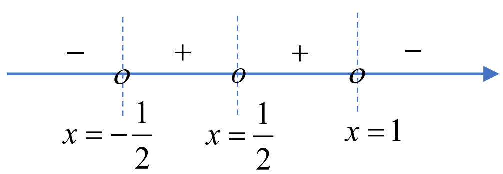
2023 H2MA Prelim Paper 1

1	Solution [6] Inequality P1 Q1	
(i)	$\frac{4x^2-4x+1}{1+x-2x^2}<0$ $\frac{(2x-1)^2}{(1-x)(2x+1)}<0$	
	Method 1: $\frac{(2x-1)^2}{(1-x)(2x+1)}<0$ Since $(2x-1)^2 \geq 0$ for all real values of x, $(1-x)(2x+1)<0$ $\therefore x<-\frac{1}{2}$ or $x>1$	
	Method 2: $\frac{(2x-1)^2}{(1-x)(2x+1)}<0$  $\therefore x<-\frac{1}{2}$ or $x>1$	
(ii)	$\frac{(2^{x+1}-1)^2}{1+2^x-2^{2x+1}}\leq 0$ $\frac{(2(2^x)-1)^2}{1+(2^x)-2(2^x)^2}\leq 0$ Replace x with 2^x,	

	$2^x < -\frac{1}{2}$ OR $2^x > 1$ OR $2^x = \frac{1}{2}$ (rejected, since 2^x is positive for all real values of x.) $x > 0$ $x = -1$	

2	Solution [6] Complex Numbers P1 Q2	
(i)	$\left\|\frac{z-1}{z^*(1+\mathrm{i})}\right\|=\frac{1}{2}$ $\left\|\frac{(a+\mathrm{i})-1}{(a+\mathrm{i})^*(1+\mathrm{i})}\right\|=\frac{1}{2}$ $\left\|\frac{(a-1)+\mathrm{i}}{(a-\mathrm{i})(1+\mathrm{i})}\right\|=\frac{1}{2}$ $\frac{\|(a-1)+\mathrm{i}\|}{\|(a-\mathrm{i})\|\|(1+\mathrm{i})\|}=\frac{1}{2}$ $\frac{\sqrt{(a-1)^2+1^2}}{\sqrt{a^2+1^2}\sqrt{1^2+1^2}}=\frac{1}{2}$ $2\sqrt{a^2-2a+1+1}=\sqrt{2}\sqrt{a^2+1}$ $4\left(a^2-2a+2\right)=2\left(a^2+1\right)$ $2a^2-4a+4=a^2+1$ $a^2-4a+3=0$ $a=1$ or 3 (rejected since $a<2$)	
(ii)	Consider $\arg\left(\frac{z-1}{z^*w}\right)^n$ $=n\arg\left(\frac{z-1}{z^*w}\right)$ $=n\left[\arg(z-1)-\arg z^*-\arg w\right]$ $=n\left[\arg(\mathrm{i})-\arg(1-\mathrm{i})-\frac{\pi}{4}\right]$ $=n\left[\frac{\pi}{2}-\left(-\frac{\pi}{4}\right)-\frac{\pi}{4}\right]$ $=\frac{n\pi}{2}$ If $\left(\frac{z-1}{z^*w}\right)^n$ is purely imaginary and negative, then:	

	$\arg\left(\frac{z-1}{z^{*}w}\right)^{n}=\frac{3\pi}{2},\frac{7\pi}{2},\frac{11\pi}{2},...$ or $\frac{3\pi}{2}+2k\pi$, $k\in\mathbb{Z}^{+}$ Thus, $\frac{n\pi}{2}=\frac{3\pi}{2},\frac{7\pi}{2},\frac{11\pi}{2},...$ $n=3,\ 7,\ 11,\ ...$ 3 smallest positive values of $n=3,7,11$	

