2020 A Level H2 Further Mathematics (9649) Paper 1 (Suggested Solution)

1 Let $z = \cos\theta + \mathrm{i}\sin\theta$. By considering z^4, prove the identity

$$\tan 4\theta \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$ [5]

[Solution]

Using de Movire's Theorem and Binomial expansion

$z^4 = (\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$ ----- (1)

Also, $(\cos\theta + \mathrm{i}\sin\theta)^4$

$$= \cos^4\theta + 4\cos^3\theta(\mathrm{i}\sin\theta) + 6\cos^2\theta(\mathrm{i}\sin\theta)^2 + 4\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4$$

$$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$$

$$= \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right) + \mathrm{i}\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right) \text{ --- (2)}$$

Comparing the real parts of (1) and (2): $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

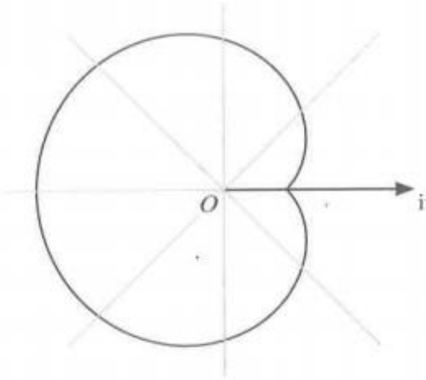
Comparing the imaginary parts of (1) and (2): $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

Therefore, $\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$ (dividing throughout by $\cos^4\theta$)

2 The diagram shows part of the kidney-shaped curve known as *Freeth's nephroid*. Its polar equation is

$$r = 1 + 2\sin\tfrac{1}{2}\theta, \quad 0 \leqslant \theta < 2\pi.$$



(i) Write down, in simplest exact form, the polar coordinates (r, θ) of the point where the curve meets the initial line, and also the point on the curve where it meets the half-line $\theta = \frac{3}{2}\pi$. [2]

The total length of this curve is denoted by L.

(ii) Find, in terms of θ, an integral which gives L and evaluate L to 3 decimal places. [4]

[Solution]

(i) When $\theta = 0$, $r = 1 + 2\sin 0 = 1$.

The point the curve meets the initial line is (1, 0)

When $\theta = \frac{3\pi}{2}$, $r = 1 + 2\sin\frac{3\pi}{4} = 1 + \sqrt{2}$. The point is $(1 + \sqrt{2}, \frac{3\pi}{2})$

(ii) $r = 1 + 2\sin\frac{\theta}{2}$

$$\frac{dr}{d\theta} = \cos\frac{\theta}{2}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 + 2\sin\frac{\theta}{2})^2 + \cos^2\frac{\theta}{2}$$ [This is not the simplest yet !]

$$= 1 + 4\sin\frac{\theta}{2} + 4\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}$$

$$= 2 + 4\sin\frac{\theta}{2} + 3\sin^2\frac{\theta}{2}$$

Thus L = $2\int_0^{\pi}\sqrt{(2 + 4\sin\frac{\theta}{2} + 3\sin^2\frac{\theta}{2})}\, d\theta$

Using a GC, L = 15.13490477 ≈ 15.135 (correct to 3 dps)

3 The sequence $\{X_n\}$ is given by $X_1 = 2$, $X_2 = 7$ and

$$X_n = 2nX_{n-1} - n(n-1)X_{n-2} \quad \text{for } n \geqslant 3.$$

By dividing the recurrence relation throughout by $n!$, use a suitable substitution to determine X_n as a function of n. [7]

[Solution]

Given $X_n = 2nX_{n-1} - n(n-1)X_{n-2}$, $n \geq 3$

$\Rightarrow \dfrac{X_n}{n!} = \dfrac{2nX_{n-1}}{n!} - \dfrac{n(n-1)X_{n-2}}{n!}$

$\Rightarrow \dfrac{X_n}{n!} = \dfrac{2X_{n-1}}{(n-1)!} - \dfrac{X_{n-2}}{(n-2)!}$

Let $U_n = \dfrac{X_n}{n!}$. Thus $U_1 = \dfrac{X_1}{1} = 2$ and $U_2 = \dfrac{X_2}{2!} = \dfrac{7}{2}$

Recurrence relation becomes: $U_n = 2U_{n-1} - U_{n-2}$ ---- (*)

The characteristic equation is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$

The general solution for (*) is: $U_n = A + Bn$

$n = 1$, $2 = A + B$ ----- (1)

$n = 2$, $\dfrac{7}{2} = A + 2B \Rightarrow 7 = 2A + 4B$ ------- (2)

