Chapter 3: Equations and Inequalities

Content Outline

Include:

- formulating an equation, a system of linear equations, or inequalities from a problem situation
- solving an equation exactly or approximately using a graphing calculator
- solving a system of linear equations using a graphing calculator
- solving inequalities of the form $\frac{\mathrm{f}(x)}{\mathrm{g}(x)} > 0$ where f(x) and g(x) are linear or quadratic expressions that are either factorisable or always positive
- concept of $|x|$ and use of relations $|x-a| < b \Leftrightarrow a-b < x < a+b$ and $|x-a| > b \Leftrightarrow x < a-b$ or $x > a+b$ in the course of solving inequalities
- solving inequalities by graphical methods

References

- Further Pure Mathematics by Brian and Mark Gaulter (Oxford)
- My First Step in using TI-84Plus CE for H1 and H2 Math by Dr Spario Soon (Interactive Math Exploration Centre)

Prerequisites

- Basic rules for manipulating inequalities
- Ability to sketch graphs of quadratic, cubic and quartic functions and use the graphing calculator to find the x-intercepts of these graphs
- Determine the equations of vertical, horizontal asymptotes and restrictions on the possible values of x and y

1. Properties of Inequalities (Recap from O Level Mathematics)

1.1 Addition & Subtraction

(a) Any quantity **can be added to or subtracted from** both sides of an inequality **without changing** the inequality sign.

$$\boxed{\text{If } a > b \text{, then } a+c > b+c \text{ and } a-c > b-c}$$

For example, If $x > 3$, then $x + 7 > 3 + 7$
and $x - 8 > 3 - 8$

(b) **Inequalities of the same kind can be added together**.

$$\boxed{\text{If } a > b \text{ and } c > d \text{, then } a + c > b + d}$$

For example, If $x > 3$ and $y > 4$, then $x + y > 3 + 4$

(c) **NEVER SUBTRACT** inequalities.
For example, If $x > 2$ & $y > 0$, then $x - y$ may not be greater than 2

Counter example: For $x = 3$ and $y = 4$,

$$x - y = 3 - 4 = -1 \not> 2$$

1.2 Multiplication & Division

(a) Both sides of an inequality **can be multiplied or divided by any positive quantity without changing** the inequality sign.

If $c > 0$ and $a > b$, then $ac > bc$ and $\dfrac{a}{c} > \dfrac{b}{c}$

For example: If $x > 3$, then $x \times 2 > 3 \times 2$
and $\dfrac{x}{2} > \dfrac{3}{2}$

(b) Both sides of an inequality **can be multiplied or divided by any negative quantity but** the inequality **sign must be reversed**.

If $d < 0$ and $a > b$, then $ad < bd$ and $\dfrac{a}{d} < \dfrac{b}{d}$

For example: If $x > 3$, then $-2x < -2(3)$
and $\dfrac{x}{-2} < \dfrac{3}{-2}$

Note:
Greater care must be taken when dealing with **logarithmic terms**.

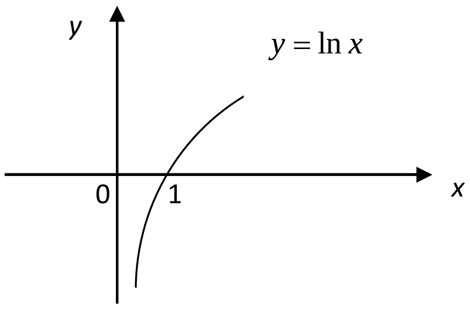
When solving the inequality,
$n \ln 0.2 < \ln 5$

$n > \dfrac{\ln 5}{\ln 0.2}$.(Since $\ln 0.2 < 0$)

When solving the inequality,
$n \ln 2 < \ln 5$

$n < \dfrac{\ln 5}{\ln 2}$.(Since $\ln 2 > 0$)

<u>Recall:</u>
$\ln x > 0$ if $x > 1$ and
$\ln x < 0$ if $0 < x < 1$
so $\ln 2 > 0$ but $\ln 0.2 < 0$



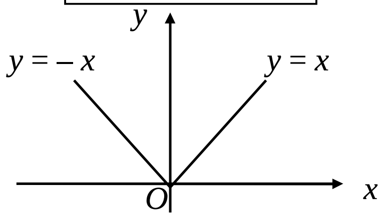
1.3 Inequalities involving Modulus Function

Recall definition of $|x|$,

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

Hence $|x| \ge 0$ for all real values of *x*.

