HILLGROVE SECONDARY SCHOOL
PRELIMINARY EXAMINATION 2023
SECONDARY FOUR (EXPRESS)
[MARK SCHEME]

CANDIDATE NAME	( )	CLASS	-
CENTRE NUMBER	S	INDEX NUMBER	

Additional Mathematics **4049/01**

Paper 1 **23 August 2023**

2 hours 15 minutes

Candidates answer on the Question Paper. **10.45 a.m. – 1.00 p.m.**

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use	
TOTAL	**90**

Parent's/ Guardian's Signature: ___________________

Setter: Mdm Lee Li Lian

This document consists of **18** printed pages, including this page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 The equation of a curve is $y = 4x^2 - 16x + 19$.

(a) By expressing $4x^2 - 16x + 19$ in the form $a(x+b)^2 + c$, where *a*, *b* and *c* are constants, find the coordinates of the stationary point on the curve. [2]

$$\begin{aligned} 4x^2 - 16x + 19 &= 4(x^2 - 4x) + 19 \\ &= 4\left[(x-2)^2 - 4\right] + 19 \\ &= 4(x-2)^2 - 16 + 19 \\ &= 4(x-2)^2 + 3 \text{ [A1]} \end{aligned}$$

$\therefore$ Coordinates of the stationary point is (2, 3). [A1]

(b) The line $y = 4x + 3$ intersects the curve at the points *A* and *B*. Find the value of *k* for which the distance *AB* can be expressed as $3\sqrt{k}$. [4]

$y = 4x^2 - 16x + 19$ ....(1)

$y = 4x + 3$ ....(2)

Substitute (1) into (2):

$4x^2 - 16x + 19 = 4x + 3$

$4x^2 - 20x + 16 = 0$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$ [M1]

$x = 1$ or $x = 4$

$y = 7$ or $x = 19$

Coordinates of *A* and *B* are (1, 7) and (4, 19). [A1]

$$\begin{aligned} \text{Distance of } AB &= \sqrt{(4-1)^2 + (19-7)^2} \\ &= \sqrt{153} \text{ [A1]} \\ &= 3\sqrt{17} \end{aligned}$$

$\therefore k = 17$ [A1]

2 It is given that $\left(\sqrt{5}\right)^9 - \left(\sqrt{5}\right)^7 + \left(\sqrt{5}\right)^5 - \left(\sqrt{5}\right)^3 + 105\left(\sqrt{5}\right) = 5^k$. By factorisation, find the value of k. [4]

$$\left(\sqrt{5}\right)^9 - \left(\sqrt{5}\right)^7 + \left(\sqrt{5}\right)^5 - \left(\sqrt{5}\right)^3 + 105\left(\sqrt{5}\right) = \sqrt{5}\left[\left(\sqrt{5}\right)^8 - \left(\sqrt{5}\right)^6 + \left(\sqrt{5}\right)^4 - \left(\sqrt{5}\right)^2 + 105\right] \text{ [M1]}$$

$$= \sqrt{5}(5^4 - 5^3 + 5^2 - 5 + 105) \text{ [M1]}$$

$$= \sqrt{5}(625 - 125 + 25 - 5 + 105)$$

$$= \sqrt{5}(625)$$

$$= 5^{\frac{1}{2}} \times 5^4$$

$$= 5^{4\frac{1}{2}} \text{ [A1]}$$

$$\therefore k = 4\frac{1}{2} \text{ [A1]}$$

3 The loudness of a sound can be measured using the equation $L = 10\lg\dfrac{I}{I_0}$, where I is the intensity of sound to be measured and I_0 is the intensity of sound that can barely be heard, also known as the threshold of hearing. The unit of L is the decibel (dB).

(a) Given that the loudness of a scream is 110 dB, find the ratio of the intensity of the scream to the threshold of hearing. [2]

$L = 10\lg\dfrac{I}{I_0}$

When $L = 110$,

$10\lg\dfrac{I}{I_0} = 110$

$\lg\dfrac{I}{I_0} = 11$

$\dfrac{I}{I_0} = 10^{11}$ [A1]

$\therefore$ Ratio is $10^{11}:1$ [A1]

(b) 130 dB is the pain threshold (the maximum level of sound we can hear without feeling intense pain and instantly damaging our hearing). [2]

Explain the impact, on hearing, the loudness of the sound of an unknown object falling from the sky onto Earth, if it has a sound intensity of $10^{-10.5}$ units and threshold of hearing of 10^{-25} units.