(2) - 2(1): $3 = 2B \Rightarrow B = \dfrac{3}{2}$ and thus $A = \dfrac{1}{2}$

Thus $U_n = \dfrac{1}{2} + \dfrac{3}{2}n \Rightarrow \dfrac{X_n}{n!} = \dfrac{1}{2} + \dfrac{3}{2}n \Rightarrow X_n = \dfrac{n!}{2}(3n+1)$, $n = 1, 2, \ldots$

4 (i) Show that the equation $f(x) = 0$, where $f(x) = x^3 - 9x - 14$, has a root α in the interval $[3, 4]$. [1]

(ii) In order to find an approximation, β, to α, one stage of the linear interpolation process is used on the interval $[3, 4]$. State, with a brief justification, the value of β that will be obtained. [2]

(iii) (a) Show how consideration of $f''(x)$ in the interval $[3, 4]$ enables you to determine whether β is an under-estimate or an over-estimate of α. [3]

(b) Use a second stage of the linear interpolation process to find a second approximation, γ, to α, giving your answer to 3 significant figures. [2]

[Solution]

(i) $f(3) = 3^3 - 9(3) - 14 = -14 < 0$

$f(4) = 4^3 - 9(4) - 14 = 14 > 0$

There is a **change of sign in** the interval [3, 4] and f is a **continuous function**. So there is a root (or at least one) in the interval [3, 4]

(ii) Linear interpolation is based on drawing a chord joining the points (3, −14) and (4, 14) and the intersection between the chord and the x-axis. So an approximation must be in (3, 4). So we can take $\beta = 3.5$ as an approximation of α.

(iii)(a) $f'(x) = 3x^2 - 9$

$f''(x) = 6x$

For $x \in (3, 4)$, $f'(x) > 0$ and $f''(x) > 0$.

So, f is increasing and concave upwards in (3, 4). Thus β is an under-estimate of α

(b) So $\beta < \alpha < 4$

2nd stage of linear interpolation: $$\beta_2 = \frac{3.5|f(4)| + 4|f(3.5)|}{|f(4)| + |f(3.5)|}$$

$$= \frac{59.5}{16.625} = 3.57894 \approx 3.58 \text{ (3 sf)}$$

5 (i) The points O, P_1, P_2 and P_3 in the complex plane represent the complex numbers $z_0 = 0 + 0\text{i}$, $z_1 = 3 + \text{i}$, $z_2 = 2 + \text{i}$ and $z_3 = z_1 z_2$ respectively.

On a single Argand diagram, draw the line segments OP_1, OP_2 and OP_3 and deduce the result

$$\tfrac{1}{4}\pi = \tan^{-1}\left(\tfrac{1}{3}\right) + \tan^{-1}\left(\tfrac{1}{2}\right).$$ [3]

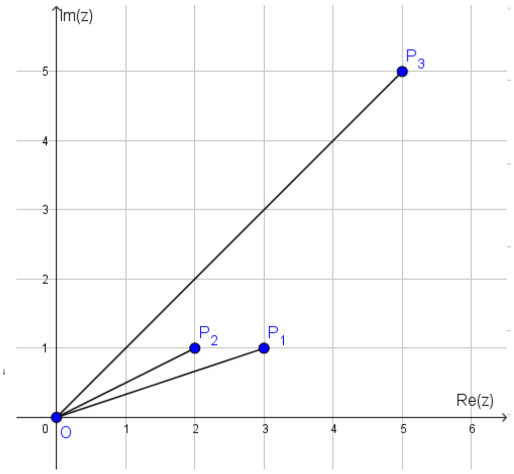
(ii) Let n and k be positive integers. Prove that $\tan^{-1}\left(\frac{1}{n}\right) = \tan^{-1}\left(\frac{1}{n+k}\right) + \tan^{-1}\left(\frac{k}{n^2+nk+1}\right)$. [3]

(iii) Show that $\quad w_1 = n + k + i \qquad w_2 = (n^2 + nk + 1) + ki$

$$\tfrac{1}{4}\pi = 2\tan^{-1}\left(\tfrac{1}{3}\right) + \tan^{-1}\left(\tfrac{1}{12}\right) + \tan^{-1}\left(\tfrac{1}{32}\right) + \tan^{-1}\left(\tfrac{1}{46}\right) + \tan^{-1}\left(\tfrac{1}{173}\right).$$ [3]

[Possible Solution]

(i) $z_3 = z_1 z_2 = (3 + i)(2 + i) = 5 + 5i$

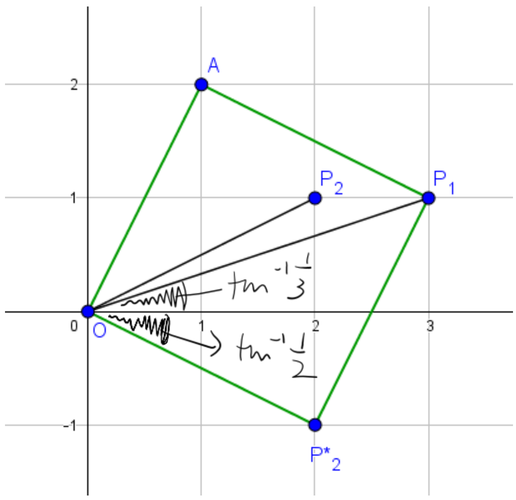


Note: The word 'deduce" forces us to use the diagram to obtain $\dfrac{\pi}{4} = \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3}$. If not arg z_3 = arg z_1 + arg z_2 will do !