Graph of $y = |x|$



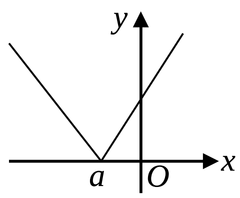
Then, definition of $|x-a|$:

$$|x-a| = \begin{cases} x-a & \text{if } x-a \ge 0 \\ -(x-a) & \text{if } x-a < 0 \end{cases}$$

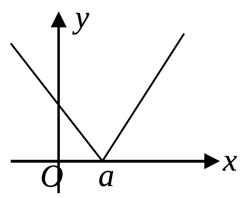
Hence $|x-a| \ge 0$ for all real values of *x*.

Graph of $y = |x-a|$

For $a < 0$



For $a > 0$

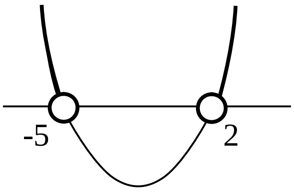


	$\lvert x-a \rvert < b$	$\lvert x-a \rvert > b$
$b > 0$	$\lvert x-a \rvert < b$ $\Rightarrow -b < x-a < b$ $\Rightarrow a-b < x < a+b$ For example, $\lvert x-2 \rvert < 5$ $\Rightarrow -5 < x-2 < 5$ $\Rightarrow -3 < x < 7$	$\lvert x-a \rvert > b$ $\Rightarrow x-a < -b$ or $x-a > b$ $\Rightarrow x < a-b$ or $x > a+b$ For example, $\lvert x-2 \rvert > 5$ $\Rightarrow x-2 < -5$ or $x-2 > 5$ $\Rightarrow x < -3$ or $x > 7$
$b < 0$	No solution for *x*	$x \in \mathbb{R}$

2 Solving inequalities of the form $(x-a)(x-b)......>0$
(Including cases involving <, ≥ and ≤)

Example 1: (Recap from Additional Mathematics)
Solve the inequality $(x+5)(x-2)>0$

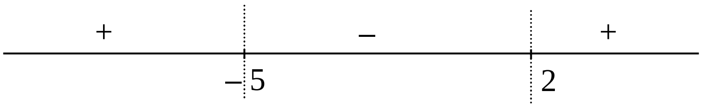
<u>Method 1: Graphical Sketch</u>
Sketch the graph of the LHS of the inequality with the graphic calculator (GC).



From the question, we need the LHS of the inequality to be '> 0' (i.e. positive). Indicate with a hollow circle on the sketch to show that $x=-5$ and 2 are not required in the answer.

Hence, the answer is $x < -5$ or $x > 2$.

<u>Method 2: Sign Test</u>
The above problem could also be solved with a line diagram. Equate each factor to zero to find the critical values of *x*. Hence, critical values of *x* are – 5 and 2



To determine the sign in each zone:
i) choose a value of *x* in any one zone
ii) substitute the value into the LHS of the inequality to determine the sign (+/–)
iii) the signs will then alternate between each zone **unless there are repeated factors**.

E.g.
Choose $x=-10$, then $(x+5)(x-2)$ has
Choose $x=0$, then $(x+5)(x-2)$ has
Choose $x=10$, then $(x+5)(x-2)$ has

Hence, $x < -5$ or $x > 2$.

3 Solving inequalities of the form $\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)} > 0$ (including cases involving $<$, $\geq$ and $\leq$)

3.1 Solving inequalities where f(x) and g(x) are linear or are factorisable (By Sign Test)

Use of sign test

□ Factorise f(x) and g(x) completely into linear factors.
□ Equate each factor to zero to find the critical values of x.
□ Use the sign test to determine the regions that correspond to the inequality.

Note 1: The sign test must be properly presented.

Note 2: For quadratic factors which are difficult to factorise, we use **completing the square** on the factor, or the quadratic formula to find the roots.

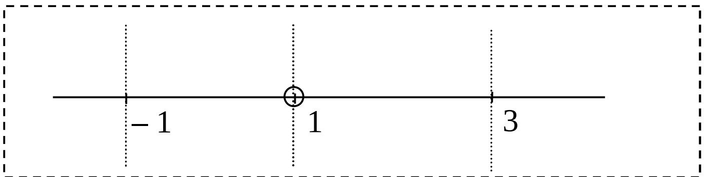
Example 2:

Without using calculator, solve the inequality $\dfrac{x^2-2x-3}{x-1} \geq 0$.

