Inequalities

1

$\dfrac{x+4}{3+2x-x^2}<1$ ----------- (*)

$\dfrac{x+4}{3+2x-x^2}-1<0$

$\dfrac{x+4}{3+2x-x^2}-\dfrac{3+2x-x^2}{3+2x-x^2}<0$

$\dfrac{x^2-x+1}{3+2x-x^2}<0$

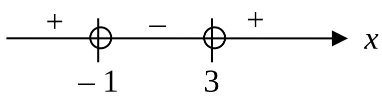
$\dfrac{x^2-x+1}{x^2-2x-3}>0$

Since $x^2-x+1=\left(x-\dfrac{1}{2}\right)^2+\dfrac{3}{4}>0$ for any real x

(**OR** coefficient of $x^2=1>0$ and discriminant, $b^2-4ac=(-1)^2-4(1)(1)=-3<0$ imply that $x^2-x+1>0$ for any real x)

$\therefore \dfrac{1}{x^2-2x-3}>0$

$\dfrac{1}{(x+1)(x-3)}>0$



$\therefore x<-1$ or $x>3$

By replacing x with $\left(-x^2\right)$ in (*), we have:

$\dfrac{-x^2+4}{3-2x^2-x^4}<1$

$\dfrac{x^2-4}{x^4+2x^2-3}<1$ which is what we need to solve.

From earlier part, $-x^2<-1$ or $-x^2>3$ (N.A., since $-x^2\le 0$ for all real x)

$x^2>1$

$x<-1$ or $x>1$

2i

$4x^2+4x+3=(2x+1)^2+2>0 \quad \forall x\in\mathbb{R}$

Alternative Method

Discriminant = $4^2-4(4)(3)=-32$

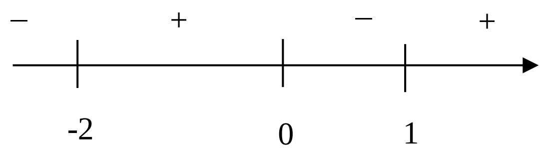
Since coefficient of x^2 is positive and discriminant < 0, $4x^2+4x+3$ is positive

ii

$$\frac{4x^3+4x^2+3x}{x^2+x-2}<0$$

$$\frac{x(4x^2+4x+3)}{(x+2)(x-1)}<0$$

$$\frac{x}{(x+2)(x-1)}<0 \quad \text{since } (2x+1)^2+2>0 \quad \forall x\in\mathbb{R}$$



Therefore $x<-2$ or $0<x<1$

iii

Let $y=|\ln x|$

$$\frac{4y^3+4y^2+3y}{y^2+y-2}\ge 0$$

$-2\le y\le 0$ or $y>1$

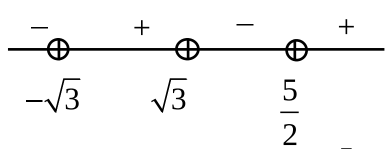
$-2\le|\ln x|\le 0$ or $|\ln x|>1$

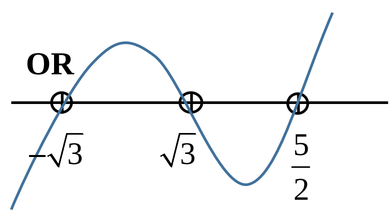
$x=1$ or $\ln x<-1$ or $\ln x>1$

$x=1$ or $0<x<e^{-1}$ or $x>e$

3

$$\frac{2w-5}{w^2-3}>0$$





$-\sqrt{3}<w<\sqrt{3}$ or $w>\frac{5}{2}$

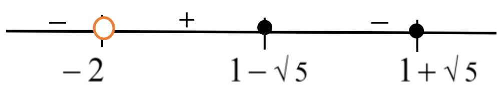
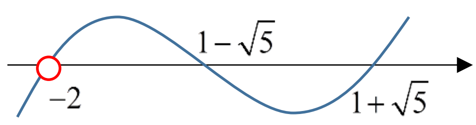
$$\frac{(2|y|-5)\sin x}{y^2-3}\le 0,$$