Consider the $(z_2)^* = 2 - i$ and $1 + 2i$

In the diagram, $OP^*_2 P_1 A$ is a square with diagonal OP_1

$\angle P_1OP^*_2 = \dfrac{\pi}{4}$ and the angle between OP_2^* and the positive x-axis is $\alpha = \tan^{-1}\dfrac{1}{2}$



The angle between OP_1 and the positive x-axis is $\beta = \tan^{-1}\left(\frac{1}{3}\right)$.

Thus $\frac{\pi}{4} = \alpha + \beta = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$

(ii)

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$x = \tan A \Rightarrow A = \tan^{-1} x$$
$$y = \tan B \Rightarrow B = \tan^{-1} y$$
$$A + B = \tan^{-1}\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right)$$
$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

It is known that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Similarly, $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Thus $\tan^{-1}\left(\frac{1}{n+k}\right) + \tan^{-1}\left(\frac{k}{n^2+nk+1}\right) = \tan^{-1}\left(\frac{\frac{1}{n+k} + \frac{k}{n(n+k)+1}}{1 - \frac{1}{n+k}\left(\frac{k}{n(n+k)+1}\right)}\right)$

$$= \tan^{-1}\left(\frac{n^2+nk+1+(n+k)k}{(n+k)(n^2+nk+1)-k}\right) = \tan^{-1}\left(\frac{n^2+2nk+k^2+1}{n(n+k)^2+n+k-k}\right)$$

$$= \tan^{-1}\left(\frac{(n+k)^2+1}{n((n+k)^2+1)}\right) = \tan^{-1}\frac{1}{n}$$

Alternatively use $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ will be less tedious

Consider $\tan^{-1}\frac{1}{n} - \tan^{-1}\left(\frac{1}{n+k}\right) = \tan^{-1}\left(\frac{\frac{1}{n} - \frac{1}{n+k}}{1 + \frac{1}{n}\left(\frac{1}{n+k}\right)}\right)$

$$= \tan^{-1}\left(\frac{\frac{n+k-n}{n(n+k)}}{\frac{n(n+k)+1}{n(n+k)}}\right) = \tan^{-1}\left(\frac{k}{n^2+nk+1}\right) \text{ (shown)}$$

(ii) Using (ii), $\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2+1} + \tan^{-1}\frac{1}{2^2+2+1} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$

$$\tan^{-1}\frac{1}{7} = \tan^{-1}\frac{1}{7+5} + \tan^{-1}\frac{5}{7^2+35+1} = \tan^{-1}\frac{1}{12} + \tan^{-1}\frac{1}{17}$$

$$\tan^{-1}\frac{1}{17} = \tan^{-1}\frac{1}{17+15} + \tan^{-1}\frac{15}{17^2+17(15)+1} = \tan^{-1}\frac{1}{32} + \tan^{-1}\frac{3}{109}$$

and $\tan^{-1}\frac{1}{46} + \tan^{-1}\frac{1}{173} = \tan^{-1}\left(\frac{\frac{1}{46}+\frac{1}{173}}{1-\frac{1}{46}\frac{1}{173}}\right) = \tan^{-1}\left(\frac{219}{7957}\right) = \tan^{-1}(\frac{3}{109})$

Thus $\frac{\pi}{4} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{12} + \tan^{-1}\frac{1}{32} + \tan^{-1}\frac{3}{109}$

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{12} + \tan^{-1}\frac{1}{32} + \tan^{-1}\frac{1}{46} + \tan^{-1}\frac{1}{173}$$

6 The sequence $\{E_n\}$ is defined for $n \geqslant 1$ by the first-order recurrence relation

$$E_n = 5(E_{n-1})^3 - 3(E_{n-1})$$

together with the initial term $E_0 = 1$.

(i) Calculate E_1, E_2 and E_3. [1]

The sequence of Fibonacci numbers, $\{F_n\}$, is defined for $n \geqslant 1$ by the second-order recurrence relation

$$F_{n+1} = F_n + F_{n-1}$$

together with the initial terms $F_0 = 0$ and $F_1 = 1$.