Solution:

$$\frac{x^2-2x-3}{x-1} \geq 0 \quad \Rightarrow x \neq 1$$

$$\frac{(x+1)(x-3)}{x-1} \geq 0$$



(check signs of zones with a value in each region)

Hence, $-1 \leq x < 1$ or $x \geq 3$.

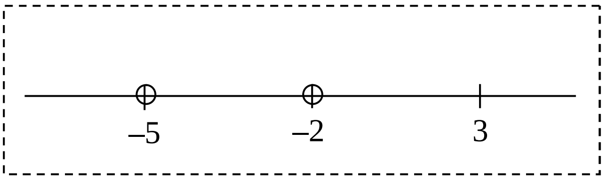
Note: $x \neq 1$ because $\dfrac{x^2-2x-3}{x-1}$ is undefined when $x=1$.

Example 3:

Solve the inequality $\dfrac{x-3}{x^2+7x+10} \geq 0$ algebraically.

Solution:

$$\frac{x-3}{(x+5)(x+2)} \geq 0 \quad \text{(Note } x \neq -5 \text{ and } x \neq -2\text{)}$$



$\therefore$ The answer is

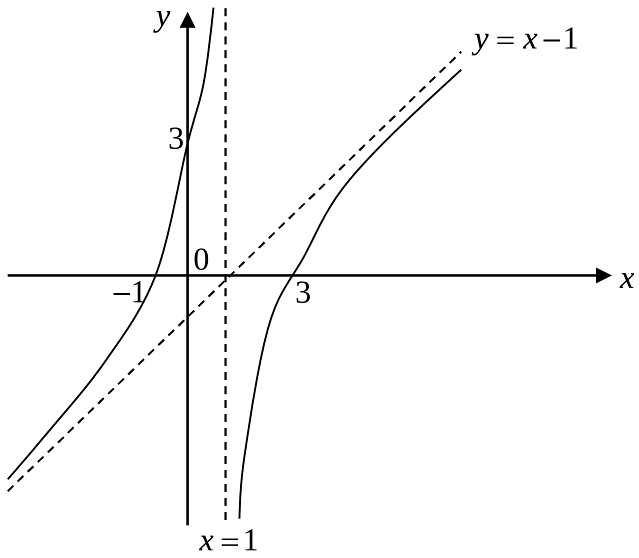
3.2 Solving inequalities where f(x) and g(x) are linear or are factorisable (By Graphical Method)

Use of graphical method

- □ Sketch graph using GC.
- □ Read required range of x.

Example 4:

Sketch the graph of $y=\dfrac{x^2-2x-3}{x-1}$. Hence, solve the inequality $\dfrac{x^2-2x-3}{x-1}\geq 0$.



Solution:

$$\frac{x^2-2x-3}{x-1}\geq 0 \quad \Rightarrow x\neq 1$$

Hence, $-1\leq x<1$ or $x\geq 3$

Note 1:
Notice that the vertical asymptote $x=1$ is not shown on the GC. You need to be mindful of the presence of vertical asymptote.

Note 2:
For inequalities involving $\geq$ and $\leq$, we need to be careful on whether x can be equal to the critical values. In the above example, $\dfrac{x^2-2x-3}{x-1}\geq 0$, $x\neq 1$ as this value of x will result in the denominator being zero. Hence, the answer is $-1\leq x<1$ or $x\geq 3$.

Example 5:

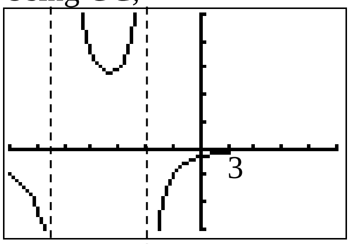
Solve the inequality $\dfrac{x-3}{x^2+7x+10} \geq 0$.

