2024 Nanyang Junior College Preliminary Examination H2 Mathematics Paper 1

1 Find the complex numbers v and w which satisfy the following simultaneous equations.

$$v^2 - \mathrm{i}w + 2 = 0$$

$$v = w + \mathrm{i} + 3$$

Give your answers in the form $a + \mathrm{i}b$, where a and b are real numbers. [4]

2 The path traced by a moving particle is a curve C, given by

$$x = a\theta - a\sin\theta \quad \text{and} \quad y = a - a\cos\theta,$$

where a is a positive constant and $0 \le \theta < 2\pi$.

(a) Show that the gradient of C at a point with parameter θ, is $\cot\left(\frac{1}{2}\theta\right)$. [2]

(b) Find the equation of the tangent to C at the point where $\theta = \frac{\pi}{3}$ and show that this tangent passes through the point $\left(\frac{1}{3}\pi a,\ 2a\right)$. [4]

3 **(a)** Given that $y = \left(\sin^{-1} x\right)^2$, show that $\left(1 - x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} = 2$. [2]

(b) Hence obtain the Maclaurin expansion of y in terms of x, up to and including the term in x^4. [3]

(c) By using $x = \frac{1}{2}$, find an approximation for π in surd form. Find the percentage error of this approximation and comment on its accuracy. [3]

4 It is given that $\mathrm{f}(x) = \sqrt{3}\cos x + \sin x$.

(a) Write $\mathrm{f}(x)$ as $R\cos(x - \alpha)$, where R and α are constants to be found. [2]

(b) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{6}} \left(\frac{1}{\mathrm{f}(x)}\right)^2 \mathrm{d}x$. [2]

(c) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{12}} \frac{1}{\mathrm{f}(2x)}\,\mathrm{d}x$. [3]

5 **(a)** Find $\displaystyle\int \frac{4x - 1}{x^2 + 4x + 4}\,\mathrm{d}x$, giving your answer in simplest form. [4]

(b) Find the value of $\displaystyle\int_0^1 \frac{|4x - 1|}{x^2 + 4x + 4}\,\mathrm{d}x$, expressing your answer in the form $p \ln q + r$, where p, q and r are exact constants in simplest form to be found. [3]

6 **(a)** An arithmetic series has first term a and common difference d, where $d \neq 0$. The 1st, 6th and 14th terms of this series are the 1st, 2nd and 3rd terms of a geometric series. Find d in terms of a. [3]

(b) A geometric series has first term b, where $b > 0$, and common ratio 0.5.