Since $\sin x<0$ for $\pi<x\le\frac{3\pi}{2}$, $\quad \frac{2|y|-5}{y^2-3}\ge 0$

From above, $\quad -\sqrt{3}<|y|<\sqrt{3}$ or $|y|\ge\frac{5}{2}$

$\quad 0\le|y|<\sqrt{3}$ or $|y|\ge\frac{5}{2}$

	$-\sqrt{3} < y < \sqrt{3}$ or $y \geq \frac{5}{2}$ or $y \leq -\frac{5}{2}$

4i	$4 - x \geq \frac{4}{x+2}$ $\frac{4}{x+2} + x - 4 \leq 0$ $\quad$ $y = x - 2$ $y = \frac{5}{x}$ $\frac{4 + (x+2)(x-4)}{x+2} \leq 0$ $\frac{x^2 - 2x - 4}{x+2} \leq 0$ $\frac{(x-1)^2 - 5}{(x+2)} \leq 0$ $\quad$ $a^2 - b^2 = (a+b)(a-b)$ $\frac{\left(x - [1 - \sqrt{5}]\right)\left(x - [1 + \sqrt{5}]\right)}{x+2} \leq 0$  $\therefore x < -2$ or $1 - \sqrt{5} \leq x \leq 1 + \sqrt{5}$ **<u>Alternative</u>** $4 - x \geq \frac{4}{x+2}$ Multiply by $(x+2)^2$ on both sides, $(4-x)(x+2)^2 - 4(x+2) \geq 0, \quad x \neq -2$ $(x+2)[(4-x)(x+2) - 4] \geq 0$ $(x+2)(-x^2 + 2x + 4) \geq 0$ $\quad$ $x \neq -2$ $(x+2)(x^2 - 2x - 4) \leq 0$ $(x+2)\left(x - [1 - \sqrt{5}]\right)\left(x - [1 + \sqrt{5}]\right) \leq 0$  $\therefore x < -2$ or $1 - \sqrt{5} \leq x \leq 1 + \sqrt{5}$
ii	$4 - x \geq \frac{4}{x+2}$

$$4-\left(|x|-1\right)\geq \frac{4}{\left(|x|-1\right)+2}$$

$$5-|x|\geq \frac{4}{|x|+1}$$

Since $2-\sqrt{5}<0$, $|x|$ is always greater than $2-\sqrt{5}$ for all real x

Hence, replace x with $\left(|x|-1\right)$ in earlier sol:

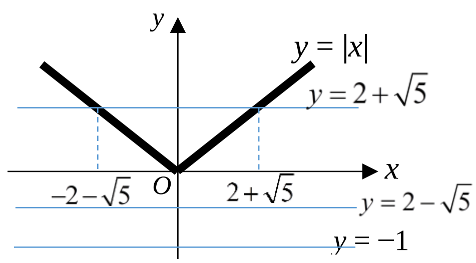
$\therefore |x|-1<-2$ or $1-\sqrt{5}\leq |x|-1\leq 1+\sqrt{5}$

$\Rightarrow$ $|x|<-1$ or $2-\sqrt{5}\leq |x|\leq 2+\sqrt{5}$

No solution as $|x|\geq 0$ for all real x | $2-\sqrt{5}\leq |x|$ and $|x|\leq 2+\sqrt{5}$

$x\in\mathbb{R}$ as $|x|\geq 0$ and $-(2+\sqrt{5})\leq x\leq 2+\sqrt{5}$

i.e. $-2-\sqrt{5}\leq x\leq 2+\sqrt{5}$



5

$$\frac{3x}{x^2-6x+11}>2$$

$$3x>2(x^2-6x+11)$$

$$\left(x-\frac{11}{2}\right)\left(x-2\right)<0$$

$$2<x<\frac{11}{2}$$

Let $y=2e^x, 2<2e^x<\frac{11}{2}\Rightarrow 0<x<\ln\frac{11}{4}$

6

$\ln\left(2x-\frac{a}{x}\right)\geq 0$ where $a>1$.

