1(a) Let x, y and z denote the number of wins, draws and losses by Lucy's favorite team this season respectively.

$x + y + z = 38$ ...Eq(1)

$3x + y + 0z = 54$ ...Eq(2)

$x + y - 2z = 8$ ...Eq(3)

From GC, $x = 13$, $y = 15$ and $z = 10$

$\therefore$ Lucy's favorite team won 13 games this season.

(b) Points scored by Mark's favorite team $= 3(13 - 2) + (15 + 5)$

$= 53 < 54$

Hence Lucy's favorite team performed better this season.

<u>Alternatively</u>,

Since Mark's favorite team won 2 games fewer and drew 5 games more,

this team scored $-3(2) + 5\ (1) = -1$ point more.

Hence Lucy's favorite team performed better this season.

2(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x)$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right)$$
$$= \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x \text{ (Shown)}$$

(b)

$$\sin 2x \cot x$$
$$= 2\sin x \cos x\left(\frac{\cos x}{\sin x}\right)$$
$$= 2\cos^2 x \text{ (Shown)}$$

(c)

$$\int_{\frac{\pi}{30}}^{\frac{\pi}{15}} \operatorname{cosec}(10x)\tan(5x)\ \mathrm{d}x$$

$$= \int_{\frac{\pi}{30}}^{\frac{\pi}{15}} \frac{1}{\sin(10x)\cot(5x)}\ \mathrm{d}x$$

$$= \int_{\frac{\pi}{30}}^{\frac{\pi}{15}} \frac{1}{2\cos^2 5x}\ \mathrm{d}x$$

$$= \frac{1}{2}\int_{\frac{\pi}{30}}^{\frac{\pi}{15}} \sec^2 5x\ \mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{1}{5}\tan 5x\right]_{\frac{\pi}{30}}^{\frac{\pi}{15}}$$

$$= \frac{1}{10}\left[\tan\frac{\pi}{3} - \tan\frac{\pi}{6}\right]$$

$$= \frac{1}{10}\left[\sqrt{3} - \frac{1}{\sqrt{3}}\right]$$

$$= \frac{1}{10}\left[\frac{2}{\sqrt{3}}\right] = \frac{\sqrt{3}}{15}$$

3(a)

$$\sin\theta = \frac{d}{PQ}$$

$$PQ = d\ \operatorname{cosec}\theta$$

$$PQ + QR + RS = 2a$$

$$QR = 2a - 2(d\operatorname{cosec}\theta)$$

$$QR = 2(a - d\operatorname{cosec}\theta)$$

$$\text{Area} = \frac{1}{2}d(QR + PS)$$

$$= \frac{1}{2}d\left[2(a - d\operatorname{cosec}\theta) + 2d\cot\theta + 2(a - d\operatorname{cosec}\theta)\right]$$

$$= \frac{1}{2}d\left[4a - 4(d\operatorname{cosec}\theta) + 2d\cot\theta\right]$$

$$= 2ad + d^2(\cot\theta - 2\operatorname{cosec}\theta)$$

(b)

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = -d^2\operatorname{cosec}^2\theta + 2d^2\operatorname{cosec}\theta\cot\theta$$

When $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$, $-d^2\operatorname{cosec}^2\theta + 2d^2\operatorname{cosec}\theta\cot\theta = 0$

$$\operatorname{cosec}\theta\left(\operatorname{cosec}\theta - 2\cot\theta\right) = 0$$
$$\operatorname{cosec}\theta = 2\cot\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}$$

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = -d^2\operatorname{cosec}^2\theta + 2d^2\operatorname{cosec}\theta\cot\theta$$

θ	1.046	$\frac{\pi}{3} = 1.0472$	1.048
$\frac{\mathrm{d}A}{\mathrm{d}\theta}$	$0.002769\,d^2$	0	$-0.0018519\,d^2$
Slope	/	–	\

Using the first derivative test, $\theta = \frac{\pi}{3}$ gives a maximum value for A.

<u>Alternative method</u>

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = -d^2\operatorname{cosec}^2\theta + 2d^2\operatorname{cosec}\theta\cot\theta$$
$$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = 2d^2\operatorname{cosec}^2\theta\cot\theta - 2d^2\left(\operatorname{cosec}^3\theta + \operatorname{cosec}\theta\cot^2\theta\right)$$

At $\theta = \frac{\pi}{3}$, $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = -2.309d^2 \quad > 0$

$\therefore \theta = \frac{\pi}{3}$ gives a maximum value for A.