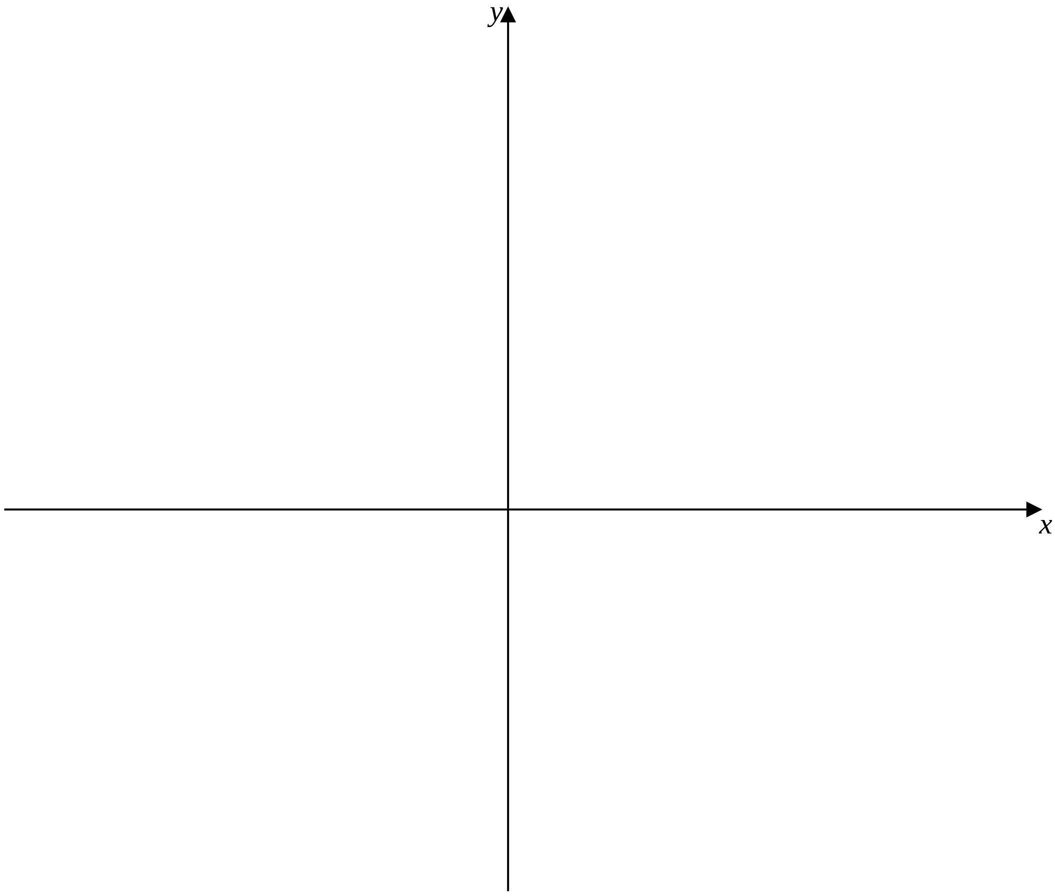
(i) Find the sum to infinity of this series in terms of b. [1]

(ii) Find the smallest possible value of n for which the sum of the first n terms of the series differs from the sum of the first $2n$ terms of the series by less than $0.004b$. [4]

7 An isosceles triangle ABC, with $AB = AC$, is inscribed in a fixed circle of radius 1 unit and centre O. It is given that angle $BOC = 2\theta$, where θ is acute. Using calculus, show that the area of the triangle ABC is a maximum when it is equilateral. State the maximum value of this area exactly. [8]

8 A curve C has equation $y = ax + b + \dfrac{1}{x-a}$, where a and b are positive real constants such that $a > 1$ and $x \neq a$.

(a) Sketch C on the axes below stating the equations of any asymptotes and the coordinates of the point where C crosses the y-axis. [4]



(b) On the same axes, sketch the graph of $(x-a)^2 + \left(y - a^2 - b\right)^2 = r^2$, where r is a positive constant, such that it intersects C at more than 2 points. $y = \pm\sqrt{r^2 - (x-a)^2} + a^2 + b$ [2]

(c) By stating the values of a and b, solve the inequality $2x+1+\dfrac{1}{x-2}>\sqrt{16-(x-2)^2}+5$. [3]

9 A curve R has equation $y=x^2\ln x$ for $x>0$. R crosses the x-axis at point A and has a turning point at B.

(a) State the coordinates of A. [1]

(b) Find the exact coordinates of B. [3]

(c) Find the exact area of the region bounded by R and the line segment AB. [5]

10 The functions f and g are defined by

$$\mathrm{f}: x \mapsto 2x+1, \quad x\in\mathbb{R},$$

$$\mathrm{g}: x \mapsto \frac{ax-1}{bx-a}, \quad x\in\mathbb{R}, \quad x\neq\frac{a}{b},$$

where a and b are positive constants.

(a) Find fg(1) in terms of a and b. You do not need to simplify your answer. [2]

(b) Explain why the function gf does not exist. [1]

(c) Find $\mathrm{g}^{-1}(x)$. [2]

(d) Hence, or otherwise, find $\mathrm{g}^2(x)$. [1]

(e) Given that $a>1$ and $b=1$, describe fully a sequence of transformations which transforms the curve $y=\mathrm{g}(x)$ **onto** the curve $y=\dfrac{1}{x}$. [4]

11 Ethylene is a gaseous plant hormone that plays an important role in inducing the ripening process for bananas. As a banana matures, ethylene is produced as a signal to induce fruit ripening. The amount of ethylene that a banana produces can be used to determine its ripeness.

