 km apart. Taking the origin as the mid-point of *A* and *B*, a ship is found to be on a hyperbola H_1 with *A* and *B* as the foci, as shown below.~~



~~Radio signals are transmitted at a constant speed of 0.3 km per microsecond (μs) from the stations. It is found that the TDOA of signals from stations *A* and *B* to the ship is 140 μs.~~

~~(i) Find the equation of H_1. [5]~~

~~Another transmitter station *C* is 40 km away from station *B* and is located on the line *AB* produced. Taking the origin as the mid-point of *B* and *C* instead, the same ship is found to be on a hyperbola H_2 with *B* and *C* as the foci, as shown below.~~



~~The equation of H_2 is $\frac{x^2}{324} - \frac{y^2}{76} = 1$.~~

~~(ii) Find the coordinates of the location of the ship, with reference to the coordinate axes with the origin as the mid-point of *A* and *B*. [3]~~

[**Turn over**

7. For $0 \le \theta \le \pi$, find the range of values θ for which $\sqrt{3} - 2\sin 2\theta \ge 0$. [2]

A curve C has polar equation $r = \sqrt{3} - 2\sin 2\theta,\ 0 \le \theta \le \pi$ and $r \ge 0$.

(i) Sketch C, indicating clearly the polar coordinates of the axial intercepts and any tangents to the curve at the pole. [3]

(ii) Find the exact area of the region bounded by C and the line $y = -x$. [3]

8. A curve C has parametric equations $x = 4\sin t,\ y = \cos 2t$ for $0 \le t \le \dfrac{\pi}{2}$.

Sketch C, indicating clearly the coordinates of the end points. [1]

(i) Let A be the area of the surface of revolution formed when this arc is rotated through 2π radians about the y-axis. Show that $A = 32\pi \int_0^{\frac{\pi}{2}} \sin t \cos t \sqrt{1 + \sin^2 t}\ \mathrm{d}t$. Hence find A. [3]

(ii) The region bounded by the arc, the line $y = -1$ and the y-axis is rotated through 2π radians about the y-axis. Using the shell method, find the exact volume of the solid generated. [4]

9. The Kentucky Department of Fish and Wildlife Reserve models the population growth of white-tailed deer in the state of Kentucky using the following differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{5} P(N - P),$$

where P is the population of white-tailed deer in thousands and t is the time in years.

(i) State what N represents and explain what it refers to in the context of the model. [2]

(ii) Sketch, on the same diagram, the curves that represent the behaviour of the population P with time when the initial population P_0, in thousands, is such that

(a) $0 < P_0 < N$,

(b) $P_0 > N$. [2]

The Kentucky Department of Fish and Wildlife Reserve would like to set guidelines for hunting in the state, stipulating that white-tailed deer can only be hunted at a constant rate of $100H$ thousands per year.

(iii) Write down a revised differential equation taking into account the hunting of white-tailed deer. Suppose $P_0 = 150$ and $N = 900$, determine the maximum value of H so that the population does not become extinct. [5]

10. The function $y = \mathrm{f}(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\mathrm{e}^{-x} - (x+5)y}{x+3},$$

with $y = \frac{2}{3}$ when $x = 0$.

(i) Use two steps of Euler method to determine an approximation to y when $x = 0.5$. [2]

(ii) Use one step of Improved Euler method to determine an alternative approximation to y when $x = 0.5$. [1]

(iii) Discuss the relative merits of the two methods employed to obtain these approximations. [2]

(iv) Show that $y = \frac{p(x)}{\mathrm{e}^{x}(x+3)^2}$ where $p(x)$ is a polynomial in terms of x. [4]

11. **(a)** Referred to the origin O, the points P and Q are such that $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$ and $\mathbf{p} \cdot \mathbf{q} = 0$. The point R lies on PQ such that $PR : RQ = \lambda : 1 - \lambda$. Given that $\overrightarrow{OR}$ is perpendicular to $\overrightarrow{PQ}$, show that $\lambda = \frac{|\mathbf{p}|^2}{|\mathbf{p}|^2 + |\mathbf{q}|^2}$. [4]

(b) Two planes, π_1 and π_2, have the following equations

$$\mathbf{r}.\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 10 \text{ and } \mathbf{r}.\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = -7 \text{ respectively.}$$

(i) Given that π_1 and π_2 meet in line l_1, find a vector equation of l_1. [1]

(ii) l_1 lies in plane p with equation $x + ay - z = b$. Find the values for a and b. [3]

(iii) The line l_2 joins the points A and B with coordinates (1, 4, 0) and (1, 0, 5) respectively. Find an equation of the line of reflection of l_2 in the plane π_2. [5]

12. A top fashion brand wants to promote its new spring collection. As part of the marketing drive, the brand creates a post on a social media platform featuring some of the new clothes and accessories, with a call-to-action for users to share the post on their own accounts. In order to incentivise users to participate, the brand offers a voucher for 20% off any purchase to everyone who shares the post to at least 5 other persons.

Suppose that any user who shares the post shares to exactly 5 other persons.

On Day 1 of the marketing drive, 20 users shared the post with their friends. For every subsequent day, each of the friends who receives the post the day before shares it with his or her friend. The brand also continues to share the post to new users and that results in k new users sharing the post each day.

Let u_t denote the number of times the post is shared by users on the social media platform on Day t of the marketing drive.

(i) Show that $u_3 = 2500 + 30k$. Write down a recurrence relation for u_t in terms of u_{t-1} and hence determine u_t as a function of t. State an assumption that has been made. [5]

(ii) Find the range of values of k so that the daily number of post shared can reach 75000 within the first 5 days of the marketing drive. [2]

(iii) For $k = 15$, find the number of days needed for the total number of post shared from Day 1 of the marketing drive to exceed 100 million. [2]

Let v_t denote the number of times another post by the fashion brand is shared by users on the social media platform on Day t of another marketing drive. Suppose v_t satisfies the following recurrence relation,

$$v_t = 5v_{t-1} + v_{t-2},\ t \geq 3,\ v_1 = 100,\ v_2 = 550.$$

(iv) Find v_t in terms of t. (Answer need not be in exact form.) [3]

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