(ii) Use *Binet's formula*, $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, where $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$, to calculate F_9 and F_{27}. [1]

(iii) Conjecture and prove, for $n \geqslant 0$, an expression for E_n in terms of the Fibonacci numbers. [7]

[Solution]

Given $E_n = 5(E_{n-1})^3 - 3E_{n-1}$ and $E_0 = 1$

(i) $E_1 = 5 - 3 = 2$

$E_2 = 5(2^3) - 3(2) = 34$

$E_3 = 5(34^3) - 3(34) = 196418$

(ii) $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ were $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$

Using a GC, $F_9 = \frac{1}{\sqrt{5}}(\alpha^9 - \beta^9) = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^9 - (\frac{1-\sqrt{5}}{2})^9 \right) = 34$

$$F_{27} = \frac{1}{\sqrt{5}}(\alpha^{27} - \beta^{27}) = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^{27} - (\frac{1-\sqrt{5}}{2})^{27} \right) = 196418$$

(iii) Note that $E_0 = 1 = F_1$

$E_1 = 2 = F_3$

$E_2 = 34 = F_9 = F_{3^2}$

$E_3 = 196418 = F_{27} = F_{3^3}$

Conjecture: $E_n = F_{3^n}$ for $n = 0, 1, 2, 3, \ldots$

Let P(n) be the statement $E_n = F_{3^n}$ for $n = 0, 1, 2, 3, \ldots$ where $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$

P(0), P(1), P(2) and P(3) are true. Assume P(k) is true for some positive integer k, that is $E_k = F_{3^k} = \frac{1}{\sqrt{5}}(\alpha^{3^k} - \beta^{3^k})$.

$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2} \Rightarrow \alpha\beta = \frac{1}{4}(1-5) = -1|$

To prove P(k+1) is true that is $E_{k+1} = F_{3^{k+1}} = \frac{1}{\sqrt{5}}(\alpha^{3^{k+1}} - \beta^{3^{k+1}})$

LHS = $E_{k+1} = 5(E_k)^3 - 3E_k$

$= E_k(5(E_k)^2 - 3) = F_{3^k}[5(F_{3^k})^2 - 3]$

$$= \frac{1}{\sqrt{5}}(\alpha^{3^k} - \beta^{3^k})\left[5(\frac{\alpha^{3^k} - \beta^{3^k}}{\sqrt{5}})^2 - 3\right]$$

$$= \frac{1}{\sqrt{5}}(\alpha^{3^k} - \beta^{3^k})\left[\alpha^{2(3^k)} + \beta^{2(3^k)} - 2(\alpha^{3^k}\beta^{3^k}) - 3\right]$$

$$= \frac{1}{\sqrt{5}}(\alpha^{3^k} - \beta^{3^k})\left[\alpha^{2(3^k)} + \beta^{2(3^k)} - 2(\alpha\beta)^{3^k} - 3\right]$$

$$= \frac{1}{\sqrt{5}}(\alpha^{3^k} - \beta^{3^k})\left[\alpha^{2(3^k)} + \beta^{2(3^k)} - 1\right] \quad \text{as } (\alpha\beta)^{3^k} = (-1)^{3^k} = -1$$

$$= \frac{1}{\sqrt{5}}(\alpha^{3(3^k)} + \alpha^{3^k}\beta^{2(3^k)} - \alpha^{3^k} - \beta^{3^k}\alpha^{2(3^k)} - \beta^{3(3^k)} + \beta^{3^k})$$

$$= \frac{1}{\sqrt{5}}(\alpha^{3^{k+1}} - \beta^{3^{k+1}} + \alpha^{3^k}\beta^{3^k}\beta^{3^k} - \alpha^{3^k} - \beta^{3^k}\alpha^{3^k}\alpha^{3^k} + \beta^{3^k})$$

$$= \frac{1}{\sqrt{5}}(\alpha^{3^{k+1}} - \beta^{3^{k+1}} - \beta^{3^k} - \alpha^{3^k} + \alpha^{3^k} + \beta^{3^k})$$

$$= \frac{1}{\sqrt{5}}(\alpha^{3^{k+1}} - \beta^{3^{k+1}}) = F_{3^{k+1}}$$

P(k) is true $\Rightarrow$ P(k+ 1) is true. Thus by induction ….