Solution:

$$\frac{x-3}{x^2+7x+10} \geq 0$$

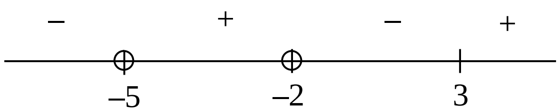
$$\frac{x-3}{(x+2)(x+5)} \geq 0$$

Using GC,



$x = -5 \quad x = -2$

OR



$\therefore$ The answer is

Note: You should know both graphical and sign test methods and apply the more efficient method or as required by the question. If exact answer is required, no GC method is allowed (refer to Example 8)

What happens if the RHS of the inequality is not zero?

Example 6:

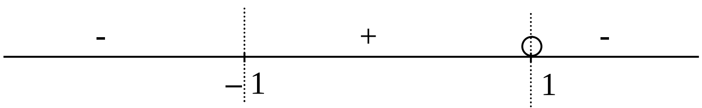
Solve the inequality $\dfrac{x-5}{x-1} \geq 3$.

Solution:

$$\Rightarrow \frac{x-5-3(x-1)}{x-1} \geq 0 \text{ (note } x \neq 1)$$

$$\Rightarrow \frac{-2x-2}{x-1} \geq 0$$

$$\Rightarrow \frac{-2(x+1)}{x-1} \geq 0$$



Example 7

Solve the inequality $1 < \dfrac{x+5}{2x+3} < 4$.

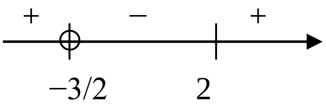
<u>Solution</u> :

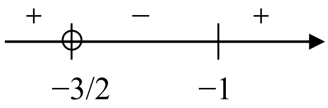
$$1 < \frac{x+5}{2x+3} \qquad \text{and} \qquad \frac{x+5}{2x+3} < 4$$

$$1 - \frac{x+5}{2x+3} < 0 \qquad\qquad \frac{x+5-4(2x+3)}{2x+3} < 0$$

$$\frac{(2x+3)-(x+5)}{2x+3} < 0 \qquad\qquad \frac{-7x-7}{2x+3} < 0$$

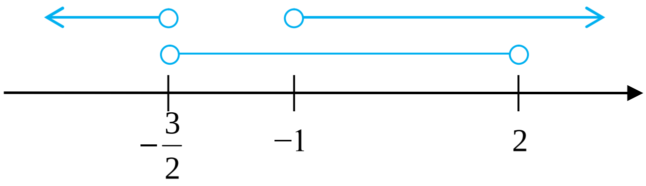
$$\frac{x-2}{2x+3} < 0 \qquad\qquad \frac{x+1}{2x+3} > 0$$





and

Since x satisfy both inequalities, we find the intersection of the 2 solution sets.



Example 8:

Without using a calculator, solve the inequality $\dfrac{2x-1}{x^2-4x+2} \le 0$.

Solution:

Observe that x^2-4x+2 is not easily factorized,

<u>*Completing the square method:*</u>

x^2-4x+2

Hence,

$$\frac{2x-1}{x^2-4x+2} \le 0$$

OR

Using formula method:

To find the critical values from the denominator, solve $x^2-4x+2=0$.

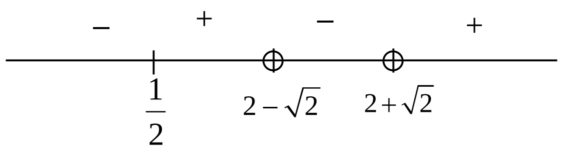
$$x =$$

$$x = \frac{4 \pm \sqrt{8}}{2}$$

$$x = 2 \pm \sqrt{2}$$

Hence,

$$\frac{2x-1}{x^2-4x+2} \le 0$$



Therefore solution is $x \le \frac{1}{2}$ or $2-\sqrt{2} < x < 2+\sqrt{2}$.

Example 9:

Solve algebraically the following inequality $\frac{x^2}{x+2} \le 2$.

Solution:

$$\frac{x^2}{x+2} \le 2$$

Completing the square method:

$x^2 - 2x - 4$

$= (x-1)^2 - (\sqrt{5})^2$

$=$

OR

Using formula method:

To find the critical values from the denominator, solve $x^2 - 2x - 4 = 0$.

