2007 H2 Mathematics Prelim Paper 2 solutions

No.	Solutions
1a	$\frac{dy}{dx}=\frac{1}{1+(\ln x^3)^2}\left(\frac{3}{x}\right)=\frac{3}{x\left(1+9(\ln x)^2\right)}$
1b	$\frac{dx}{dt}=\frac{-2}{t^2}+2t, \quad \frac{dy}{dt}=2t-1$ $\frac{dy}{dx}=(2t-1)\left(\frac{t^2}{2t^3-2}\right)=\frac{(2t-1)t^2}{2t^3-2}$ At intersection, $2t^3-2=0 \Rightarrow t=1$ $x=\frac{2}{1}+1^2=3$ $p=3$
2	$2r+3=2(r+1)+r-(r-1)$ $\sum_{r=1}^{n}(2r+3)2^r=\sum_{r=1}^{n}\left[2^r\left(2(r+1)+r+(r-1)\right)\right]$ $=\sum_{r=1}^{n}\left[(r+1)2^{r+1}+r2^r-(r-1)2^r\right]$ $=\sum_{r=1}^{n}\left[(r+1)2^{r+1}+r2^r-2(r-1)2^{r-1}\right]$ $=\sum_{r=1}^{n}\left[f(r+1)+f(r)-2f(r-1)\right]$ where $f(r)=r2^r$ $=f(2)+f(1)-2f(0)$ $+f(3)+f(2)-2f(1)$ $+f(4)+f(3)-2f(2)$ $+$.............................. $+$... $+f(n-2)+f(n-3)-2f(n-4)$ $+f(n-1)+f(n-2)-2f(n-3)$ $+f(n)+f(n-1)-2f(n-2)$ $+f(n+1)+f(n)-2f(n-1)$ $=f(n+1)+2f(n)-f(1)-2f(0)$ $=(n+1)2^{n+1}+2n2^n-2$ $=2^n(2n+2+2n)-2$ $=2^n(4n+2)-2$ $=(2n+1)2^{n+1}-2$