7 Use the substitution $u = y\cos x$ to find the general solution of the differential equation

$$\frac{d^2y}{dx^2}\cos x - 2\frac{dy}{dx}\sin x + ay\cos x = 0$$

in each of the cases

- $a = -1$,
- $a = 3$,

giving each answer for y in the form $y = \mathrm{f}(x)$. [11]

$u = y\cos x$ ----- (1)

$$\frac{du}{dx} = \frac{dy}{dx}\cos x - y\sin x$$

$$\frac{d^2u}{dx^2} = \frac{d^2y}{dx^2}\cos x - \frac{dy}{dx}\sin x - \frac{dy}{dx}\sin x - y\cos x$$

$$= \frac{d^2y}{dx^2}\cos x - 2\frac{dy}{dx}\sin x - y\cos x \text{ ----- (2)}$$

Substitute eqn (1) and (2) into DE:

$$\frac{d^2y}{dx^2}\cos x - 2\frac{dy}{dx}\sin x + ay\cos x = 0 \text{ becomes}$$

$$\left(\frac{d^2y}{dx^2}\cos x - 2\frac{dy}{dx}\sin x - y\cos x\right) + y\cos x + ay\cos x = 0$$

$$\frac{d^2u}{dx^2} + (a+1)u = 0$$

When $a = -1$,

$$\frac{d^2u}{dx^2} = 0$$

$$\frac{du}{dx} = A_1$$

$$u = \int A_1\, dx = A_1x + A_2$$

$$y\cos x = A_1x + A_2$$

$$\therefore y = (A_1x + A_2)\sec x$$

When $a = 3$,

$$\frac{d^2u}{dx^2} + 4u = 0$$

Auxiliary equation: $m^2 + 4 = 0 \Rightarrow m = \pm 2\mathrm{i}$

General Solution: $u = C_1\cos 2x + C_2\sin 2x$

$$y\cos x = C_1\cos 2x + C_2\sin 2x$$

$$\therefore y = (C_1\cos 2x + C_2\sin 2x)\sec x$$

8 The size of the population of ground-foraging rodents on a nature reserve is denoted by N. At a given point in time $t = 0$, the number of these rodents is N_0. When left unchecked, the rate of growth of the population is proportional to N and, under these circumstances, it is known that the size of the population will double every year.

(i) Use this information, determine the exact value of the constant of proportionality a for which

$$\frac{dN}{dt} = aN,$$

where time t is measured in years. [3]

However, because of the destructive nature of these rodents' eating habits, the population is carefully controlled. This process introduces a negative term into the growth rate equation that is proportional to N^2. In the long-term, it is intended to allow the population to stabilise at $N = 1000$.

(ii) Write down a revised differential equation for N and evaluate the second constant of proportionality involved in it. [3]

(iii) Given that $N_0 = 50$, determine N as a function of t. [6]

(iv) Find, in exact form, the time predicted by this model for the population of rodents to reach 950 and comment on the validity of any modelling assumptions used. [3]

(i)	$\frac{dN}{dt} = aN$ $\frac{1}{N}\frac{dN}{dt} = a$ $\int \frac{1}{N}\,dN = \int a\,dt$ $\ln N = at + C_1$ (since $N > 0$) When $t = 0$, $N = N_0$ $\ln N_0 = C_1$ When $t = 1$, $N = 2N_0$ $\ln 2N_0 = a + \ln N_0 \Rightarrow a = \ln 2$ What is the interpretation of a here? a refers to the growth rate of the rodents.
(ii)	Revised DE: $\frac{dN}{dt} = aN - bN^2 = a\left(1 - \frac{b}{a}N\right)N$ Now a is known as the intrinsic growth rate, that is the growth rate in the absence of any limiting factors. To find equilibrium solutions, let $\frac{dN}{dt} = 0$, $aN - bN^2 = 0$ $N(a - bN) = 0$

	$N=0$ or $N=\dfrac{a}{b}$ Since population stabilise to N = 1000 in the long term, $\dfrac{a}{b}=1000 \Rightarrow b=\dfrac{a}{1000}=\dfrac{\ln 2}{1000}$
(iii)	$\dfrac{dN}{dt}=\dfrac{\ln 2}{1000}N(1000-N)$ $\dfrac{1}{N(1000-N)}\dfrac{dN}{dt}=\dfrac{\ln 2}{1000}$ $\int \dfrac{1}{N(1000-N)}dN=\int \dfrac{\ln 2}{1000}dt$ $\dfrac{1}{1000}\int \dfrac{1}{N}+\dfrac{1}{(1000-N)}dN=\int \dfrac{\ln 2}{1000}dt$ $\dfrac{1}{1000}\left[\ln\lvert N\rvert-\ln\lvert 1000-N\rvert\right]=\dfrac{\ln 2}{1000}t+C_2$ $\ln\left\lvert\dfrac{N}{1000-N}\right\rvert=t\ln 2+1000C_2$ $\dfrac{N}{1000-N}=\pm e^{t\ln 2+1000C_2}$ $\dfrac{N}{1000-N}=Ae^{t\ln 2}$, where $A=e^{1000C_2}$ When t = 0, N = 50, $\dfrac{50}{1000-50}=A \Rightarrow A=\dfrac{1}{19}$ $\dfrac{N}{1000-N}=\dfrac{1}{19}e^{t\ln 2}$ $\dfrac{1000-N}{N}=19e^{-t\ln 2}$ $\dfrac{1000}{N}=1+19e^{-t\ln 2}$ $N=\dfrac{1000}{1+19e^{-t\ln 2}}$ $=\dfrac{1000}{1+19\left(e^{-\ln 2}\right)^t}$ $=\dfrac{1000}{1+19\left(2^{-t}\right)}$ or $\dfrac{1000\left(2^t\right)}{2^t+19}$
(iv)	When N = 950, $950=\dfrac{1000\left(2^t\right)}{2^t+19}$ $950\left(2^t\right)+18050=1000\left(2^t\right)$ $2^t=361$