$2x-\frac{a}{x}\geq 1$

$\frac{2x^2-x-a}{x}\geq 0$

$$\frac{2\left(x-\frac{1+\sqrt{1+8a)}}{4}\right)\left(x-\frac{1-\sqrt{1+8a)}}{4}\right)}{x}\geq 0$$

$\frac{1-\sqrt{1+8a}}{4}\leq x<0$ *or* $x\geq\frac{1+\sqrt{1+8a}}{4}$

7

$\frac{x^2+2x+3}{ax^2-(a+1)x+1}<0$

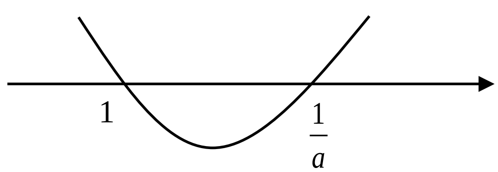
$\Rightarrow\frac{(x+1)^2-1+3}{(ax-1)(x-1)}<0$

$\Rightarrow\frac{(x+1)^2+2}{(ax-1)(x-1)}<0$

Since $(x+1)^2+2>0$ for all $x\in\mathbb{R}$,

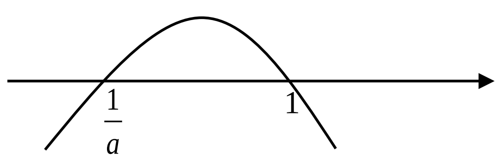
$\Rightarrow(ax-1)(x-1)<0$

If a is positive i.e. $0<a<1$



$1<x<\frac{1}{a}$

If a is negative i.e. $a<0$



$x<\frac{1}{a}$ or $x>1$.

8

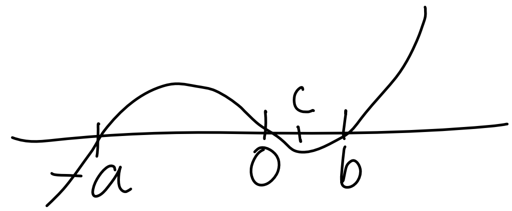
$$\frac{(x+a)(x-b)}{(x)(x-c)^2} < 0, \quad x \neq c,\ x \neq 0$$

Since $(x-c)^2 > 0$ as $x \neq c$,

$$\frac{(x+a)(x-b)}{x} < 0$$

$$x(x+a)(x-b) < 0$$

$$x < -a \quad \text{or} \quad 0 < x < b \quad \text{and} \quad x \neq c$$



Replace x by $\ln x$.

$$\ln x < -a \quad \text{or} \quad 0 < \ln x < b \quad \text{and} \quad \ln x \neq c$$

$$0 < x < e^{-a} \quad \text{or} \quad 1 < x < e^{b} \quad \text{and} \quad x \neq e^{c}$$

9

$$\frac{x^2 - x + 2}{x^2 + (1-a)x - a} \leq 0$$

$$\frac{(x-\frac{1}{2})^2 + \frac{7}{4}}{(x-a)(x+1)} \leq 0$$

Since numerator is always positive,

$$\frac{1}{(x-a)(x+1)} \leq 0 \Rightarrow (x-a)(x+1) < 0 \Rightarrow -1 < x < a$$