Given that $I = 10^{-10.5}$ and $I_0 = 10^{-25}$

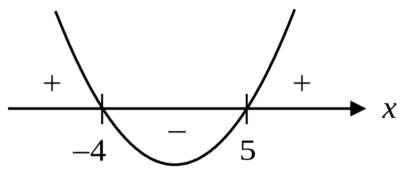
$L = 10\lg\dfrac{10^{-10.5}}{10^{-25}}$

$\quad = 10\lg 10^{14.5}$

$\quad = 145$ [A1]

Since the sound of the unknown object falling has a loudness of 145dB which exceeds the pain threshold, this can cause damage to our hearing. [A1]

4 Given that the range of values of x where $x^2 + ax < b$ is $-4 < x < 5$, find the value of a and of b. [4]



From the diagram,

$$(x+4)(x-5) < 0 \text{ [M1]}$$

$$x^2 - 5x + 4x - 20 < 0$$

$$x^2 - x < 20 \text{ [A1]}$$

Comparing $x^2 + ax < b$ with $x^2 - x < 20$, $a = -1$ [A1] and $b = 20$ [A1]

5 The line $y = mx + c$ is drawn on the same axes as the curve $y = 4x - 2x^2$.

(a) Given that the line is a tangent to the curve when $c = \frac{1}{2}$, find the possible values of m. [3]

$y = mx + \frac{1}{2}$.................................(1)

$y = 4x - 2x^2$.............................(2)

Substitute (1) into (2):

$mx + \frac{1}{2} = 4x - 2x^2$

$2x^2 - 4x + mx + \frac{1}{2} = 0$

$2x^2 + (m-4)x + \frac{1}{2} = 0$ [M1]

$a = 2, b = m - 4, c = \frac{1}{2}$

Since the line is a tangent to the curve,

Discriminant = 0

$(m-4)^2 - 4(2)\left(\frac{1}{2}\right) = 0$ [M1]

$(m-4)^2 = 4$

$m - 4 = 2$ or $m - 4 = -2$

$m = 6$ or $m = 2$ [A1]

(b) The line $y = mx + c$ has a negative y-intercept. Do the line and the curve have 0, 1 or 2 points of intersections? Show your working clearly. [3]

$y = mx + c$.................................(3)

Substitute (3) into (2):

$mx + c = 4x - 2x^2$

$2x^2 - 4x + mx + c = 0$

$2x^2 + (m-4)x + c = 0$

Discriminant = $(m-4)^2 - 4(2)c$

$= (m-4)^2 - 8c$ [A1]

Since $(m-4)^2 \geq 0$ and $c < 0$, $\underline{\text{discriminant} > 0}$. [A1]

$\therefore$ The line and the curve have 2 points of intersection. [A1]

6 A polynomial, P, is $3x^3+5x^2-x+k$, where k is a constant.

(a) Find the value of k given that P leaves a remainder of 24 when divided by $x-2$. [2]

Let $P=\mathrm{f}(x)=3x^3+5x^2-x+k$

Given that $\mathrm{f}(2)=24$

$3(2)^3+5(2)^2-2+k=24$ [M1]

$24+20-2+k=24$

$k+42=24$

$k=-18$ [A1]

(b) In the case where $k=-3$, the quadratic expression $3x^2+ax^2-3$ is a factor of P. Find the value of the constant a. [4]

$$\begin{aligned}3x^3+5x^2-x-3&=(3x^2+ax-3)(x+b)\ \text{[M1]}\\&=3x^3+3bx^2+ax^2+abx-3x-3b\\&=3x^3+(3b+a)x^2+(ab-3)x-3b\ \text{[M1]}\end{aligned}$$

Equating constants,

$-3b=-3$

$b=1$ [A1]

Equating coefficients of x^2,

$3b+a=5$

$3(1)+a=5$

$a=2$ [A1]

<u>Alternative Method</u>

Equating coefficients of x:

$ab-3=-1$

$a-3=-1$

$a=2$

7 **(a)** Divide $2x^3+4x-2$ by x^3+2x. [1]

$$\begin{array}{r} 2 \\ x^3+2x \overline{\big)\, 2x^3+0x^2+4x-2} \\ \underline{-(2x^3 \qquad +4x)} \\ -2 \end{array}$$

$\therefore \dfrac{2x^3+4x-2}{x^3+2x}=2-\dfrac{2}{x^3+2x}$ [A1]

(b) Express $\dfrac{2x^3+4x-2}{x^3+2x}$ in partial fractions. [5]

$$\frac{2x^3+4x-2}{x^3+2x}=2+\frac{-2}{x(x^2+2)}$$

Let $\dfrac{-2}{x(x^2+2)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+2}$ [A1]

$-2=A(x^2+2)+x(Bx+C)$

Let $x=0$: $2A=-2$

$A=-1$ [A1]

Comparing coefficients of x^2: $-1+B=0$

$B=1$ [A1]

Comparing coefficients of $x: C=0$ [A1]

$\therefore \dfrac{2x^3+4x-2}{x^3+2x}=2-\dfrac{1}{x}+\dfrac{x}{x^2+2}$ [A1]

(c) Hence, find $\displaystyle\int \frac{2x^3+4x^2-2}{x^3+2x}\,dx$. [2]

$$\int \frac{2x^3+4x-2}{x^3+2x}\,dx=\int 2-\frac{1}{x}+\frac{x}{x^2+2}\,dx$$

$$= 2x-lnx+\frac{1}{2}ln(x^2+2)+c \text{ [A2: 1 mark per pair]}$$

where c is a constant.

8 The line $3y+4x=12$ meets the x-axis at the point A and the y-axis at the point B.

(a) Write the coordinates of A and of B. [2]

When $y=0,\ 4x=12$

$x=3$

When $x=0,\ 3y=12$

$y=4$

$\therefore$ Coordinates of $A=(3,0)$ [B1] and $B=(0,4)$ [B1]

The perpendicular bisector of AB meets the line $y=x$ at the point C.

(b) Find the coordinates of C. [4]

$$\text{Midpoint of } AB=\left(\frac{3+0}{2},\frac{0+4}{2}\right)$$

$$=\left(1\frac{1}{2},2\right) \text{ [A1]}$$

$$\text{Gradient of } AB=\frac{4-0}{0-3}$$

$$=-\frac{4}{3}$$

$$\text{Gradient of the perpendicular bisector of } AB=\frac{3}{4}$$

Equation of the perpendicular bisector of AB:

$$\frac{y-2}{x-1\frac{1}{2}}=\frac{3}{4} \text{ [M1]}$$

$$4y-8=3x-4\frac{1}{2}$$

$$4y=3x+3\frac{1}{2} \text{ ....(1) [A1]}$$

Substitute $y=x$ into (1):

$$4x=3x+3\frac{1}{2}$$

$$x=3\frac{1}{2}$$

$$\therefore \text{Coordinates of } C=\left(3\frac{1}{2},3\frac{1}{2}\right) \text{ [A1]}$$

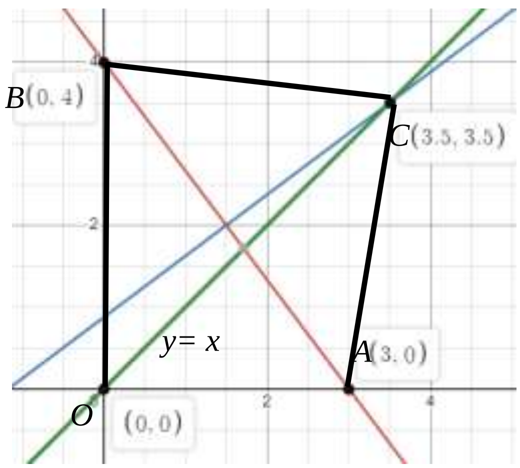
(c) Find the area of quadrilateral $OACB$, where O is the origin. [2]

$$\text{Area of } OACB = \frac{1}{2}\begin{vmatrix} 0 & 3 & 3\frac{1}{2} & 0 & 0 \\ 0 & 0 & 3\frac{1}{2} & 4 & 0 \end{vmatrix} \text{ [M1]}$$