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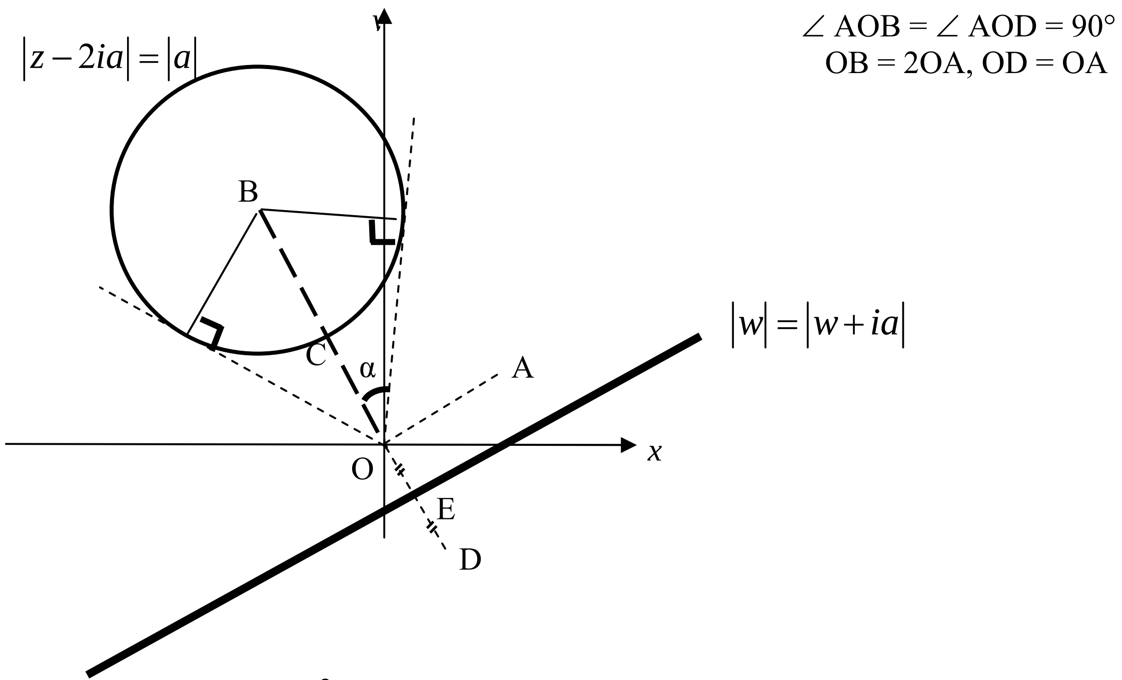
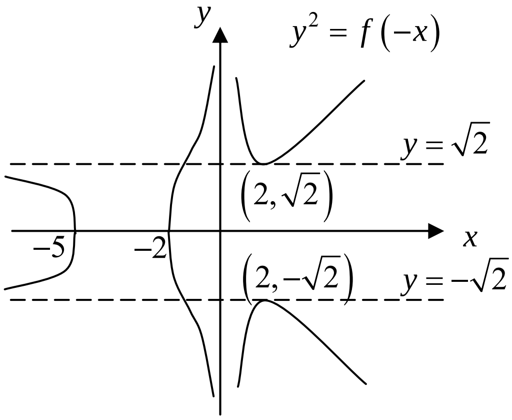
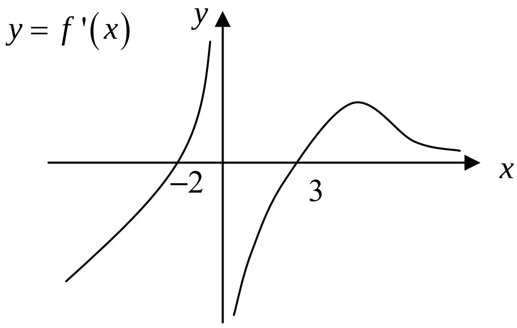
	$\sum_{r=n}^{2n}(2r+3)2^{r-1}=\frac{1}{2}\sum_{r=n}^{2n}(2r+3)2^r$ $=\frac{1}{2}\left[\sum_{r=1}^{2n}(2r+3)2^r-\sum_{r=1}^{n-1}(2r+3)2^r\right]$ $=\frac{1}{2}\left[(4n+1)2^{2n+1}-2-(2(n-1)+1)2^n+2\right]$ $=(4n+1)2^{2n}-(2n-1)2^{n-1}$ $=2^{n-1}\left[(4n+1)2^{n+1}-2n+1\right]$
3a	
3a	Minimum |z - w| = EC = $\frac{3}{2}\|a\|$
3b	$\arg\left(\frac{1}{z}\right)=-\arg z$ $\sin\alpha=\frac{\|a\|}{2\|a\|}=\frac{1}{2}\Rightarrow\alpha=\frac{\pi}{6}$ $\arg a+\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\le\arg z\le\arg a+\left(\frac{\pi}{2}+\frac{\pi}{6}\right)$ $\Rightarrow\arg a+\frac{\pi}{3}\le\arg z\le\arg a+\frac{2\pi}{3}\Rightarrow -\arg a-\frac{2\pi}{3}\le\arg\left(\frac{1}{z}\right)\le-\arg a-\frac{\pi}{3}$
4a	$x^2-y^2-4y-5=0$ $x^2-\left[y^2+4y+5\right]=0$ $x^2-\left[(y+2)^2-4+5\right]=0$ $x^2-(y+2)^2-1=0$ $x^2-(y-(-2))^2=1$ $\Rightarrow$ A translation of – 2 units in the y – direction.
4bi)	

No.	Solutions
4bii)	 
5i	$\Pi_1: \quad \boldsymbol{r} \bullet \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} = 15$ Distance of A from $\Pi_1 = \left\| \frac{\begin{pmatrix} 6 \\ 2 \\ -6 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}}{\sqrt{50}} - \frac{15}{\sqrt{50}} \right\|$ $= \left\| \frac{4}{\sqrt{50}} - \frac{15}{\sqrt{50}} \right\| = \frac{11}{\sqrt{50}}$ $\frac{15}{\sqrt{50}} > \frac{4}{\sqrt{50}} \Rightarrow A$ and O are on the same side of Π_1
5ii	Vector parallel to $\Pi_2 = \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$ Normal vector of Π_2 is $\begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -10 \end{pmatrix} = \begin{pmatrix} 4 \\ -12 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -6 \\ -1 \end{pmatrix}$ Angle between Π_1 and Π_2

