Name:____________________________________ Class:_____________

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2023

FURTHER MATHEMATICS
Higher 2

9649/01

Paper 1

14 September 2023
3 hours

Additional materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the booklets. If you need any additional answer paper ask the invigilator for a continuation booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages and **1** blank page.

1 **(a)** Given that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with a common eigenvector $\mathbf{x}$ corresponding to eigenvalues λ and μ respectively, show that $\lambda + \mu$ is an eigenvalue of the matrix $\mathbf{A}+\mathbf{B}$ with $\mathbf{x}$ as a corresponding eigenvector. [2]

(b) The matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -4 & -6 & -6 \\ 5 & 11 & 10 \end{pmatrix}$ has $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ as an eigenvector. Find the corresponding eigenvalue. [1]

(c) The other two eigenvalues of $\mathbf{A}$ are 1 and 2, with corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ respectively. A square matrix $\mathbf{B}$ has an order 3 and has eigenvalues 2, 3 and 1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ respectively. Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A}+\mathbf{B})^3 = \mathbf{PDP}^{-1}$. [3]

2 On the first day of January 2023, a freshwater prawn farmer bought some prawns to start an aquaculture business. It is known that the population of prawns increases by 20% every month. On the last day of every month, the farmer sells 40% of the prawns and bought another 14 000 prawns.

Let p_n $(n \geq 1)$ denote the number of prawns in the farm n months after he starts his business.

(a) Write down an expression for p_{n+1} in terms of p_n. [2]

(b) Given that the farmer bought 220 000 prawns on the first day of January 2023, determine p_n in terms of n. [3]

According to a business analyst, the business will no longer be profitable if the total number of prawns falls below 60 000. If he continues with his present business model without taking any additional measures to rescue his farm, how many months of operation should pass before the farmer considers closing his business? [2]

3 **(a)** Solve the equation $z^5 + 16 + 16\sqrt{3}\,\mathrm{i} = 0$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [5]

(b) Show these roots on an Argand diagram. Identify the root for which $|2 - z|$ is the least. [3]

4 An ellipse has equation $x^2 + 2y^2 = 2$. It is given that a point P has coordinates (x_1, y_1) and is external to the ellipse.

(a) Write down the equation of the line that has gradient *m* and passing through *P*. [1]

(b) Show that the *x*-coordinates of the points of intersection of the line and the ellipse are given by the roots of the quadratic equation

$$\left(1 + 2m^2\right)x^2 + 4m\left(y_1 - mx_1\right)x + 2y_1^{\,2} + 2m^2x_1^{\,2} - 4mx_1y_1 - 2 = 0.$$ [2]

(c) Given that the line is a tangent to the ellipse, show that

$$\left(x_1^{\,2} - 2\right)m^2 - 2x_1y_1m + y_1^{\,2} - 1 = 0.$$ [2]

(d) Hence show that the locus of the points *P* at which the two tangents to the ellipse are perpendicular is a circle, stating its centre and radius. [3]

[You may use without proof the formulae:

If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0, a \neq 0$,

then $$\alpha + \beta = -\frac{b}{a},$$

$$\alpha\beta = \frac{c}{a}.]$$

5 The linear transformations $T_1 : \mathbb{R}^4 \to \mathbb{R}^4$ and $T_2 : \mathbb{R}^4 \to \mathbb{R}^4$ are represented by the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively, where

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 7 & 8 \\ 1 & 7 & 11 & 13 \\ 1 & 2 & 5 & 5 \end{pmatrix} \text{ and } \mathbf{M}_2 = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 5 & 1 & -3 & -3 \\ 3 & -1 & -1 & -1 \\ 13 & -1 & -6 & -6 \end{pmatrix}.$$

(a) Find a basis for R_1, the range space of T_1. [2]

(b) Find a basis for K_2, the null space of T_2. Hence, show that K_2 is a subspace of R_1. [5]

The set of vectors which belong to R_1 but do not belong to K_2 is denoted by W.

(c) State whether W is a vector space, justifying your answer. [1]

The linear transformation $T_3 : \mathbb{R}^4 \to \mathbb{R}^4$ is the result of applying T_1 and then T_2, in that order.

(d) Find the dimension of the null space of T_3. [2]

6 The curve C is defined parametrically by

$$x = at^2, \quad y = 2at,$$

where a is a positive constant and $t \in \mathbb{R}$.

(a) Find, in terms of a and t, the distance between $P\left(at^2, 2at\right)$ and $F\left(a, 0\right)$. [2]

(b) By considering the distance of P from the line $x = -a$, or otherwise, describe the curve C. [2]

(c) Find the equation of the tangent to C at P. Show that this tangent meets the x-axis at $Q\left(-at^2, 0\right)$. [4]

(d) Let D be the foot of perpendicular of P to the line $x = -a$. With the aid of a sketch of C and the points F, P, Q and D, show that $\angle DPQ = \angle FPQ$. (Do **not** quote the Reflection Property of C.) [3]

7 **(a)** It is given that $z = \cos\theta + \mathrm{i}\sin\theta$.