	$t=\log_2 361 = 2\log_2 19$ or $\dfrac{\ln 361}{\ln 2}$ or $\dfrac{\lg 361}{\lg 2}$ The model does not factor in limitations on space, food supply, or other resources which will reduce the growth rate of the rodents.

9 In a multi-stage experimental process, liquids in vats are mixed together. Initially, vat A_0 contains 700 litres of liquid, vat B_0 contains 200 litres of liquid and vat C_0 contains 400 litres of liquid. Thereafter, at stage n of the process, empty vats A_n, B_n and C_n are given liquids transferred from vats A_{n-1}, B_{n-1} and C_{n-1} in the following way:

- vat A_n is given one-third of the contents of each of the vats A_{n-1}, B_{n-1} and C_{n-1};
- vat B_n is given one-third of the original contents of vat B_{n-1} along with the remaining contents of vat C_{n-1};
- vat C_n is given one-half of the original contents of vat A_{n-1} along with liquid equivalent, in volume and composition, to one-half of the original contents of vat B_{n-1}.

(i) By modelling the process in matrix form, determine in exact form the volume of liquid that is in

(a) vats A_1, B_1 and C_1 after 1 stage of the process,

(b) vats A_2, B_2 and C_2 after 2 stages of the process. [6]

(ii) The process is left to run indefinitely. Describe the long-term results. [2]

(iii) **(a)** Now suppose that the matrix is used with each element rounded to 1 significant figure. Show that the matrix with rounded elements predicts a very different outcome to the original. [2]

(b) By calculating the eigenvalues of the two matrices, explain why the outcomes are so different. [5]

(iv) You are now given that the liquids in vats A_0, B_0 and C_0 at the start of the process are of different types. Show how to modify the modelling of the process so that the output indicates the amount of each type of liquid that is present in each vat. [2]

[Solution]

Given $A_0 = 700$, $B_0 = 200$ and $C_0 = 400$

(i)(a) $A_n = \frac{1}{3}A_{n-1} + \frac{1}{3}B_{n-1} + \frac{1}{3}C_{n-1}$

$B_n = \frac{1}{3}B_{n-1} + \frac{2}{3}C_{n-1}$

$C_n = \frac{1}{2}A_{n-1} + \frac{1}{2}B_{n-1}$

Let $\mathbf{X} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

The above equation can be written as $\begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \end{pmatrix}$

After one stage $\begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 700 \\ 200 \\ 400 \end{pmatrix} = \begin{pmatrix} \frac{1300}{3} \\ \frac{1000}{3} \\ 450 \end{pmatrix}$

(b) $\begin{pmatrix} A_2 \\ B_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1300}{3} \\ \frac{1000}{3} \\ 450 \end{pmatrix} = \begin{pmatrix} \frac{3650}{9} \\ \frac{3700}{9} \\ \frac{2300}{6} \end{pmatrix}$

(ii) Using a GC, consider after stage 20, $\begin{pmatrix} A_{20} \\ B_{20} \\ C_{20} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^{20} \begin{pmatrix} 700 \\ 200 \\ 400 \end{pmatrix} = \begin{pmatrix} 400.000 \\ 400.000 \\ 400.000 \end{pmatrix}$

$$\begin{pmatrix} A_{50} \\ B_{50} \\ C_{50} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^{50} \begin{pmatrix} 700 \\ 200 \\ 400 \end{pmatrix} = \begin{pmatrix} 400.000 \\ 400.000 \\ 400.000 \end{pmatrix}$$

In the long run, each vat tends to 400 litres.

(iii)(a) Suppose $\mathbf{M} = \begin{pmatrix} 0.3 & 0.3 & 0.3 \\ 0 & 0.3 & 0.7 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ (each element is rounded to 1 sf)

$$\begin{pmatrix} A_{20} \\ B_{20} \\ C_{20} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 & 0.3 \\ 0 & 0.3 & 0.7 \\ 0.5 & 0.5 & 0 \end{pmatrix}^{20} \begin{pmatrix} 700 \\ 200 \\ 400 \end{pmatrix} = \begin{pmatrix} 216.416 \\ 247.639 \\ 238.311 \end{pmatrix}$$

$$\begin{pmatrix} A_{50} \\ B_{50} \\ C_{50} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 & 0.3 \\ 0 & 0.3 & 0.7 \\ 0.5 & 0.5 & 0 \end{pmatrix}^{50} \begin{pmatrix} 700 \\ 200 \\ 400 \end{pmatrix} = \begin{pmatrix} 97.084 \\ 111.090 \\ 106.906 \end{pmatrix}$$

In the case, in the long run the amount in each vat is a lot lesser than the original and they are not equal and it seems that they take a longer time to stablise.