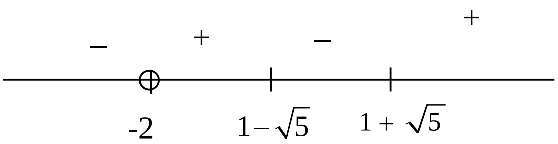
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{20}}{2}$$

$$x = 1 \pm \sqrt{5}$$

Hence,

$$\frac{\left(x-\left(1+\sqrt{5}\right)\right)\left(x-\left(1-\sqrt{5}\right)\right)}{x+2} \le 0$$



Therefore solution is $x < -2$ or $1-\sqrt{5} \le x \le 1+\sqrt{5}$

3.3 Solving inequalities where f(x) and g(x) are not factorisable

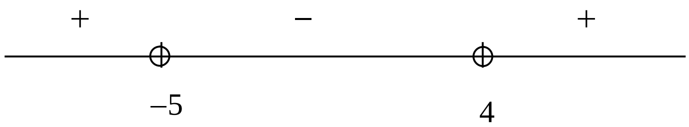
So far, we have solved inequalities by considering all the factors of the numerator, f(x), and the denominator, g(x). However, not all inequalities will have numerators, f(x), and denominators, g(x), that are factorisable.

When f(x) and g(x) are not factorisable, it means that f(x) or g(x) is always positive/negative

Example 10:

Find the set of values of x such that $\dfrac{x^2+1}{(x-4)(x+5)} < 0$.

Solution:
Since $x^2 \geq 0$ for all real values of *x*, $x^2 + 1 > 0$ for all real values of *x*.
Therefore, we only consider $(x-4)(x+5) < 0$.



Hence, the solution is

Example 11:
Given that *x* is real, prove that $2x^2 - 6x + 5$ is always positive. Hence find the set of values of *x* for which $\dfrac{2x^2 - 6x + 5}{x^2 - 25} \geq 0$.

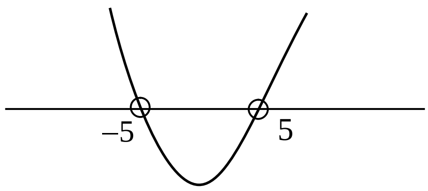
Solution:
To show that $2x^2 - 6x + 5 > 0$ *for all real values of x:*

Completing the square method:

Since $\left(x - \dfrac{3}{2}\right)^2 \geq 0$, $2\left(x - \dfrac{3}{2}\right)^2 + \dfrac{1}{2} > 0$ for all real values of *x*.

Since $2x^2 - 6x + 5 > 0$ for all real values of *x*, we only consider
$x^2 - 25 > 0$, i.e. $(x-5)(x+5) > 0$
(note that $x \neq 5$ and $x \neq -5$)

Using graphical sketch,

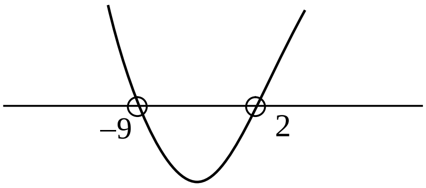


Example 12:
Find the set of values of *x* such that $\dfrac{-3x^2 + 3x - 1}{(x-2)(x+9)} < 0$.

Solution :
Completing the square method:

Since $-3\left(x-\frac{1}{2}\right)^2 \leq 0$, $-3\left(x-\frac{1}{2}\right)^2 - \frac{1}{4} < 0$ for all real values of x.

Since $-3x^2+3x-1<0$ for all real values of x, we only consider $(x-2)(x+9)>0$.



Using graphical sketch, $x<-9$ or $x>2$

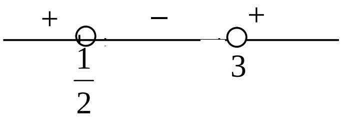
Example 13:

Solve the inequality $\frac{2x^2+3x+5}{(2x-1)(x-3)} \geq 0$. Hence solve $\frac{2x^2+3|x|+5}{(2|x|-1)(|x|-3)} \geq 0$

Solution:

Completing the square method:

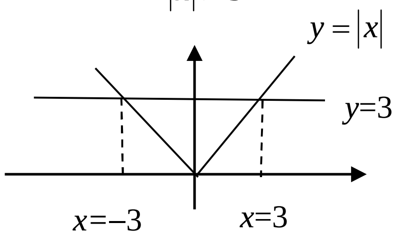
Therefore, we consider only

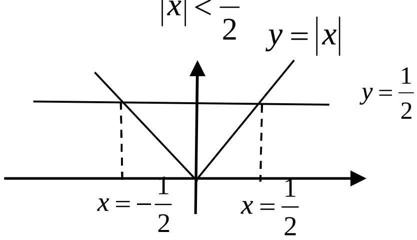


Hence,.