At $\theta = \frac{\pi}{3}$, $A = 2ad + d^2\left(\cot\frac{\pi}{3} - 2\operatorname{cosec}\frac{\pi}{3}\right)$

$$= 2ad + d^2\left(\frac{1}{\sqrt{3}} - 2\left(\frac{2}{\sqrt{3}}\right)\right)$$
$$= 2ad - d^2\sqrt{3}$$

4 (a)

$l_1: \frac{x+1}{4} = \frac{2-y}{3} = z$

$\Rightarrow l_1 : \mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$

$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ are position vectors of 2 points on l_1, substituting into p_2,

$-\alpha + 6 = 7$

$3\alpha - 3 + \beta = 7$

$\begin{pmatrix} \alpha \\ 3 \\ \beta \end{pmatrix} . \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} = 0$
$4\alpha + \beta = 9$

Solving, $\alpha = -1, \beta = 13$

(b)

$\mathbf{n}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

p_1 : $\mathbf{r}. \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$

$x + y - z = 0$

$-3x + 3y + 2z = 7$

Solving using GC, line of intersection is $\mathbf{r} = \frac{7}{6} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}, \ \mu \in \mathbb{R}$.

Alternative method

$\mathbf{r}. \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = 7$

Since P1 is on P2, sub P1 into equation of P2

$$\begin{pmatrix} 3t \\ s-t \\ s+2t \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = 7$$

$$-9t + 3s - 3t + 2s + 4t = 7$$

$$-8t + 5s = 7$$

$$s = \frac{7+8t}{5}$$

Sub s into equation P1

$$\mathbf{r} = \frac{7+8t}{5}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\mathbf{r} = \frac{7}{5}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{8t}{5}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\mathbf{r} = \frac{7}{5}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ \frac{3}{5} \\ \frac{18}{5} \end{pmatrix}$$

$$\mathbf{r} = \frac{7}{5}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} \quad \text{where } \mu \in \mathbb{R}$$

(c)

$$\text{Distance} = \left| \left[\begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right] \cdot \frac{\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}}{\sqrt{10}} \right|$$

$$= \left| \begin{pmatrix} 1 \\ -6 \\ 0 \end{pmatrix} \cdot \frac{\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}}{\sqrt{10}} \right|$$

$$= \frac{6}{\sqrt{10}}$$

5 (a)

$$\frac{3}{2r-1}-\frac{4}{2r+1}+\frac{1}{2r+3}$$
$$=\frac{3(2r+1)(2r+3)-4(2r-1)(2r+3)+(2r+1)(2r-1)}{(2r-1)(2r+1)(2r+3)}$$
$$=\frac{12r^2+24r+9-16r^2-16r+12+4r^2-1}{(2r-1)(2r+1)(2r+3)}$$
$$=\frac{8r+20}{(2r-1)(2r+1)(2r+3)}$$

(b)

$$\sum_{r=1}^{n}\frac{2r+5}{(2r-1)(2r+1)(2r+3)}$$
$$=\frac{1}{4}\sum_{r=1}^{n}\frac{8r+20}{(2r-1)(2r+1)(2r+3)}$$
$$=\frac{1}{4}\sum_{r=1}^{n}\left(\frac{3}{2r-1}-\frac{4}{2r+1}+\frac{1}{2r+3}\right)$$
$$=\frac{1}{4}\left[\begin{array}{ccccccc} & \frac{3}{1} & - & \frac{4}{3} & + & \frac{1}{5} \\ + & \frac{3}{3} & - & \frac{4}{5} & + & \frac{1}{7} \\ + & \frac{3}{5} & - & \frac{4}{7} & + & \frac{1}{9} \\ & & & \vdots & & \\ + & \frac{3}{2n-5} & - & \frac{4}{2n-3} & + & \frac{1}{2n-1} \\ + & \frac{3}{2n-3} & - & \frac{4}{2n-1} & + & \frac{1}{2n+1} \\ + & \frac{3}{2n-1} & - & \frac{4}{2n+1} & + & \frac{1}{2n+3} \end{array}\right]$$
$$=\frac{1}{4}\left[3-\frac{4}{3}+1+\frac{1}{2n+1}-\frac{4}{2n+1}+\frac{1}{2n+3}\right]$$
$$=\frac{2}{3}+\frac{1}{4}\left(\frac{1}{2n+3}-\frac{3}{2n+1}\right)$$
$$=\frac{2}{3}-\frac{2n+1-3(2n+3)}{(2n+1)(2n+3)}$$
$$=\frac{2}{3}-\frac{n+2}{(2n+1)(2n+3)}$$