No.	Solutions

$$= \cos^{-1}\left|\frac{\begin{pmatrix}5\\-4\\3\end{pmatrix}\bullet\begin{pmatrix}2\\-6\\-1\end{pmatrix}}{\sqrt{50}\sqrt{41}}\right| = \cos^{-1}\left|\frac{31}{\sqrt{50\times 41}}\right| = 46.8°$$

5iii

Equation of Π_2 is $\boldsymbol{r}\bullet\begin{pmatrix}2\\-6\\-1\end{pmatrix} = \begin{pmatrix}6\\2\\-6\end{pmatrix}\bullet\begin{pmatrix}2\\-6\\-1\end{pmatrix}$ i.e. $\boldsymbol{r}\bullet\begin{pmatrix}2\\-6\\-1\end{pmatrix} = 6$

Π_1 : $5x-4y+3z=15$ ------ ①

Π_2 : $2x-6y-z=6$ ------ ②

Π_3 : $x+8y+az=b$ ------ ③

For line of intersection of Π_1 and Π_2

$$A=\begin{bmatrix}5 & -4 & 3 & 15\\ 2 & -6 & -1 & 6\end{bmatrix}$$

rref $A=\begin{bmatrix}1 & 0 & 1 & 3\\ 0 & 1 & 1/2 & 0\end{bmatrix}$

$x+z=3 \quad \Rightarrow x=3-z$

$y+\frac{1}{2}z=0 \quad \Rightarrow y=-\frac{1}{2}z$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3-z\\-\frac{1}{2}z\\z\end{pmatrix}=\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}$$

$$\boldsymbol{r}=\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}$$

Let $\mu=0,1$, two points on the common line are (3,0,0), and (1 , −1 , 2)

Substitute into $x+8y+az=b$:

$3+8(0)+0a=b \Rightarrow b=3$

& $1-8+2a=b \Rightarrow 2a-b=7$

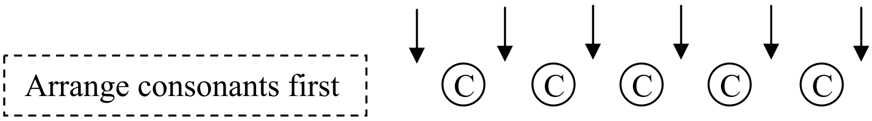
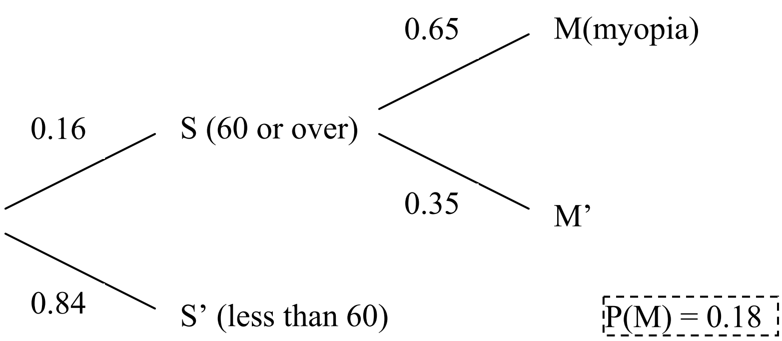
$2a=10 \Rightarrow a=5$

<u>Alternatively</u>

Since the common line lies in Π_3,

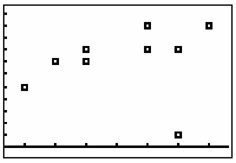
$$\left[\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}\right]\begin{pmatrix}1\\8\\a\end{pmatrix}=b \text{ for all } \mu$$