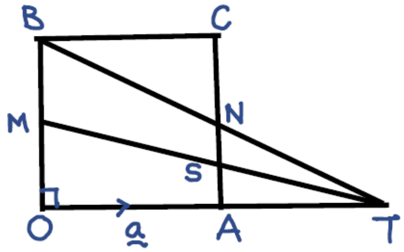
3	Solution [7] Summation P1 Q3	
(i)	$u_{n-1}-2u_n+u_{n+1}$ where $n\geq 2$ $=\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}$ $=\frac{n(n+1)-2(n-1)(n+1)+(n-1)n}{(n-1)n(n+1)}$ $=\frac{n^2+n-2n^2+2+n^2-n}{n\left[(n-1)(n+1)\right]}$ $=\frac{2}{n(n^2-1)}$ $=\frac{2}{n^3-n}$ $A=2$	
(ii)	$\sum_{n=2}^{N}\frac{1}{n^3-n}$ $=\frac{1}{2}\sum_{n=2}^{N}\frac{2}{n^3-n}$ $=\frac{1}{2}\sum_{n=2}^{N}\left(u_{n-1}-2u_n+u_{n+1}\right)$ $=\frac{1}{2}\left[\begin{array}{ccc} u_1 & -2u_2 & \cancel{+u_3} \\ +u_2 & \cancel{-2u_3} & +u_4 \\ \cancel{+u_3} & -2u_4 & +u_5 \\ \ldots & \ldots & \ldots \\ +u_{N-3} & -2u_{N-2} & \cancel{+u_{N-1}} \\ +u_{N-2} & \cancel{-2u_{N-1}} & +u_N \\ \cancel{+u_{N-1}} & -2u_N & +u_{N+1} \end{array}\right]$ $=\frac{1}{2}\left(u_1-u_2-u_N+u_{N+1}\right)$ $=\frac{1}{2}\left(1-\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$ $=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$	

(iii)	$\sum_{n=2}^{\infty}\frac{1}{n^3-n}=\lim_{N\to\infty}\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$ Since $\frac{1}{N}\to 0$ as $N\to\infty$, $\frac{1}{N+1}\to 0$ as $N\to\infty$ $\therefore$ It converges.	
(iv)	$\sum_{n=3}^{N}\frac{1}{(n-2)(n-1)(n)}$ $=\sum_{n+1=3}^{n+1=N}\frac{1}{(n+1-2)(n+1-1)(n+1)}$ (Replace n with $n+1$.) $=\sum_{n=2}^{N-1}\frac{1}{(n-1)(n)(n+1)}$ $=\sum_{n=2}^{N-1}\frac{1}{n^3-n}$ $=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N-1}+\frac{1}{N}\right)$	
	(Alternative, but not recommended) $\sum_{n=2}^{N}\frac{1}{n^3-n}=\sum_{n=2}^{N}\frac{1}{n(n-1)(n+1)}$ $=\sum_{n-1=2}^{n-1=N}\frac{1}{(n-1)(n-2)(n)}$ (n replaced by $n-1$) $=\sum_{n=3}^{N+1}\frac{1}{(n-1)(n-2)(n)}$ From (ii), $\sum_{n=2}^{N}\frac{1}{n^3-n}=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$ Thus, $\sum_{n=3}^{N+1}\frac{1}{(n-1)(n-2)(n)}=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$ $\therefore\sum_{n=3}^{N}\frac{1}{(n-1)(n-2)(n)}$ $=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N-1}+\frac{1}{N-1+1}\right)$	

	$=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N-1}+\frac{1}{N}\right)$	

4	Solution [13] Complex Numbers P1 Q4	
(i)	From: $x^4-4x^3+6x^2-ax+b=0$ Given that $x=x_0$ is a root, then: ${x_0}^4-4{x_0}^3+6{x_0}^2-ax_0+b=0$ ---eqn (1) Consider applying conjugate on both sides: $\left({x_0}^4-4{x_0}^3+6{x_0}^2-ax_0+b\right)^*=(0)^*$ $\left({x_0}^4\right)^*-\left(4{x_0}^3\right)^*+\left(6{x_0}^2\right)^*-\left(ax_0\right)^*+(b)^*=0$ Since coefficients are all real, then $(a)^*=a$ and $(b)^*=b$ $\left(x_0{}^*\right)^4-4\left(x_0{}^*\right)^3+6\left(x_0{}^*\right)^2-a\left(x_0{}^*\right)+b=0$ Therefore, $x_0{}^*$ is a root as well. Alternatively, Substitute $x={x_0}^*$ into LHS of eqn (1): $\left(x_0{}^*\right)^4-4\left(x_0{}^*\right)^3+6\left(x_0{}^*\right)^2-a\left(x_0{}^*\right)+b$ $=\left({x_0}^4\right)^*-4\left({x_0}^3\right)^*+6\left({x_0}^2\right)^*-a\left(x_0\right)^*+b$ $=\left({x_0}^4-4{x_0}^3+6{x_0}^2-ax_0+b\right)^*$ since *a, b* are real $=(0)^*=0$ Thus $x_0{}^*$ is also a root.	
(ii)	Using Remainder Theorem: Since $x=2-\mathrm{i}$ is a root of the equation, $(2-\mathrm{i})^4-4(2-\mathrm{i})^3+6(2-\mathrm{i})^2-a(2-\mathrm{i})+b=0$ $-7-24\mathrm{i}-4(2-11\mathrm{i})+6(3-4\mathrm{i})-2a+a\mathrm{i}+b=0$ $3-4\mathrm{i}-2a+a\mathrm{i}+b=0$ Comparing the real and imaginary parts, $3-2a+b=0$ and $-4+a=0$ $a=4$ and $b=5$ $x^4-4x^3+6x^2-4x+5=0$	