10

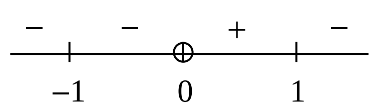
$$\frac{1}{x} + 1 \leq x(1+x)$$

$$\frac{1}{x} + 1 - x(1+x) \leq 0$$

$$\frac{1 + x - x^2(1+x)}{x} \leq 0$$

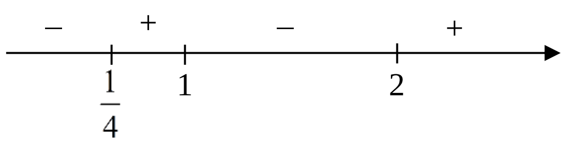
$$\frac{(1+x)(1-x^2)}{x} \leq 0$$

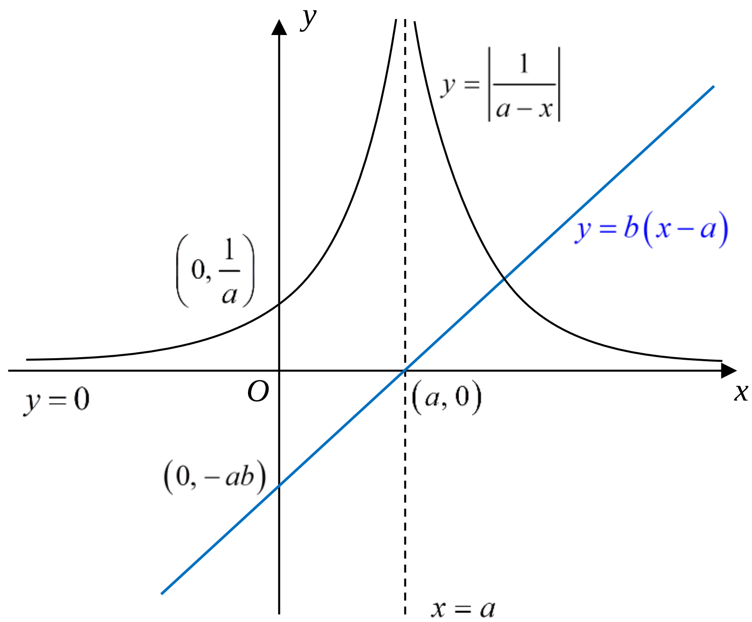
$$\frac{(1+x)^2(1-x)}{x} \leq 0$$



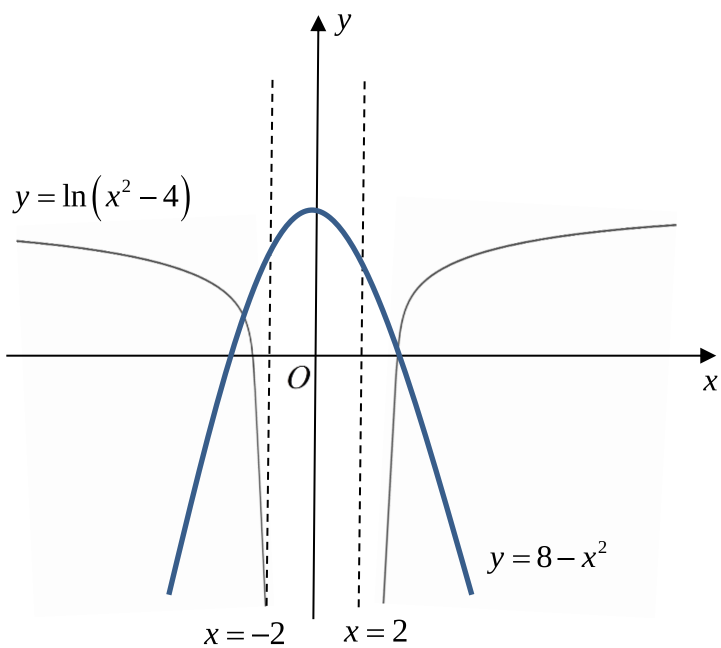
$$x < 0 \quad \text{or} \quad x \geq 1$$

	$1-e^{2x}+e^{-x}-e^{x}\le 0$ $1+\dfrac{1}{e^{x}}\le e^{x}\left(1+e^{x}\right)$ Replace x by e^{x}, $e^{x}\ge 1$ or $e^{x}<0$ (reject) $x\ge 0$

11	$\dfrac{3}{1-x}\le 5-4x$ $\dfrac{3}{1-x}-\left(5-4x\right)\le 0$ $\dfrac{3-5+9x-4x^{2}}{1-x}\le 0$ $\dfrac{4x^{2}-9x+2}{x-1}\le 0$ $\dfrac{\left(4x-1\right)\left(x-2\right)}{x-1}\le 0$  $\therefore\ x\le\dfrac{1}{4}$ or $1<x\le 2$ Let $x=\sin y$, $\therefore\ \sin y\le\dfrac{1}{4}$ or $1<\sin y\le 2$ (rejected) $\therefore\ -\pi\le y\le 0.253$ or $2.89\le y\le\pi$