$$= \frac{1}{2}\left\{\left[0+3\left(3\frac{1}{2}\right)+3\frac{1}{2}(4)+0\right]-0\right\}$$

$$= \frac{1}{2}\left(\frac{49}{2}\right)$$

$$= 12\frac{1}{4} \text{ units}^2 \text{ [A1]}$$



Alternative Method

$$\text{Area of } OACB = \frac{1}{2}\times 4\times 3\frac{1}{2} + \frac{1}{2}\times 3\times 3\frac{1}{2} \text{ [M1]}$$

$$= 7 + 5\frac{1}{4}$$

$$= 12\frac{1}{4} \text{ units}^2 \text{ [A1]}$$

9 **(a)** Write down the first three terms in the expansion of $\left(2-\frac{x}{4}\right)^n$, where n is a positive integer greater than 2, in ascending powers of x. [4]

$$\left(2-\frac{x}{4}\right)^n = 2^n + \binom{n}{1}2^{n-1}\left(-\frac{x}{4}\right)^1 + \binom{n}{2}2^{n-2}\left(-\frac{x}{4}\right)^2 + \ldots \text{ [M1]}$$

$$= 2^n - n2^{n-1}\left(\frac{x}{2^2}\right) + \frac{n(n-1)}{2}2^{n-2}\left(\frac{x^2}{4^2}\right) + \ldots$$

$$= 2^n - n\left(2^{n-3}\right)x + n(n-1)2^{n-7}x^2 + \ldots \text{ [A3]}$$

The first two non-zero terms in the expansion of $(2+x)\left(2-\frac{x}{4}\right)^n$ in ascending powers of x are $a+bx^2$, where a and b are constants.

(b) Find the value of n. [2]

$$(2+x)\left(2-\frac{x}{4}\right)^n = (2+x)\left[2^n - n\left(2^{n-3}\right)x + n(n-1)2^{n-7}x^2 + \ldots\right]$$

Equating coefficients of x,

$2\left[-n\left(2^{n-3}\right)\right] + 2^n = 0$ [M1]

$2^n - n\left(2^{n-2}\right) = 0$

$2^n\left(1-\frac{n}{4}\right) = 0$

$2^n = 0$ (N.A.) or $1-\frac{n}{4} = 0$

$n = 4$ [A1]

(c) Hence, find the value of a and of b. [2]

Equating constants,

$a = 2(2^4)$

$= 32$ [B1]

Equating coefficients of x^2,

$b = 4(3)2^{-2} - 4(2^1)$

$= -5$ [A1]

10 A curve is such that $\frac{d^2y}{dx^2} = 3e^{-x} + 8e^{2x}$. The curve intersects the y-axis at $P(0, -5)$ and has a gradient of 5 at P. Find the equation of the curve. [7]

$\frac{d^2y}{dx^2} = 3e^{-x} + 8e^{2x}$

$\frac{dy}{dx} = -3e^{-x} + 4e^{2x} + c$ [M1]

When $x = 0$, $\frac{dy}{dx} = 5$,

$-3 + 4 + c = 5$ [M1]

$c = 4$ [A1]

$\frac{dy}{dx} = -3e^{-x} + 4e^{2x} + 4$ [A1]

$y = 3e^{-x} + 2e^{2x} + 4x + c$ [M1]

When $x = 0$, $y = -5$,

$3 + 2 + c = -5$

$c = -10$ [A1]

$\therefore$ Equation of the curve is $y = 3e^{-x} + 2e^{2x} + 4x - 10$ [A1]

11 **(a)** Show that $6\sin^2 x - 4\cos^2 x$ can be written as $a + b\cos 2x$, where a and b are integers. [2]

$$6\sin^2 x - 4\cos^2 x = 6\left(\frac{1-\cos 2x}{2}\right) - 4\left(\frac{1+\cos 2x}{2}\right) \text{ [M1]}$$
$$= 3 - 3\cos 2x - 2 - 2\cos 2x$$
$$= 1 - 5\cos 2x \text{ [A1]}$$