	Using Factor Theorem: $x^4-4x^3+6x^2-4x+5$ $\equiv\left[x-(2-\text{i})\right]\left[x-(2+\text{i})\right]\left(x^2+Ax+B\right)$ Comparing constant term: $5=B(2+\text{i})(2-\text{i})\Rightarrow B=1$ Substitute $x=2$: $16-32+24-8+5$ $\equiv\left[2-(2-\text{i})\right]\left[2-(2+\text{i})\right](4+2A+B)$ $5=4+2A+1\Rightarrow A=0$ For $x^2+Ax+B=0\Rightarrow x^2+1=0$ $x=\text{i}$ or $-\text{i}$ The roots are: $x=2-\text{i},\ 2+\text{i},\ -\text{i},\ \text{i}$	•
(iii)	$by^4-ay^3+6y^2-4y+1=0$ $1-4y+6y^2-ay^3+by^4=0$ Divide throughout by y^4, $\left(\frac{1}{y}\right)^4-4\left(\frac{1}{y}\right)^3+6\left(\frac{1}{y}\right)^2-a\left(\frac{1}{y}\right)+b=0$ Replace x with $\frac{1}{y}$, Then $\frac{1}{y}=2-\text{i},\ 2+\text{i},\ -\text{i},\ \text{i}$ $y=\frac{2}{5}+\text{i}\frac{1}{5},\ \frac{2}{5}-\text{i}\frac{1}{5},\ \text{i},\ -\text{i}$	

5	Solution [7] Abstract Vectors P1 Q5	
(i)	 By Ratio Theorem, $\overrightarrow{OS}=\frac{\overrightarrow{OC}+3\overrightarrow{OA}}{1+3}$ $\overrightarrow{OS}=\frac{1}{4}\overrightarrow{OC}+\frac{3}{4}\overrightarrow{OA}$ $\overrightarrow{OS}=\frac{1}{4}(\mathbf{a}+\mathbf{b})+\frac{3}{4}\mathbf{a}$ Since $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}$ $\overrightarrow{AC}=\overrightarrow{OB}=\mathbf{b}$ $\overrightarrow{OS}=\mathbf{a}+\frac{1}{4}\mathbf{b}$	
(ii)	$\overrightarrow{MS}=\overrightarrow{OS}-\overrightarrow{OM}$ $=\left(\mathbf{a}+\frac{1}{4}\mathbf{b}\right)-\frac{1}{2}\mathbf{b}=\mathbf{a}-\frac{1}{4}\mathbf{b}$ $l_{MS}:\mathbf{r}=\overrightarrow{OM}+\lambda\overrightarrow{MS}$ $\mathbf{r}=\frac{1}{2}\mathbf{b}+\lambda\left(\mathbf{a}-\frac{1}{4}\mathbf{b}\right),\ \lambda\in\mathbb{R}$ $\overrightarrow{BN}=\overrightarrow{ON}-\overrightarrow{OB}$ $=\frac{1}{2}\left[\mathbf{a}+(\mathbf{a}+\mathbf{b})\right]-\mathbf{b}=\mathbf{a}-\frac{1}{2}\mathbf{b}$ $l_{BN}:\mathbf{r}=\overrightarrow{OB}+\mu\overrightarrow{BN}$ $\mathbf{r}=\mathbf{b}+\mu\left(\mathbf{a}-\frac{1}{2}\mathbf{b}\right),\ \mu\in\mathbb{R}$	
(iii)	At T, $\frac{1}{2}\mathbf{b}+\lambda\left(\mathbf{a}-\frac{1}{4}\mathbf{b}\right)=\mathbf{b}+\mu\left(\mathbf{a}-\frac{1}{2}\mathbf{b}\right)$	