Replace x with $|x|$

$|x|>3$ or $|x|<\frac{1}{2}$



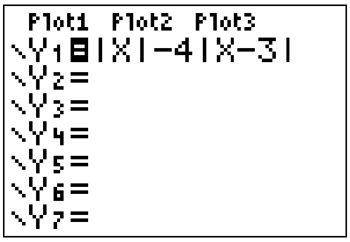
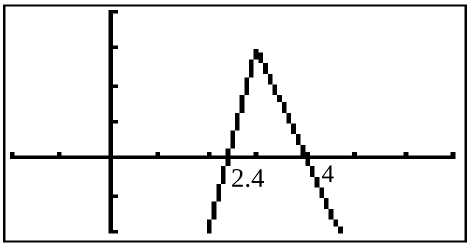


4. Solving inequalities involving other types of functions using a graphic calculator (e.g. modulus, exponential, logarithmic, trigonometric etc)

Example 14:
Solve the inequality $|x| < 4|x-3|$.

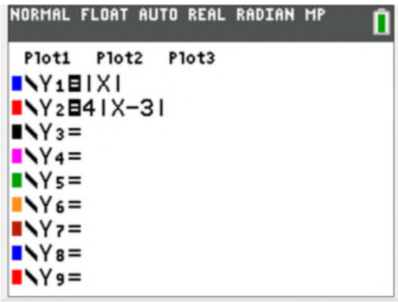
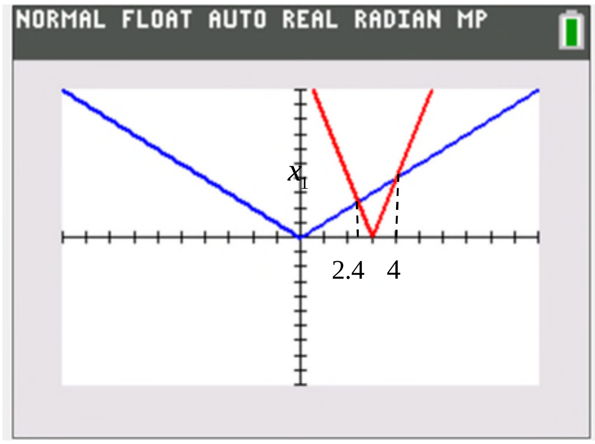
Solution :
<u>Method 1</u>
$|x| - 4|x-3| < 0$

Sketch the graph of $y = |x| - 4|x-3|$

Steps/Keystrokes/Explanations	Screen Display
1. $Y_1 = \|x\| - 4\|x-3\|$ (To enter the absolute function, press [alpha][window] followed by "1" OR choose [math], then choose NUM followed by '1')	
2. Press [graph] to display the graph. (You may change the window setting for a bigger diagram)	
3. Use [2nd] [calc] and choose 2 to find $x_1 = 2.4$ and $x_2 = 4$	Hence $y = \|x\| - 4\|x-3\| < 0$ when $x < 2.4$ or $x > 4$.

Method 2

Sketch the graphs of $y = |x|$ and $y = 4|x-3|$

Steps/Keystrokes/Explanations	Screen Display
1. $Y_1 = \mid x \mid$ 2. $Y_2 = 4 \mid x - 3 \mid$ (To enter the absolute function, press $\boxed{\text{alpha}}\boxed{\text{window}}$ followed by “1” OR choose $\boxed{\text{math}}$, then choose NUM followed by ‘1’)	
3. Press $\boxed{\text{graph}}$ to display the graph. (You may change the window setting for a bigger diagram)	
4. Use $\boxed{\text{2nd}}$ [calc] and choose 5 to find x_1 and x_2	To solve $\|x\| < 4\|x-3\|$ Locate the region where Y_1 is below Y_2 Hence $x < 2.4$ or $x > 4$.

Note: The graph and the intersecting values must be clearly presented.

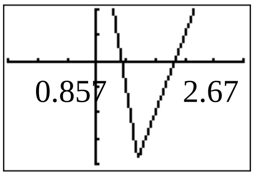
Example 15:
Solve the inequality $|7-5x|>2x+1$.