Scientists set up an experiment to measure the amount of ethylene produced by a banana. The amount of ethylene, x parts per million (ppm), produced by a banana is modelled such that, at any time t hours, the rate of increase in the amount of ethylene is $k(160-x)$, for some positive constant k.

Initially no ethylene is present and it is found that there are 40 ppm of ethylene at time $t=12$.

(a) Write down and solve a differential equation involving x and t. Find the time it takes for the amount of ethylene to reach 100 ppm. [4]

To reduce the amount of ethylene produced, ethylene absorbers can be used. The scientists repeat the experiment by adding ethylene absorbers at a rate of d ppm per hour. Initially, no ethylene is present and the rate of increase in the amount of ethylene, $k(160-x)$, is the same as that found in part **(a)**.

(b) Write down and solve a differential equation, giving x in terms of t and d. [6]

(c) Find the value of d for which the amount of ethylene will be kept at a maximum of 10 ppm in the long run. [2]

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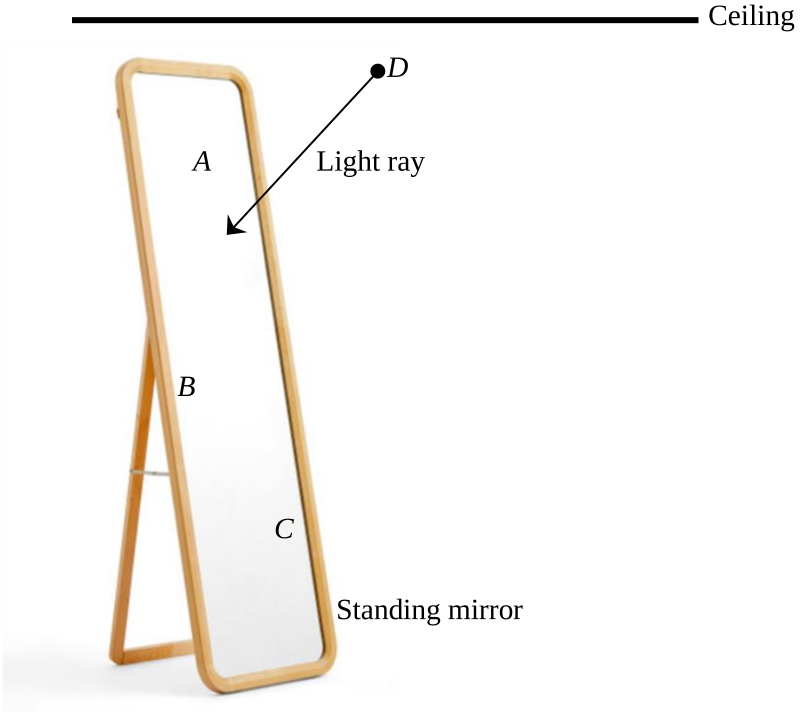

Credit: Image taken from https://www.fortytwo.sg/theola-standing-mirror-birch.html

In a room, there is a standing mirror and a light source at D below the ceiling. The three points A, B and C on the mirror, and D have position vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -\mathbf{j} + 6\mathbf{k}$, $\mathbf{c} = -\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{d} = 5\mathbf{i} - \mathbf{j} + 8\mathbf{k}$ respectively with respect to a fixed point of reference.

(a) Show that the mirror can be modelled as a plane with cartesian equation $3x - 3y + 2z = 15$. [2]

(b) Find the exact position vector of point F, the foot of perpendicular from D to the mirror. [3]

A light ray is sent in the direction of $\begin{pmatrix} -6 \\ -2 \\ -13 \end{pmatrix}$ from D and hits the mirror at point E.

(c) Find the acute angle between the light ray and the mirror. [2]

(d) Show that the coordinates of E is $\left(2, -2, \frac{3}{2}\right)$. [2]

(e) The light ray from D is reflected in the mirror. The reflected ray and the light ray lie in the same plane. The angle between the light ray and the mirror is the same as the angle between the reflected ray and the mirror. Find a vector equation of the reflected ray. [3]