$3 - 2\mu - 8\mu + 2\mu a = b$

No.	Solutions
	3- μ (10 – 2a) = b b = 3, a = 5
6i	E E I I I G L T M S  Number of arrangements $= 5! \times {}^{6}C_5 \times \frac{5!}{3!2!} = 7200$
6ii	All letters different: ${}^{7}p_3 = 210$ A pair of identical letters: ${}^{6}C_1 \times \frac{3!}{2!} \times 2 = 36$ Three I's : 1 Total number of ways = 210+36+1= 247
7a	 Required Probability = $P(S'\cap M)$ = P (M) – $P(S\cap M)$ = 0.18 – 0.16 X 0.65 = 0.076
7b	Required Probability = P (S / M') $= \frac{P(S\cap M')}{P(M')}$ $= \frac{0.16\times 0.35}{1-0.18}$ $= 0.06829... = 0.0683$
7c	$P(S\cup M)$ $= P(S)+P(M)-P(S\cap M)$ $= 0.16+0.18-0.16\times 0.65$ $= 0.236$
8a	let X = number of red pens in sample of 10 $X \sim B(10, 0.35)$ Required probability = $P(X > 5) = 1 - P(X \le 5)$ = 0.0949

No.	Solutions
8b	P(at most n more pens) > 0.98 $(0.35)+(0.65)(0.35)+(0.65)^2(0.35)+....+(0.65)^{n-1}(0.35)>0.98$ $\dfrac{0.35\left(1-(0.65)^n\right)}{1-0.65}>0.98$ $(0.65)^n<1-0.98$ $n>\dfrac{\ln(0.02)}{\ln(0.65)}$ $n > 9.08$ least $n = 10$
9a	Let A = number of calls Alice receives in 30 minutes Let B = number of calls Brenda receives in 30 minutes $A \sim Po(0.5)$ and $B \sim Po(1.8)$ $\therefore\ A+B \sim Po(2.3)$ required probability $=\left[P(A+B<5)\right]^3$ $=\left[P(A+B\le 4)\right]^3$ $= 0.91625^3 = 0.769$
9b	required probability $= P(A\le 1/A+B<5)$ $=\dfrac{P(A=0).P(0\le B\le 4)+P(A=1).P(0\le B\le 3)}{P(A+B<5)}$ $=\dfrac{0.60653\times 0.96359+0.30327\times 0.89129}{0.91625} = 0.933$
9c	Let W= number of calls Alice received in one day $W \sim Po(8)$ $P(W<6)=P(W\le 5)$ $= 0.191$ Let C = number of days out of 60 days, with less than 6 calls per day $C \sim B(60, 0.19124)$ since np > 5 and n(1-p) >5, $C \sim N(11.4744,\ 9.28)$ approximately $P(10\le C<20)=P(9.5<C<19.5) = 0.737$
10ai	$\bar{x}=\dfrac{-18.7}{20}+80=79.065$ $S_x^{\ 2}=\dfrac{1}{19}[102.5-\dfrac{(-18.7)^2}{20}]=4.4745=(2.11530)^2$ $H_o\ :\mu=80,\quad H_1\ :\mu<80$ If H_o is true, the test statistics is $T=\dfrac{\overline{X}-80}{\frac{2.1153}{\sqrt{20}}}\sim t(19)$.