Solution :

Method 1

let $y=|7-5x|-(2x+1)$

from graph, $y>0$ when

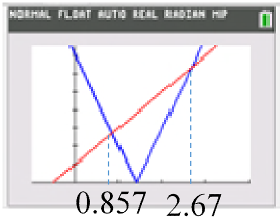


Method 2

Plot

Locate the region where $y_1>y_2$

from graph,

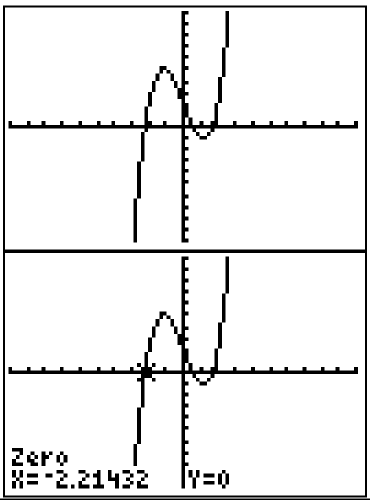
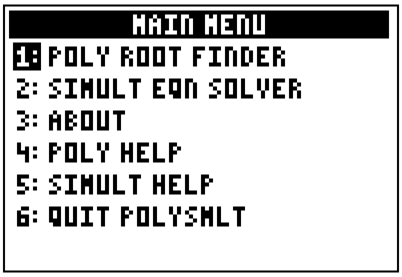


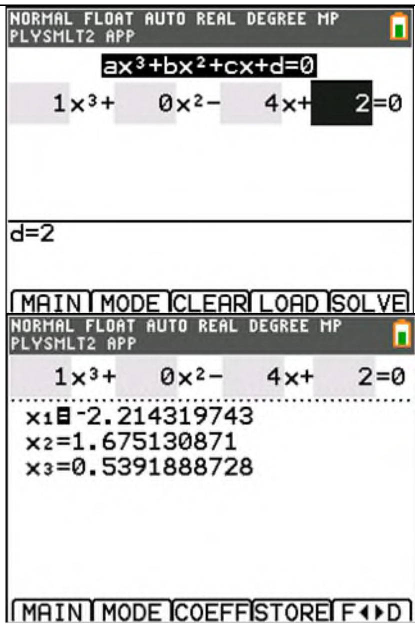
5. Solving an equation using a GC

The numerical solution (i.e. the roots or x-intercepts) of an equation, $y=0$ or $\text{f}(x)=0$, where y or $\text{f}(x)$ is a polynomial (e.g. $2x^3-5x^2+3=0$), may be found using the "Poly Root Finder" function in a GC.

Example 16:
Using a GC, find the numerical solution of the equation $x^3-4x+2=0$.

Solution:
Either the graphical method or GC APPS can be used.

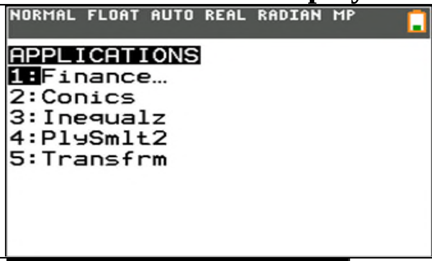
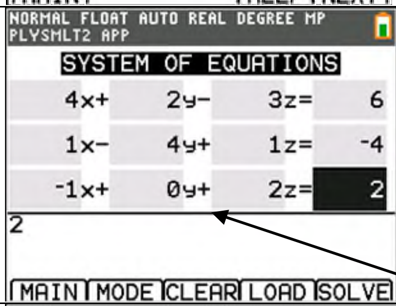
Graph	APPS PlySmlt2
Sketch the graph of $y=x^3-4x+2$ using the GC. 	Using the GC APPs PlySmlt2, go to Poly Root Finder. 

The solutions of the equation are $x = -2.21$, 0.539 and 1.68 .	 The solutions of the equation are $x = -2.21$, 0.539 and 1.68 .