	Since **a** and **b** are non parallel and non zero, comparing coefficients of **a,** $\lambda = \mu$ Comparing coefficients of **b,** $\frac{1}{2} - \frac{\lambda}{4} = 1 - \frac{\mu}{2}$ Solving, $\mu = 2 = \lambda$ $\therefore \overrightarrow{OT} = \frac{1}{2}\mathbf{b} + 2\left(\mathbf{a} - \frac{1}{4}\mathbf{b}\right) = 2\mathbf{a}$ Since $\overrightarrow{OT} = 2\overrightarrow{OA}$, then *O*, *T* and *A* are colliear points with *O* (or *A* or *T*) as the common point.	

6	Solution [7]	
	$Let\ y=\dfrac{ax+b}{x-a}$. $xy-ay=ax+b$ $x(y-a)=ay+b$ $x=\dfrac{ay+b}{y-a}$ $\therefore f^{-1}(x)=\dfrac{ax+b}{x-a}$ f is self-inverse.	
	Symmetrical about $y = x$.	
	$R_f=\mathbb{R}\setminus\{a\}$ Since $D_g=\mathbb{R}, R_f\subseteq D_g$. $\therefore$ gf exists. $D_{gf}=D_f=\mathbb{R}\setminus\{a\}$ $\mathbb{R}\setminus\{a\}\xrightarrow{f}\mathbb{R}\setminus\{a\}\xrightarrow{g}(0,c)\cup(c,e)$ or $(0,e)\setminus\{c\}$	

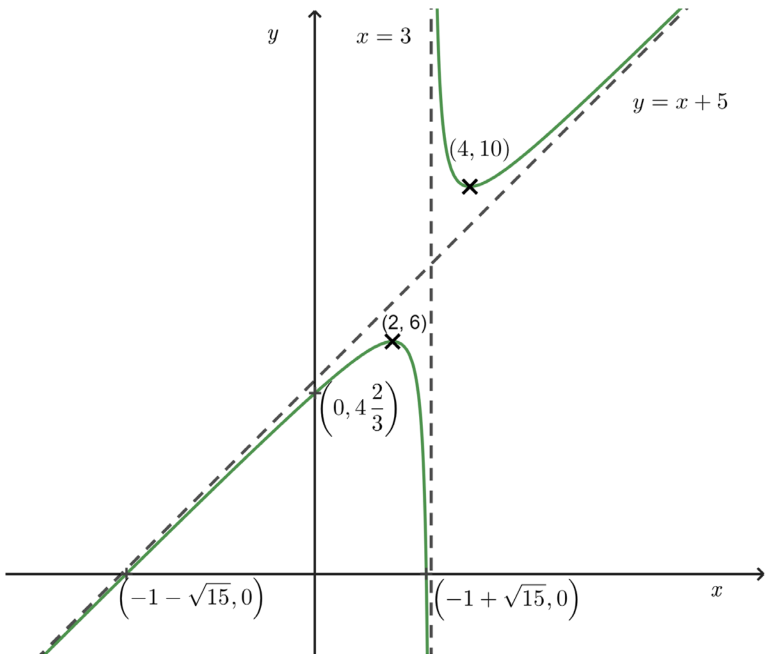
7	Solution [8] AP GP	
(a)	Let first term be $u_1 = a$ and common difference d. Middle term $u_{\frac{n+1}{2}} = a + \left(\frac{n+1}{2} - 1\right)d$ Last term $u_n = a + (n-1)d$ Given $a + a + \left(\frac{n+1}{2} - 1\right)d = a + (n-1)d$ $a + \frac{nd}{2} + \frac{d}{2} - d = nd - d$ $\frac{nd}{2} = a + \frac{d}{2}$ $n = \frac{2a+d}{d}$ $n = \frac{2a}{d} + 1$ **OR** $d = \frac{2a}{n-1}$ Sub $n = \frac{2a}{d} + 1$ into the middle term $u_{\frac{n+1}{2}} = a + \left(\frac{n+1}{2} - 1\right)d$: $u_{\frac{n+1}{2}}$ $= a + \left(\frac{\frac{2a}{d} + 1 + 1}{2} - 1\right)d$ $= a + \left(\frac{2 + \frac{2a}{d}}{2} - 1\right)d$ $= a + \left(1 + \frac{a}{d} - 1\right)d$ $= a + \left(\frac{a}{d}\right)d$ $= 2a$	