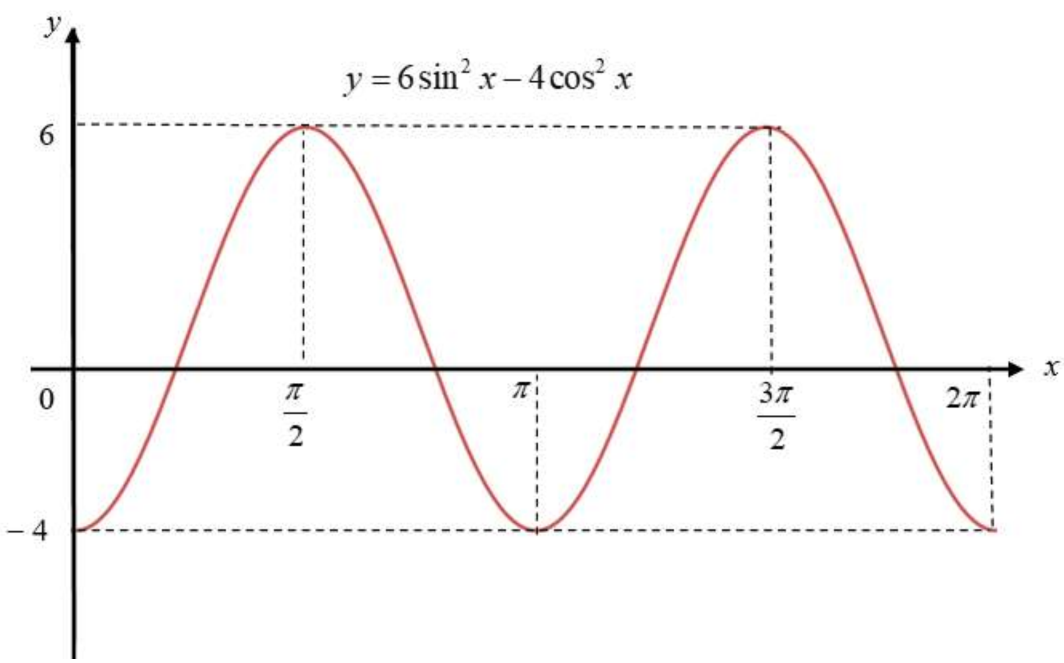
Hence,

(b) state the period and amplitude of $6\sin^2 x - 4\cos^2 x$, [2]

Period = π [B1] Accept 180°

Amplitude = 5 [B1]

(c) Sketch the graph of $y = 6\sin^2 x - 4\cos^2 x$ for $0 \le x \le 2\pi$ radians. [3]



B1: Correct shape from $0 \le x \le 2\pi$ with $(0,-4)$ and $(2\pi,-4)$

B1: Minimum points $(\pi,-4)$, $(0,-4)$ and $(2\pi,-4)$

B1: Maximum points $\left(\frac{\pi}{2},6\right)$ and $\left(\frac{3\pi}{2},6\right)$

12 Liquid is poured, at a constant rate of $25\pi\ \text{cm}^3/\text{s}$, into a hemispherical bowl of radius r cm.

When the depth of the liquid directly below the centre of the bowl is x cm, the volume, $V\,\text{cm}^3$, of the liquid in the bowl is given by $V = \frac{1}{3}\pi x^2(3r - x)$.

It is given that the radius of hemispherical bowl of radius is 12 cm, find

(a) the time taken for the depth of the liquid directly below the centre of the bowl to reach 6 cm, [3]

$V = \frac{1}{3}\pi x^2(3r - x)$

Given that $r = 12$, $V = \frac{1}{3}\pi x^2(36 - x)$ [M1]

When $x = 6$, $V = \frac{1}{3}\pi(6)^2(36 - 6)$

$= 360\pi$ [A1]

$\therefore$ Time taken $= \frac{360\pi}{25\pi}$

$= 14.4$ seconds [A1]

(b) the rate of change of the depth of liquid directly below the centre of the bowl at this time. [4]

$V = \frac{1}{3}\pi x^2(36 - x)$

$= 12\pi x^2 - \frac{1}{3}\pi x^3$

$\frac{dv}{dx} = 24\pi x - \pi x^2$ [A1]

Given that $x = 6$ and $\frac{dv}{dt} = 25\pi$,

$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$

$25\pi = \left[24\pi(6) - \pi(6)^2\right] \times \frac{dx}{dt}$ [M1]

$= 108\pi \times \frac{dx}{dt}$ [M1]