(i) Prove that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. [2]

(ii) Hence, by considering the binomial expansion of $\left(z + \dfrac{1}{z}\right)^6$, express $\cos^6\theta$ in the form

$$\frac{1}{32}\left(p + q\cos 2\theta + r\cos 4\theta + s\cos 6\theta\right),$$

where *p*, *q*, *r* and *s* are integers to be determined. [4]

(b) By considering $\sum_{n=1}^{N} z^{2n-1}$, where $z = \mathrm{e}^{i\theta}$, show that

$$\sum_{n=1}^{N} \cos(2n-1)\theta = \frac{\sin 2N\theta}{2\sin\theta},$$

where $\sin\theta \neq 0$. [6]

8 **(a)** By considering the images of $(1,0)$ and $(0,1)$, show that the matrix associated with an anticlockwise rotation of points through an angle θ about the origin is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$ [2]

The conic C_1 has equation $x^2 + y^2 - 6xy - 4 = 0$.

C_1 is rotated $45°$ anticlockwise about the origin to obtain the conic C_2.

(b) Show that the equation of C_2 is $x^2 - \dfrac{y^2}{2} = 1$. [5]

(c) State whether C_2 is a parabola, an ellipse or a hyperbola. [1]

(d) For C_2, find the

(i) eccentricity,

(ii) coordinates of the foci,

(iii) equations of the directrices. [4]

9 It is given that the curve C is defined for $-\ln 3 \le x \le \ln 3$ and has equation

$$y = \frac{e^{x} + e^{-x}}{2}.$$

(a) Find the exact length of C. [3]

A region R bounded by C and the line $y = \frac{5}{3}$ is rotated π radians about the y-axis to form a solid S.

(b) Using shell method, show that the volume of the solid S can be expressed as

$$V = \frac{\pi}{3}\left[p(\ln 3)^2 + q(\ln 3) + r \right],$$

where p, q and r are constants to be determined. [5]

The y-ordinate of the centroid, also known as the centre of gravity, of a solid generated by rotating a region bounded by a curve $y = f(x)$, the lines $y = a$ and $y = b$, about the y-axis is given by

$$\bar{y} = \frac{\pi}{V}\int_a^b x^2 y \, dy,$$

where V is the volume of the solid generated.

(c) Show that the centroid of S is given by

$$\bar{y} = \frac{\pi}{4V}\int_0^{\ln 3} x^2 \left(e^{2x} - e^{-2x}\right) dx.$$ [2]

An object is made such that it has a base in the shape of S as shown in Figure 1 below.

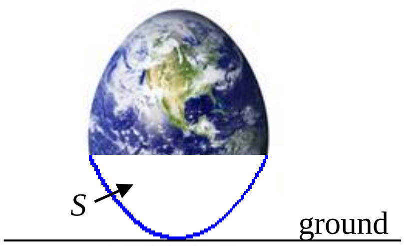

Figure 1

(d) Given that the weight of the materials used to make the part of the object above the base S is negligible, find the numerical value of the height of the centroid of the object from the ground. [2]

10 Two ornithologists A and B are investigating the change in the population of a certain species of birds in a particular region. They started tracking the population of the birds at the end of year 2022 when there were 500 birds.

Ornithologist A proposed to model the bird population using the following variation of the *logistic equation*, namely, the recurrence relation

$$C_{n+1} = 2C_n\left(1 - \frac{\sqrt{C_n}}{4}\right), \qquad \text{for } n \geq 0,$$

where C_n (in hundreds) represents the size of the bird population n years after the start of the project. This model predicts that the population approaches a limiting value of L hundred.

(a) Find the value of L. [3]

(b) By sketching the graphs of $y = 2x\left(1 - \frac{\sqrt{x}}{4}\right)$ and $y = x$ for $0 \leq x \leq 6$, show that

(i) if $0 < C_n < L$, then $C_n < C_{n+1} < L$, [3]

(ii) if $C_n > L$, then $L < C_{n+1} < C_n$. [1]

(c) Use the results in part **(b)** to deduce the behaviour of the bird population after 2022. [2]

Ornithologist B proposed another population model for the bird population, denoted by P (in hundreds) using the following logistic growth model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2P\left(1 - \frac{P}{k}\right),$$

where k is a positive constant.

Given that the equilibrium population value of this model is also L hundred,

(d) find the value of k, [1]

(e) state and explain the significance of k in this context, [2]

(f) sketch a graph to illustrate the behaviour of P for $t \geq 0$. [2]

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