12	
	Method 1: Simplifying the modulus using the graph From the graph, the root of $\left\lvert\frac{1}{a-x}\right\rvert=b(x-a)$ is equal to the root of $-\left(\frac{1}{a-x}\right)=b(x-a)$ $\Rightarrow \frac{1}{b}=(x-a)^2 \Rightarrow x=a\pm\frac{1}{\sqrt{b}}$. Since $x>a$, $x=a+\frac{1}{\sqrt{b}}$. -- **Method 2**: Solve by squaring both sides $\left\lvert\frac{1}{a-x}\right\rvert=b(x-a)\Rightarrow\left(\frac{1}{a-x}\right)^2=b^2(x-a)^2$ $\Rightarrow\left(\frac{1}{b}\right)^2=(x-a)^2(a-x)^2=(x-a)^4$ $\Rightarrow\frac{1}{b}=(x-a)^2$ (since $b>0$) $\Rightarrow x=a\pm\frac{1}{\sqrt{b}}$. Since $x>a$, $x=a+\frac{1}{\sqrt{b}}$.
	From the sketch in part **(i)**, $x<a$ or $a<x<a+\frac{1}{\sqrt{b}}$ OR $x<a+\frac{1}{\sqrt{b}}$, $x\neq a$

13	$\frac{x+3}{1-x} \geq \frac{x}{2x+1}$ $\Rightarrow \frac{x}{2x+1} - \frac{x+3}{1-x} \leq 0$ $\Rightarrow \frac{x}{2x+1} + \frac{x+3}{x-1} \leq 0$ $\Rightarrow \frac{x(x-1)+(x+3)(2x+1)}{(2x+1)(x-1)} \leq 0$ $\Rightarrow \frac{(x+1)^2}{(2x+1)(x-1)} \leq 0$ $\therefore -\frac{1}{2} < x < 1,\ x = -1$ Replace x by $-\lvert x\rvert$ $\Rightarrow -\frac{1}{2} < -\lvert x\rvert < 1,\ -\lvert x\rvert = -1$ $\Rightarrow -1 < \lvert x\rvert < \frac{1}{2},\ \lvert x\rvert = 1$ $\Rightarrow -\frac{1}{2} < x < \frac{1}{2},\ x = \pm 1$

14	 $\ln(x^2-4) + x^2 - 8 \leq 0$ $\ln(x^2-4) \leq 8 - x^2$ The suitable additional curve is $y = 8 - x^2$. Intersection points between the 2 graphs:

	$(-2.63, 1.07)$, $(2.63, 1.07)$ (3 s.f.) The solution to the inequality $\ln(x^2-4)+x^2-8\le 0$: $-2.63\le x<-2$ or $2<x\le 2.63$ (3 s.f.)

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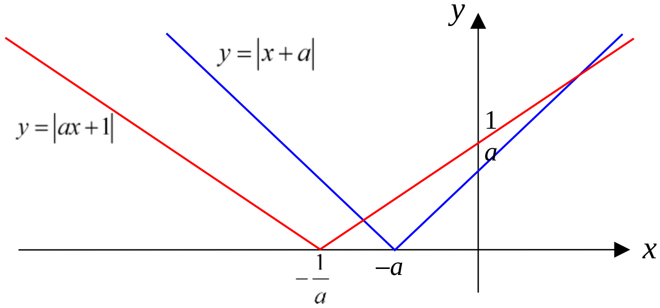
(i)



At the intersection points, $|x+a|=|ax+1|$

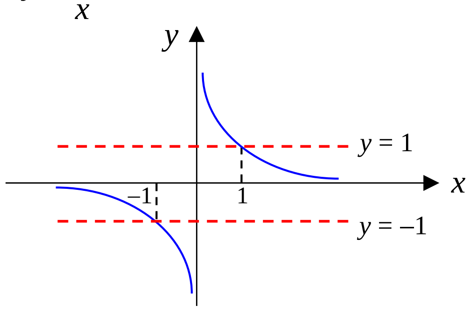
$x+a=ax+1$ or $x+a=-(ax+1)$

$x=1$ or $x=-1$

From the graph, the solution is $-1<x<1$

(ii) Replacing x by $\frac{1}{x}$, $-1<\frac{1}{x}<1$

From the graph of $y=\frac{1}{x}$,



The solution is $x<-1$ or $x>1$