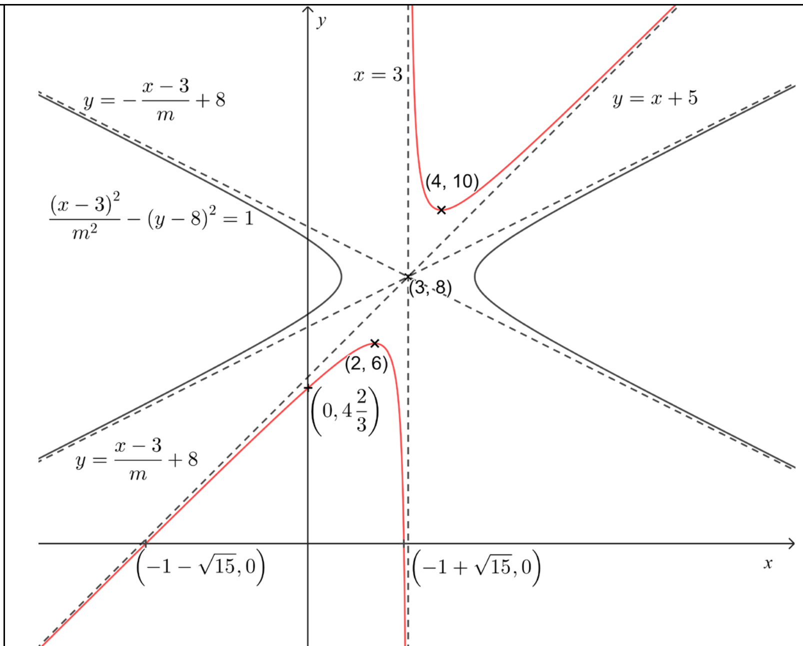
	OR Alternatively Sub $d = \frac{2a}{n-1}$ into the middle term $u_{\frac{n+1}{2}} = a + \left(\frac{n+1}{2} - 1\right)d$: $u_{\frac{n+1}{2}}$ $= a + \left(\frac{n+1}{2} - 1\right)\left(\frac{2a}{n-1}\right)$ $= a + \left(\frac{n-1}{2}\right)\left(\frac{2a}{n-1}\right)$ $= 2a$	
(b) (i)	Given $T_1 = 3p^3,\ r = \frac{p}{q^2}$ And $m = 3p^3\left(\frac{p}{q^2}\right)^{n-1}$ $m = 3p^3\left(\frac{p^{n-1}}{q^{2(n-1)}}\right)$ $m = 3\left(\frac{p^{n+2}}{q^{2n-2}}\right)$ $\ln m = \ln 3 + \ln p^{n+2} - \ln q^{2n-2}$ $\ln m = \ln 3 + (n+2)\ln p - (2n-2)\ln q$ $A = 2,\ B = 2$	
(ii)	Since it is a convergent GP, then: $\|r\| < 1$. $-1 < \frac{p}{q^2} < 1$ $q = 2p$	

	$-1<\dfrac{p}{(2p)^2}<1$ $-1<\dfrac{1}{4p}<1$ $-4<\dfrac{1}{p}<4$ $p<-\dfrac{1}{4}$ OR $p>\dfrac{1}{4}$ (Rejected, since all terms are positive.)	

8	Solution [8] Macluarin's Series	
(i)	$\left(1-x^2\right)f''(x)-xf'(x)=0$ (Given) Differentiating, $\left(1-x^2\right)f'''(x)-2xf''(x)-xf''(x)-f'(x)=0$ At $x=0$, substitute into: $\left(1-x^2\right)f''(x)-xf'(x)=0$ $f''(0)-0=0$ $f''(0)=0$ At $x=0$, substitute into: $\left(1-x^2\right)f'''(x)-2xf''(x)-xf''(x)-f'(x)=0$ $f'''(0)-0-0-f'(0)=0$ $f'''(0)=-1$ Thus, $f(x)$ $=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$ $=1+x(-1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(-1)+\ldots$ $=1-x-\frac{x^3}{6}+\ldots$	
(ii)	$f(x)=1-x-\frac{x^3}{6}+\ldots$ $f'(x)=-1-\frac{x^2}{2}+\ldots$ $\frac{f'(x)}{f(x)}=\frac{-1-\frac{x^2}{2}+\ldots}{1-x-\frac{x^3}{6}+\ldots}$ $=\left(-1-\frac{x^2}{2}+\ldots\right)\left(1-x-\frac{x^3}{6}+\ldots\right)^{-1}$	