(c)

Sum to infinity $=\frac{2}{3}$

$$\left|\frac{2}{3}-\left(\frac{2}{3}-\frac{k+2}{(2k+1)(2k+3)}\right)\right|<0.004$$

$$\frac{k+2}{(2k+1)(2k+3)}<0.004$$

k	$\frac{k+2}{(2k+1)(2k+3)}$
62	0.0040315> 0.004
63	0.0039675 < 0.004
64	0.0039056 < 0.004

From GC, the smallest value of k is 63

Alternative solution:

$$\frac{k+2}{(2k+1)(2k+3)}<0.004$$

$$k+2<0.004\,(2k+1)(2k+3)$$
$$4k^2-242k-497>0$$

$$k<-1.9884 \quad or \quad k>62.488$$

Since $k\in\mathbb{Z}$ and $k\geq 1$, the smallest value of k is 63

6(a)

$$\int \cot^2 3x \; dx$$
$$= \int \operatorname{cosec}^2 3x - 1 \; dx$$
$$= -\frac{\cot 3x}{3} - x + c$$

(b)

$$\int_2^4 \frac{x^2-4x+3}{x^2-4x+8} \; dx$$
$$= \int_2^4 \frac{x^2-4x+8}{x^2-4x+8} - \frac{5}{x^2-4x+8} \; dx$$
$$= \int_2^4 1 - \frac{5}{(x-2)^2+4} \; dx$$
$$= \left[x - \frac{5}{2}\tan^{-1}(\frac{x-2}{2}) \right]_2^4$$
$$= \left[4 - \frac{5}{2}\tan^{-1}(1) \right] - \left[2 - \frac{5}{2}\tan^{-1}(0) \right]$$
$$= 2 - \frac{5\pi}{8}$$

(c)

$$x = 2\cot\theta$$
$$\frac{dx}{d\theta} = -2\operatorname{cosec}^2\theta$$

When $x = \frac{2\sqrt{3}}{3}$, $2\cot\theta = \frac{2\sqrt{3}}{3}$, $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$

When $x = 2$, $2\cot\theta = 2$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$

$$\int_{\frac{2\sqrt{3}}{3}}^{2} \frac{4-x^2}{(4+x^2)^2} \, dx$$
$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{4-4\cot^2\theta}{(4+4\cot^2\theta)^2}(-2\operatorname{cosec}^2\theta) \; d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{4(1-\cot^2\theta)}{16(1+\cot^2\theta)^2}(2\operatorname{cosec}^2\theta) \; d\theta$$
$$= \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(1-\cot^2\theta)}{(\operatorname{cosec}^2\theta)^2}(2\operatorname{cosec}^2\theta) \; d\theta$$
$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(1-\cot^2\theta)}{(\operatorname{cosec}^2\theta)} \; d\theta$$
$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} -\left(\cos^2\theta - \sin^2\theta\right) \; d\theta$$

$$= -\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos 2\theta \ \mathrm{d}\theta$$

$$= -\frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= -\frac{1}{4}\left[\sin\frac{2\pi}{3} - \sin\frac{\pi}{2}\right]$$

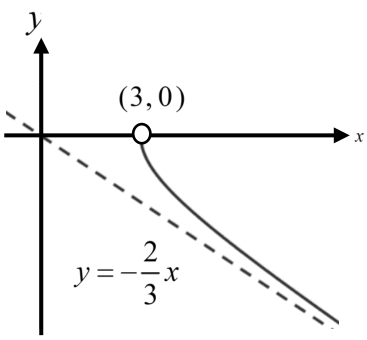
$$= -\frac{1}{4}\left[\frac{\sqrt{3}}{2} - 1\right]$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{8}$$

7(a)(i)

$$\tan^2 x + 1 = \sec^2 x$$
$$\left(-\tan x\right)^2 + 1 = \sec^2 x$$
$$\frac{y^2}{2^2} + 1 = \frac{x^2}{3^2}$$
$$\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1, \ x > 3, y < 0$$

(ii)



(b)