(b) $\mathbf{X} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.

Consider det($\mathbf{X} - \lambda\mathbf{I}$) = $\mathbf{0}$ ⇒ $\det\begin{pmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3}-\lambda & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & -\lambda \end{pmatrix} = 0$

$\left(\frac{1}{3}-\lambda\right)\left[\left(\frac{1}{3}-\lambda\right)(-\lambda)-\frac{1}{3}\right]+\frac{1}{2}\left[\frac{2}{9}-\frac{1}{3}\left(\frac{1}{3}-\lambda\right)\right]=0$ ---- this is tedious

Should use GC : Key the matrices X and I in GC

Sketch the graph of y = det($\mathbf{X} - \lambda\mathbf{I}$)

From GC, eigenvalues for $\mathbf{X}$ are $\lambda = -0.455$, 0.122 and 1

det ($\mathbf{M} - \lambda\mathbf{I}$) = 0 ⇒ $\det\begin{pmatrix} 0.3-\lambda & 0.3 & 0.3 \\ 0 & 0.3-\lambda & 0.7 \\ \frac{1}{2} & \frac{1}{2} & -\lambda \end{pmatrix} = 0$

Use a GC, eigenvalues for **M** (the rounded matrix) are : –0.472, 0.098, 0.974

The outcomes in **(iii)(a)** is very different from **(ii)** since the eigenvalues in **(iii)(a)** are all having absolute value between 0 and 1 but in the case of **(ii)**, there is an eigenvalue of 1. Thus the long term effect of **(iii)(a)** will be stabilising at 0. [WHY?]

(iii) The liquids in initial vats are all of different types, to modify modelling process so that output indicates amount of each type of liquid present in each vat, responses must indicate that the amounts in each vat should be treated separately from the outset (often done in matrix form).

For example:

Output to indicate amount of each type of liquid in each vat

eg A_{Bn} = amount of liquid from vat B at stage n:

$$\begin{pmatrix} A_{An} \\ A_{Bn} \\ A_{Cn} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} B_{An} \\ B_{Bn} \\ B_{Cn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \end{pmatrix}$$ [or use a 9 x 1 matrix and a 9 x 3 matrix]

$$\begin{pmatrix} C_{An} \\ C_{Bn} \\ C_{Cn} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \end{pmatrix}$$

Or:

For type A liquid: $\begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \end{pmatrix}$

For type B liquid: $\begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \end{pmatrix}$

For type C liquid: $\begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \end{pmatrix}$

10 Equations can have repeated roots. So, for example, the equation $(x+1)(x-3)^2 = 0$ is said to have three roots: $x = -1$ (of multiplicity 1) and $x = 3$ (a repeated root of multiplicity 2).

A student is trying to solve the equation $f(x) = 0$, where $f(x) = 3x^5 - 20x^3 + 60x - 32\sqrt{2}$.

From graphical work, the student knows that there is a positive root, R, near $x = 1$.

(i) The Newton-Raphson iterative formula can be written as $x_{n+1} = x_n - g(x_n)$, where $g(x) = \frac{f(x)}{f'(x)}$.

Write down $g(x)$ in this case. [1]

(ii) (a) The Newton-Raphson method is to be used to obtain a sequence of iterates, starting with $x_1 = 1$. Copy the table, and complete the second column of your copy in the answer booklet, giving each value of x_n correct to 3 decimal places. (Leave the third column blank for later use.) [3]

n	x_n	
1	1	
2		
3		
4		
5		
6		
7		
8		

(b) Continue this process until, correct to 3 decimal places, $x_n = x_{n+1}$. Write down the least value of n for which this is so and state the corresponding value of x_n. [2]

[Solution]

(i) $f(x) = 3x^5 - 20x^3 + 60x - 32\sqrt{2}$

$f'(x) = 15x^4 - 60x^2 + 60$

$$g(x) = \frac{f(x)}{f'(x)} = \frac{3x^5 - 20x^3 + 60x - 32\sqrt{2}}{15x^4 - 60x^2 + 60}$$

(ii) By NR-method

$$x_{n+1} = x_n - \frac{3x_n^5 - 20x_n^3 + 60x_n - 32\sqrt{2}}{15x_n^4 - 60x_n^2 + 60} = \frac{15x_n^5 - 60x_n^3 + 60x_n - (3x_n^5 - 20x_n^3 + 60x_n - 32\sqrt{2})}{15x_n^4 - 60x_n^2 + 60}$$

$$= \frac{12x_n^5 - 40x_n^3 + 32\sqrt{2}}{15x_n^4 - 60x_n^2 + 60}$$

$x_1 = 1$. Using a GC to obtain subsequent approximations.