	$=\left(-1-\frac{x^2}{2}+...\right)(1-x+...)^{-1}$ (Need not consider x^3 − term anymore.) $=\left(-1-\frac{x^2}{2}+...\right)\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2...\right)$ $=\left(-1-\frac{x^2}{2}+...\right)(1+x+x^2...)$ $=\left(-1-x-x^2-\frac{x^2}{2}+...\right)$ $=-1-x-\frac{3}{2}x^2+...$	
(iii)	$\frac{f'(x)}{f(x)}=-1-x-\frac{3}{2}x^2+...$ Integrating both sides with respect to *x*, $\int\frac{f'(x)}{f(x)}\,dx=\int -1-x-\frac{3}{2}x^2+......\,dx$ $\ln\lvert f(x)\rvert=\left[-x-\frac{1}{2}x^2+...\right]+C$ When $x=0$, $f(0)=1$: $\ln\lvert 1\rvert=C \Rightarrow C=0$ $\ln\lvert f(x)\rvert=-x-\frac{1}{2}x^2+...$	

9	Solution [10] Curve Sketching	
(i)	Since $x=3$, $c=-3$ $y=x+5+\dfrac{d}{x-3}=\dfrac{x^2+2x-15+d}{x-3}=\dfrac{ax^2+bx-14}{x-3}$ Therefore $a=1$ and $b=2$.	
(ii)	Graph of $y=\dfrac{x^2+2x-14}{x-3}$  When rounded to 3 s.f, the axial intercepts are (0, 4.67), (−4.87, 0) and (2.87, 0).	

(iii)		
(iv)	The asymptotes of $\frac{(x-3)^2}{m^2}-(y-8)^2=1$ ----(1) are $y=\frac{x-3}{m}+8$ and $y=-\frac{x-3}{m}+8$ $y=\frac{x^2+2x-14}{x-3}$ ----(2) has asymptotes $x=3$ and $y=x+5$ All the asymptotes meet at the point $(3,8)$. Observing the graphs, (1) and (2) intersect when the asymptote of (1) with positive gradient, $y=\frac{x-3}{m}+8$ must be between the asymptotes of (2) i.e. $y=x+5$ and $x=3$. Thus, gradient of $y=\frac{x-3}{m}+8$ need to be between the gradient of $y=x+5$ and gradient of the vertical line, $x=3$. $1<\frac{1}{m}$ Therefore, $0<m<1$	

10	Solution [10] Applications of Differentiation	
(i)	$x=\sqrt{4+t^2}$ $\quad y=t^2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{t}{\sqrt{4+t^2}}$ $\quad \frac{\mathrm{d}y}{\mathrm{d}t}=2t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$ $=2\sqrt{4+t^2}$ When $t=p$, $x=\sqrt{4+p^2}$, $y=p^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=2\sqrt{4+p^2}$. Equation of tangent $y-p^2=2\sqrt{4+p^2}\left(x-\sqrt{4+p^2}\right)$ $y=2\sqrt{4+p^2}\,x-8-p^2$	
(ii)	When $y=0$, $x=\frac{8+p^2}{2\sqrt{4+p^2}}$ x-coordinate of M $=\frac{1}{2}\left(\sqrt{4+p^2}+\frac{8+p^2}{2\sqrt{4+p^2}}\right)$ $=\frac{16+3p^2}{4\sqrt{4+p^2}}$ Coordinates of M are $\left(\frac{16+3p^2}{4\sqrt{4+p^2}},\frac{p^2}{2}\right)$ Parametric equation of curve traced out by M $x=\frac{16+3p^2}{4\sqrt{4+p^2}}$ $y=\frac{p^2}{2}$ $\quad (1)$ From (1), $p^2=2y$ Cartesian equation of line $x=\frac{16+3(2y)}{4\sqrt{4+2y}}$ $2x\sqrt{4+2y}=8+3y$	