When $x = 1$, $2\sin t + 1 = 1$, $\sin t = 0$, $t = 0$

When $x = 2$, $2\sin t + 1 = 2$, $\sin t = \frac{1}{2}$, $t = \frac{\pi}{6}$

$x = 2\sin t + 1$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 2\cos t$

Required Area

= Area of trapezium – $\int_1^2 y \ \mathrm{d}x$

$$= \frac{1}{2}\left(2+4\right)(1) - \int_0^{\frac{\pi}{6}} (2\cos 3t + 4\sin t)(2\cos t) \ \mathrm{d}t$$

$= 3 - \int_0^{\frac{\pi}{6}} (4\cos 3t \cos t + 8\sin t \cos t)\ \mathrm{d}t$ (shown)

$= 3 - \int_0^{\frac{\pi}{6}} (2\cos 4t + 2\cos 2t + 4\sin 2t)\ \mathrm{d}t$

$$= 3 - \left[\frac{\sin 4t}{2} + \sin 2t - 2\cos 2t\right]_0^{\frac{\pi}{6}}$$

$$= 3 - \left[\frac{\sin\frac{2\pi}{3}}{2} + \sin\frac{\pi}{3} - 2\cos\frac{\pi}{3} - (-2)\right]$$

$$= 3 - \left[\frac{3\sqrt{3}}{4} + 1\right]$$

$$= 2 - \frac{3\sqrt{3}}{4} \text{ units}^2$$

8(a)

$$y = \frac{1}{3 + \sin 2x}$$

$$y(3 + \sin 2x) = 1$$

Differentiate implicitly w.r.t. x:

$$(3 + \sin 2x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + y(2\cos 2x) = 0$$

$$\frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\cos 2x = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2\cos 2x = 0$$

<u>Alternatively</u>

$$y = (3 + \sin 2x)^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -(3 + \sin 2x)^{-2}(2\cos 2x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -y^2(2\cos 2x)$$

Differentiate implicitly w.r.t. x:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y^2 2\sin 2x + 2(2y)\frac{\mathrm{d}y}{\mathrm{d}x}\cos 2x = 0$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4y^2\sin 2x + 4y\frac{\mathrm{d}y}{\mathrm{d}x}\cos 2x = 0$$

Given that $x=0$, $y=\frac{1}{3+0}=\frac{1}{3}$,

$$\frac{dy}{dx}=-2\left(\frac{1}{3}\right)^2\cos 0=-\frac{2}{9},$$

$$\frac{d^2y}{dx^2}=4\left(\frac{1}{3}\right)^2\sin 0-4\left(\frac{1}{3}\right)\left(-\frac{2}{9}\right)\cos 0=\frac{8}{27}$$

$$y=\frac{1}{3}-\frac{2}{9}x+\frac{x^2}{2!}\left(\frac{8}{27}\right)+\ldots\approx\frac{1}{3}-\frac{2}{9}x+\frac{4}{27}x^2$$

(b)

$$y=\frac{1}{3+\sin 2x}$$

$$\approx\frac{1}{3+(2x+\cdots)}$$

$$=(3+2x)^{-1}$$

$$=\left[3\left(1+\frac{2}{3}x\right)\right]^{-1}$$

$$=\frac{1}{3}\left(1+(-1)\left(\frac{2}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{2}{3}x\right)^2\ldots\right)\text{f}\approx\frac{1}{3}-\frac{2}{9}x+\frac{4}{27}x^2 \text{ (verified)}$$

$$=\frac{1}{3}\left(1-\frac{2}{3}x+\frac{4}{9}x^2\ldots\right)$$

(c)

$$\int_0^{1.5}\frac{1}{3+\sin 2x}\,dx$$

$$\approx\int_0^{1.5}\left(\frac{1}{3}-\frac{2}{9}x+\frac{4}{27}x^2\right)dx$$

$$=\left[\frac{1}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3\right]_0^{1.5}$$

$$=\frac{5}{12}\text{ or } 0.417\text{ (3.s.f)}$$

9 (a)

$$\frac{z_1 z_2}{{z_3}^2} = \frac{\sqrt{2}e^{i\left(\frac{\pi}{4}\right)} 2e^{-i\left(\frac{\pi}{6}\right)}}{\left(e^{i\left(\frac{\pi}{3}\right)}\right)^2}$$

$$= \frac{2\sqrt{2}e^{i\left(\frac{\pi}{12}\right)}}{e^{i\left(\frac{2\pi}{3}\right)}}$$

$$= 2\sqrt{2}e^{-i\left(\frac{7\pi}{12}\right)}$$

$$= 2\sqrt{2}\left(\cos\left(\frac{-7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right)$$