$\frac{dx}{dt} = \frac{25}{108}$ cm/s [A1]

13 (a) Show that $4\sec\theta + \tan\theta = 3\cot\theta$ can be expressed as $4\sin^2\theta + 4\sin\theta - 3 = 0$. [3]

$$4\sec\theta + \tan\theta = 3\cot\theta$$

$$\frac{4}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{3\cos\theta}{\sin\theta} \text{ [M1]}$$

$$\frac{4+\sin\theta}{\cos\theta} = \frac{3\cos\theta}{\sin\theta}$$

$$4\sin\theta + \sin^2\theta = 3\cos^2\theta$$

$$= 3(1-\sin^2\theta) \text{ [M1]}$$

$$= 3 - 3\sin^2\theta$$

$$4\sin^2\theta + 4\sin\theta - 3 = 0 \text{ (shown) [A1]}$$

(b) Hence, solve $4\sec 2x + \tan 2x = 3\cot 2x$ for $-180° < x < 180°$. [5]

$$4\sec 2x + \tan 2x = 3\cot 2x$$

$$4\sin^2 2x + 4\sin 2x - 3 = 0 \text{ [A1]}$$

$$(2\sin 2x - 1)(2\sin 2x + 3) = 0 \text{ [M1]}$$

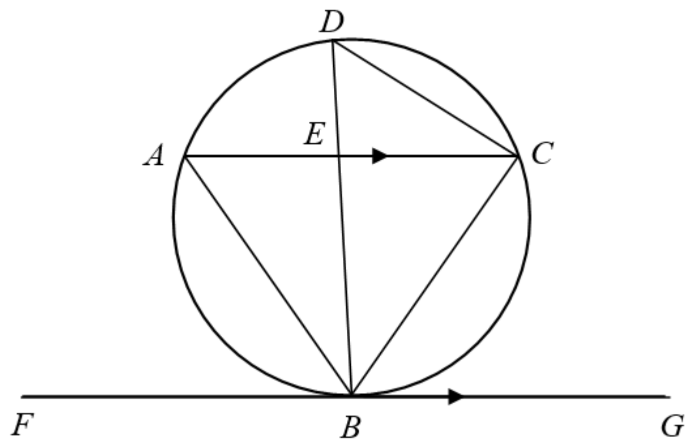
$$\sin 2x = \frac{1}{2} \text{ or } \sin 2x = -1\frac{1}{2} \text{ (no solution) [M1]}$$

Basic angle = 30°

$$2x = -330°, -210°, 30°, 150°$$

$$x = -165°, -105°, \underline{15°, 75°} \text{ [A1, } \underline{\text{A1}}\text{]}$$

14



The diagram shows triangles *ABC* and *BCD* whose vertices lie on the circumference of a circle. The chords *BD* and *AC* intersect at *E* and *AC* is parallel to *FG*. *FG* is a tangent to the circle at *B*.

Prove that

(a) ΔBCD is similar to ΔBEC, [3]

$\angle BDC = \angle CBG$ (alternate segment theorem)

$\angle BCE = \angle CBG$ (alternate angles, $AC//FG$)

$\therefore \angle BDC = \angle BCE$ [A1 with above 2 statements cited]

$\angle CBD = \angle EBC$ (common angle) [A1]

Since the corresponding angles of the triangles are equal, ΔBCD is similar to ΔBEC.
[A1]

(b) $BC^2 = BD \times BE$, [2]

Since ΔBCD is similar to ΔBEC,

$\frac{BC}{BD} = \frac{BE}{BC}$ [M1]

$BC^2 = BD \times BE$ [A1]

(c) ΔABC is an isosceles triangle. [2]

$\angle BDC = \angle BCE$ (from (a))
$\angle BDC = \angle BAC$ ($\angle$ in same segment)
$\angle BCE = \angle BAC$ [A1 with above cited]
$\therefore \Delta ABC$ is an isosceles triangle. [A1]

Alternative method,
$C\hat{B}G = A\hat{C}B$ (alternate angles, $AC // FG$)
$C\hat{B}G = B\hat{A}C$ (angles in alternate segment)
$A\hat{C}B = B\hat{A}C$ [A1 with above cited]
$\therefore \Delta ABC$ is an isosceles triangle. [A1]