n	x_n	
1	1	
2	1.150322 ≈ 1.150	
3	1.242905 ≈1.243	
4	1.301872 ≈1.302	

5	1.3401011 ≈1.340	
6	1.3651388≈1.365	
7	1.381642 ≈ 1.382	
8	1.392563 ≈1.392	
9	1.399807≈ 1.400	
10	1.404621 ≈ 1.405	
11	1.407824 ≈ 1.408	
12	1.409956 ≈1.410	
13	1.4113767 ≈ 1.411	
14	1.412322 ≈ 1.412	
15	1.4129533 ≈ 1.413	
16	1.413373 ≈ 1.413	

(b) Rounding to 3 dps, $x_{15} = x_{16} = 1.413$

(iii) Verify, by substitution, that $R = \sqrt{2}$. Supporting working must be shown. [1]

(iv) Calculate $|x_n - R|$ for $n = 1$ to 8 and enter these values, to 3 decimal places, in the third column of your table in the answer booklet. [1]

(iii) $f(x) = 3x^5 - 20x^3 + 60x - 32\sqrt{2} \Rightarrow f(\sqrt{2}) = 3(4)\sqrt{2} - 40\sqrt{2} + 60\sqrt{2} - 32\sqrt{2} = 0$

Thus $R = \sqrt{2}$

(iv)

n	x_n	$\lvert x_n - R\rvert$
1	1	0.414
2	1.150322 ≈ 1.150	0.264
3	1.242905 ≈1.243	0.171
4	1.301872 ≈1.302	0.112
5	1.3401011 ≈1.340	0.074
6	1.3651388≈1.365	0.049
7	1.381642 ≈ 1.382	0.033
8	1.392563 ≈1.392	0.022

The rate of convergence of an iterative process which converges to R is defined as C whenever there exist positive constants C and K for which $\dfrac{|x_{n+1}-R|}{|x_n-R|^C} \approx K$ for all x_n 'sufficiently close' to R.

(v) Use your answers to part **(iv)** to show that the rate of convergence of the above application of the Newton-Raphson method to the equation $f(x)=0$ is linear (i.e. $C=1$) and state a suitable value for K. [2]

The rate of convergence of the Newton-Raphson method is quadratic (i.e. $C=2$). It is also known that this is not true in the case of repeated roots. One way to restore the quadratic rate of convergence is to use the revised iterative formula

$$x_{n+1}=x_n-mg(x_n),$$

where m is the multiplicity of the root R. In general, the larger the value of C, the greater the rate of convergence and the fewer the iterations it takes to converge.

(vi) Using $x_1=1$ as an initial approximation, calculate the revised iterates obtained by using $m=2$, $m=3$ and $m=4$ in turn. In each case, continue this process until, correct to 3 decimal places, $x_n=x_{n+1}$. Explain why these revised iterates suggest that $x=\sqrt{2}$ is a repeated root of $f(x)=0$ of multiplicity 3. [3]

(v)

n	x_n	$\|x_n-R\|$	$\dfrac{\|x_{n+1}-R\|}{\|x_n-R\|}$
1	1	0.414	
2	1.150322 ≈ 1.150	0.264	
3	1.242905 ≈1.243	0.171	
4	1.301872 ≈1.302	0.112	
5	1.3401011 ≈1.340	0.074	0.662
6	1.3651388≈1.365	0.049	0.673
7	1.381642 ≈ 1.382	0.033	0.667
8	1.392563 ≈1.392	0.022	

Letting C = 1, $\dfrac{|x_{n+1}-R|}{|x_n-R|} \approx 0.67$ from n = 5 onwards, thus $k \approx 0.67$.

(vi) Using $x_{n+1}=x_n-mg(x_n)$ where $g(x)=\dfrac{f(x)}{f'(x)}=\dfrac{3x^5-20x^3+60x-32\sqrt{2}}{15x^4-60x^2+60}$

$x_1 = 1$

If $m = 2$, using GC: $\{x_n\}$ is {1, 1.301, 1.378, 1.402, 1.410, 1.413, 1.414, 1.414, …}

If $m = 3$, $\{x_n\}$ = {1..451. 1.414, 1.414, …}

If $m = 4$, $\{x_n\}$ = {1, 1.601, 1.359, 1.433, 1.408, 1.416, 1.414, 1.414, …}

When $m = 3$, it takes lesser iterative step to converge to $\sqrt{2} \approx 1.414$, so it suggests that $\sqrt{2}$ is a root of multiplicity 3.