(b) (i)

$$\frac{1+z_4}{1-z_4} = \frac{1+e^{i\theta}}{1-e^{i\theta}}$$

$$= \frac{e^{i\left(\frac{\theta}{2}\right)}\left(e^{-i\left(\frac{\theta}{2}\right)} + e^{i\left(\frac{\theta}{2}\right)}\right)}{e^{i\left(\frac{\theta}{2}\right)}\left(e^{-i\left(\frac{\theta}{2}\right)} - e^{i\left(\frac{\theta}{2}\right)}\right)}$$

$$= \frac{2\cos\frac{\theta}{2}}{-2i\sin\frac{\theta}{2}}$$

$$= \frac{1}{-i}\cot\frac{\theta}{2}$$

$$= i\cot\frac{\theta}{2}, \text{ where } k = i$$

(ii)

$$\frac{1+z_4}{1-z_4} = \frac{1+z_4}{1-z_4} \times \frac{1-z_4^*}{1-z_4^*}$$

$$= \frac{1+z_4-z_4^*-z_4z_4^*}{1-z_4-z_4^*+z_4z_4^*}$$

$$= \frac{1+2i\,\mathrm{Im}(z_4)-|z_4|^2}{1-2\,\mathrm{Re}(z_4)+|z_4|^2}$$

$$= \frac{1+2\mathrm{i}\left(\frac{\sqrt{2}}{2}\right)-1}{1-2\left(\frac{\sqrt{2}}{2}\right)+1}$$

$$= \frac{\sqrt{2}\mathrm{i}}{2-\sqrt{2}} \times \frac{2+\sqrt{2}}{2+\sqrt{2}}$$

$$= \left(1+\sqrt{2}\right)\mathrm{i} \quad \text{(shown)}$$

(iii)

$$\mathrm{i}\cot\frac{\pi}{8} = \left(1+\sqrt{2}\right)\mathrm{i}$$

$$\tan\frac{\pi}{8} = \frac{1}{1+\sqrt{2}}$$

$$= \sqrt{2}-1$$

<u>Alternative Method</u>

Consider $\frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)} = \tan\left(2\left(\frac{\pi}{8}\right)\right) = 1$

Let $x = \tan\left(\frac{\pi}{8}\right)$,

$1-x^2 = 2x$

$x^2+2x-1=0$

$x = \frac{-2 \pm \sqrt{4-4(-1)}}{2}$

$= -1 \pm \sqrt{2}$

Since $\frac{\pi}{8}$ is acute, $x = -1+\sqrt{2}$

10 (a)

$\dfrac{dx}{dt} = -kx + r$ where k is a positive constant.

$\displaystyle\int \frac{1}{-kx+r}\,dx = \int 1\,dt$

$\dfrac{1}{-k}\ln|-kx+r| = t + a$ where a is an arbitrary constant

$|-kx+r| = e^{-kt-ak}$

$x = Ce^{-kt} + \dfrac{r}{k}$ where $C = \mp\dfrac{1}{k}e^{-ak}$

When $t = 0$, $x_0 = C(1) + \dfrac{r}{k} \Rightarrow C = x_0 - \dfrac{r}{k}$

$\therefore\ x = \left(x_0 - \dfrac{r}{k}\right)e^{-kt} + \dfrac{r}{k}$ (Shown)

(b)

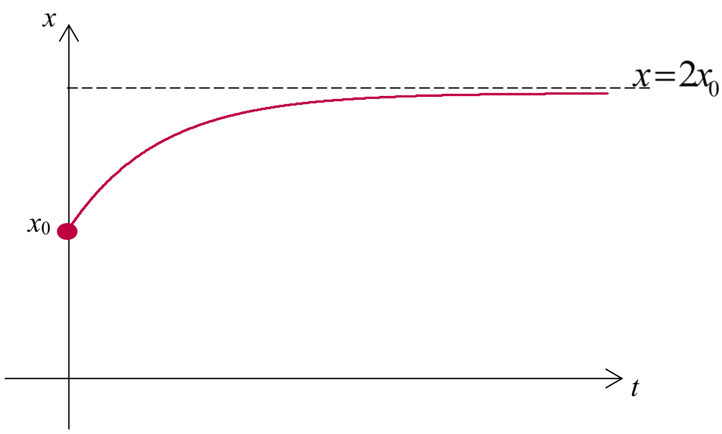
$x = (x_0 - 2x_0)e^{-kt} + 2x_0$

$= -x_0e^{-kt} + 2x_0$

As $t \to \infty$, $x \to 2x_0$

Hence the limiting value of x is $2x_0$.