6. Solving a system of linear equations using a GC: Use of APPS PlySmlt2

Example 17:
Solve the following system of linear equations.
$4a + 2b - 3c = 6$, $c + a - 4b = -4$ and $2c = 2 + a$.
Solution:
Arrange the equations such that the **unknowns are positioned in line accordingly** as shown:

$$\begin{aligned} 4a + 2b - 3c &= 6 \\ a - 4b + c &= -4 \\ -a + 0b + 2c &= 2 \end{aligned}$$

Steps/Keystrokes/Explanations	Screen Display
1. Press $\boxed{\text{apps}}$ to obtain screen shown and choose '4'	
2. Choose '2'	
3. Under **EQUATIONS** choose '3' as we have three equations Under **UNKNOWNS** choose '3' as we have three unknowns. Press **NEXT**	
4. Key in the coefficients of *a*, *b*, *c* and the numbers in the give set of simultaneous equations. Press $\boxed{\text{solve}}$ to obtain results. (If you keyed in wrong numbers and need to amend, press $\boxed{\text{mode}}$)	
5. $a = 2$, $b = 2$ and $c = 2$ (If you now realised that you keyed in wrong numbers (getting weird answers is one indicator) and need to amend, press **SYSM** to go back to previous screen)	

Note:
Coefficient of $b = 0$ in equation 3

Example 18:
Solve the following system of linear equations:
$w+y+z=3,\quad 3w+2x+5y=3-z,\quad 4y-5z+12=-7x,\quad 2w-2x=9-3y-z$

Solution :

Arrange the equations:

Ans: $w=16, x=-2, y=-7, z=-6$

7. Formulating and solving a system of linear equations from a problem situation

Many real-life problems give rise to a system of linear equations, which we can solve rather easily using a GC. Thus, the crux of solving such problems is the formation of enough relevant equations involving the unknown quantities.

Example 19:
Given that f(x) is a quadratic polynomial and the graph y = f(x) passes through the points $(-2,19)$ and (1, 4) and has a gradient 5 when x = 2, find f(x).

Solution:
$\mathrm{f}(x)=ax^2+bx+c$

Curve passes through $(-2,19)$ $\rightarrow$ substitute $x=-2, y=19$

$4a-2b+c=19$ ---(1)

Curve passes through (1, 4) $\rightarrow$ substitute $x=1, y=4$

--- (2)

$\mathrm{f}'(x)=2ax+b$

--- (3)

Using GC, a = 2, b = –3 and c = 5.
$\therefore \mathrm{f}(x)=2x^2-3x+5$

Note: **Quadratic polynomial in x :** ax^2+bx+c
Cubic polynomial in x : ax^3+bx^2+cx+d
Quartic polynomial in x : $ax^4+bx^3+cx^2+dx+e$

Example 20:
A store sells large, medium and small tins of yellow, blue and red paint. The selling price of a large, medium and small tin of any colour is $\$x$, $\$y$ and $\$z$ respectively.
The number of tins of each type that were sold one day are given in the following table.

	Large	Medium	Small
Yellow	5	3	1
Blue	2	7	5
Red	6	4	2

The total income from the sale of yellow paint was \$593, from blue paint was \$829 and from red paint was \$778.

(i) Find the price of a large, medium and small tin of paint respectively.

(ii) Mr Tan bought 3 large tins of yellow paint, 1 medium tin of red paint and 5 small tins of blue paint. How much did he pay altogether?

Solution:

i)

Using GC, $x = 72$, $y = 60$, $z = 53$

ii) He paid $3x + 1y + 5z =$

Example 21:
Ken is a real estate agent. Recently, he sold 3 flats; 1 unit in Tampines, 1 unit in Bedok and 1 unit in Jurong for a total sum of \$1,050,000. He earned commissions of 1%, 1.5% and 2% of the selling price for the unit in Tampines, Bedok and Jurong respectively. His total commissions were \$14,705. However, he had to pay a management fee of 0.5%, 0.3% and 0.25% of the selling price for the each unit in Tampines, Bedok and Jurong respectively. If he had to pay a total of \$3953.50 in management fee, find the selling price of each of the 3 units.

Solution :
Let selling price of flats in Tampines be t, Bedok be b, Jurong be j.

Total sale of flats = \$1050000

----------------- (1)

Total commissions = \$14705

Or $10t + 15b + 20j = 14705000$ ----------------------- (2)

Total fees to be paid = \$3953.50

Or $50t + 30b + 25j = 39535000$ ----------------------- (3)

Solving with GC, $t = 466000$, $b = 327000$ and $j = 257000$

The selling price of flats in Tampines, Bedok and Jurong are \$466000, \$327000 and \$257000 respectively.