No.	Solutions
	We perform a one tailed t-test at 5% level of significance and reject H_o if $p < 0.05$. Use GC with $\mu = 80,\ n = 20,\ \bar{x} = 79.065,\ S_x = (2.11530)$ we have $p = 0.03138$ As $p = 0.03138 < 0.05$, we reject H_o at 5% level of significance. And conclude that there is significant evidence that the manufacturer's claim is justified at 5% level of significance. Assume that the weights (X) follow a normal distribution.
10aii	The probability of concluding that the manufacturer is justified in his claim when actually he is not justified is 0.05
10b	σ^2 is known to be 10kg, $\bar{X} \sim N(80, 0.5)$ or $Z = \frac{\bar{X} - 80}{\sqrt{0.5}} \sim N(0,1)$ At 5% level of significance, reject H_o if P < 0.05 i.e. $P(\bar{X} < \bar{x}) < 0.05 \quad or \quad P(Z < \frac{\bar{X} - 80}{\sqrt{0.5}}) < 0.05$ Using GC, $\bar{x} < 78.84 \quad or \quad \frac{\bar{X} - 80}{\sqrt{0.5}} < -1.645 \Rightarrow \bar{x} < 78.84$ As $\bar{x} = 78.0kg < 78.84kg$, we reject H_o at 5% level of significance. Therefore H_o will also be rejected at 8% level of significance (bigger rejection region). Hence the manufacturer's claim is justified at 8% level of significance.
11a	Let X = volume of beer in large can $\Rightarrow X \sim N\left(500, 3.3^2\right)$ Let Y = volume of beer in small can $\Rightarrow Y \sim N\left(340, 2.4^2\right)$ $P\left(Y > k\right) = 0.05$ $\Rightarrow P\left(Y < k\right) = 0.95$ From GC, k = 343. 94.. = 343. 9 (or 344)
11b	$X_1 - X_2 \sim N\left(500 - 500, 3.3^2 + 3.3^2\right) \Rightarrow X_1 - X_2 \sim N\left(0, 21.78\right)$ Required probability = $P\left(\left\|X_1 - X_2\right\| \le 10\right)$ $= P\left(-10 \le X_1 - X_2 \le 10\right)$ $= 0.9678... = 0.968$
11c	Volume of beer in bottle = $E \sim N\left(4(500), 4^2(3.3^2)\right)$ Let T = $E_1 + E_2 + ... + E_6 \sim N\left(6\left(4\right)(500), 6\left(4^2\right)(3.3^2)\right)$ $\Rightarrow T \sim N\left(12000, 1045.44\right)$ $P\left(T > 12040\right) = 0.108$
11d	$\bar{X} \sim N\left(500, \frac{3.3^2}{n}\right)$ and $\bar{Y} \sim N\left(340, \frac{2.4^2}{n}\right)$ $\bar{Y} - \frac{1}{2}\bar{X} \sim N\left(340 - \frac{500}{2}, \frac{2.4^2}{n} + \frac{3.3^2}{2^2 n}\right) \quad \Rightarrow \bar{Y} - \frac{1}{2}\bar{X} \sim N(90, \frac{8.4825}{n})$

No.	Solutions
	$P\left(\overline{Y}-\frac{1}{2}\overline{X}>92\right)\le 0.01$ $\Rightarrow P\left(\overline{Y}-\frac{1}{2}\overline{X}\le 92\right)\ge 0.99$ $\Rightarrow P\left(Z<\frac{92-90}{\sqrt{\frac{8.4825}{n}}}\right)\ge 0.99$ $\Rightarrow P\left(Z<\frac{2\sqrt{n}}{\sqrt{8.4825}}\right)\ge 0.99$ From GC: $\frac{2\sqrt{n}}{\sqrt{8.4825}}\ge 2.3263$ $\Rightarrow n\ge 11.4765$ least n = 12
12ai	Use GC, r = 0.147 (3s.f.) As r is small, expect x and y to be not linearly correlated
12aii	
12aiii	As (9, 1) is an outlier (or far away from the rest of the data), the interpretation in (i) should be amended. If (9, 1) is removed, the new r = 0.823 which indicates that x and y are linearly correlated.
12bi	$\overline{x}=8,\quad \overline{y}=68$, $b=\frac{\sum xy-\frac{\sum x\sum y}{6}}{\sum x^2-\frac{\left(\sum x\right)^2}{6}}=\frac{400}{88}=\frac{50}{11}=4.54545$ The estimate least squares regression line of y on x is $y-\overline{y}=b(x-\overline{x})$ $\Rightarrow y=4.54545x+31.63636\Rightarrow y=4.55x+31.6\ (3s.f.)$
12bii	When x = 8, $y\approx$ 68 ml. The estimated evaporation loss for a drum kept in storage for eighth weeks is 68 ml
12biii	When the storage time is more than a year, x will be outside the range $1\le x\le 15$, and hence we would not expect to get good estimates from the line of regression for evaporation loss .