(c)



(d)

$1.1x_0 = -x_0e^{-0.5k} + 2x_0$

$e^{-0.5k} = 0.9$

$\therefore\ k = -2\ln 0.9 = 0.21072$

$x = -x_0 e^{2t \ln 0.9} + 2x_0$

Let $-x_0 e^{2t \ln 0.9} + 2x_0 > 0.9(2x_0)$

$-e^{2t \ln 0.9} + 2 > 1.8$

$e^{2t \ln 0.9} < 0.2$

$t > \dfrac{\ln 0.2}{2 \ln 0.9}$ = 7.6378 = 7 hours 38 mins

Hence, the time required is 5:38 pm

11 (i) (a)

700, 700+60, 700+60+60, …

This is an AP with a = 700 and d = 60

Hence the total paid after the k^{th} payment = $\frac{k}{2}\left[2(700)+(k-1)60\right]$

$$= k(700+30k-30)$$
$$= 30k^2+670k$$
$$= \$\left(30k^2+670k\right) \text{ (Shown)}$$

(b)

$30k^2+670k \geq 40000 + 4660$

$30k^2+670k-44660 \geq 0$

From the GC,

k	$30k^2 + 670k - 44660$
28	-2380
29	0
30	2440

$\therefore$ It will take Mr Kim 29 payments to fully repay his loan.

(ii) (a)

Month	Amount owed at the end of the month
Jan	$40000(1.015) - p$
Feb	$[40000(1.015) - p](1.015) - p$

$\therefore$ The amount he owes on 1st Mar 2023 = $40000(1.015)^2 - 1.015p - p$

$$= 41209 - 2.015p$$

(b)

End of	Amount owed after interest	Amount owed after payment
Jan $n=1$	$40000(1.015)$	$40000(1.015)-p$
Feb $n=2$	$40000(1.015)^2-p(1.015)$	$40000(1.015)^2-p(1.015)-p$
Mar $n=3$	$40000(1.015)^3-p(1.015)^2$ $-p(1.015)$	$40000(1.015)^3-p(1.015)^2$ $-p(1.015)-p$
…	…	…
n		$40000(1.015)^n$ $-p(1.015)^{n-1}-p(1.015)^{n-2}$ $-\cdots-p(1.015)-p$

The amount he owed at the start of the nth month (is the amount he owed at the end of the n^{th} month after interest is charged and $\$p$ payment is made)

$$= 40000(1.015)^n - p(1.015)^{n-1} - p(1.015)^{n-2} - \cdots - p(1.015) - p$$

$$= 40000(1.015)^n - p\left[(1.015)^{n-1} + (1.015)^{n-2} + \cdots + (1.015) + 1\right]$$

$$= 40000(1.015)^n - p\left[\frac{1.015^n - 1}{1.015 - 1}\right]$$

$$= 40000(1.015)^n - \frac{200}{3}p\left(1.015^n - 1\right)$$, where $\alpha = 1.015$ and $\beta = \frac{200}{3}$

(c)

$$40000(1.015)^n - \frac{200}{3}(1585)\left(1.015^n - 1\right) \geq 0$$

$$197(1.015)^n \leq 317$$

$$n\ln(1.015) \leq \ln\frac{317}{197}$$

$n \leq 31.95046$ $\Rightarrow$ Mr Kim still owes money up to the 31^{st} payment.

$\therefore$ Mr Kim will fully pay off his loan in 32 months, i.e. $k = 32$.

Under plan B, amount that Mr Kim owes at the end of 31^{st} month after interest and payment

$$= 40000(1.015)^{31} - \frac{200}{3}(1585)\left(1.015^{31} - 1\right)$$

= 1484.764837

Amount that Mr Kim needs to pay at the end of the 32^{nd} month = 1484.764837×1.015

= \$1507.04

Mr Kim will pay \$1507.04 at the end of the 32^{nd} month after interest, i.e. $m = 1507.04$