(iii)	Required area $= \int_2^3 y \, \mathrm{d}x$ $= \int_0^{\sqrt{5}} y \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t$ $= \int_0^{\sqrt{5}} \left(t^2\right)\left(\frac{t}{\sqrt{4+t^2}}\right) \mathrm{d}t$ $= \int_0^{\sqrt{5}} \frac{t^3}{\sqrt{4+t^2}} \, \mathrm{d}t \ \ (\textit{shown})$ Let $u = t^2$ then $\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$ Let $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{t}{\sqrt{4+t^2}}$ then $v = \frac{1}{2}\int 2t\left(4+t^2\right)^{-\frac{1}{2}} \, \mathrm{d}t = \left(4+t^2\right)^{\frac{1}{2}}$ $\int_0^{\sqrt{5}} \frac{t^3}{\sqrt{4+t^2}} \, \mathrm{d}t$ $= \left[t^2\sqrt{4+t^2}\right]_0^{\sqrt{5}} - \int_0^{\sqrt{5}} 2t\sqrt{4+t^2} \, \mathrm{d}t$ $= \left[5\sqrt{9} - 0\right] - \left[\frac{2}{3}\left(4+t^2\right)^{\frac{3}{2}}\right]_0^{\sqrt{5}}$ $= 15 - \frac{2}{3}\left[(9)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right]$ $= \frac{7}{3}$ units^2	

11	Solution [12] Differentiation	
(i)	A is at $(1,0)$. B is at (x,y) where $y^2=1-\frac{x^2}{4}$ $r^2=(x-1)^2+(y-0)^2$ $=(x-1)^2+1-\frac{x^2}{4}$ $=\frac{1}{4}(3x^2-8x+8)$ $F=\frac{4k}{3x^2-8x+8}$	
(ii)	$F=\frac{4k}{3x^2-8x+8}$ $\frac{dF}{dx}=-4k(3x^2-8x+8)^{-2}(6x-8)$ $=\frac{-8k(3x-4)}{(3x^2-8x+8)^2}$ At stationary point, $\frac{dF}{dx}=0$. $6x-8=0$ $x=\frac{4}{3}$ **Method 1: Using 2nd Derivative to verify nature** $\frac{d^2F}{dx^2}=\frac{(3x^2-8x+8)^2(-24k)+8k(3x-4)(2)(3x^2-8x+8)(6x-8)}{(3x^2-8x+8)^4}$ When $6x-8=0$, i.e. $x=\frac{4}{3}$: $\frac{d^2F}{dx^2}=-3.375k<0$, therefore maximum. Note: $\left.\frac{d^2F}{dx^2}\right\|_{x=\frac{4}{3}}$ can be found easily using GC.	

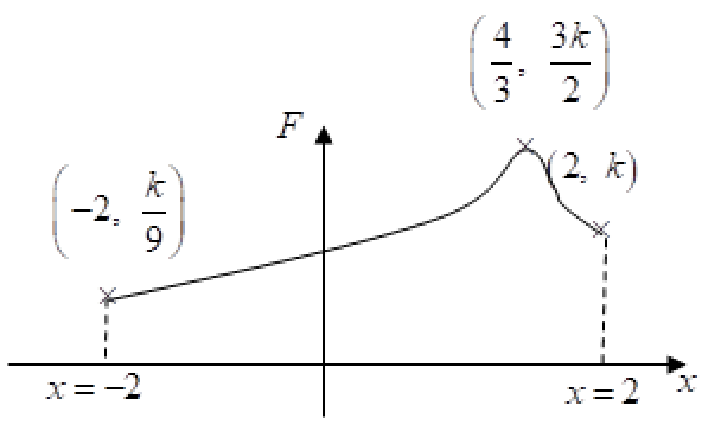
Method 2: Using 1st Derivative to verify nature

x	1.33	$\frac{4}{3}$	1.34
$\frac{dF}{dx}$	**0.00112*k***	**0**	**-0.0225*k***
	/	-	\

$$F_{max} = \frac{4k}{3\left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) + 8}$$

$$= \frac{3k}{2}$$

(iii)



Note that $y^2 = 1 - \frac{x^2}{4} \Rightarrow x = \pm 2\sqrt{1 - y^2}$

Therefore $-2 \le x \le 2$

(iv) Minimum F occurs when B is farthest from A.
i.e. when B is $(-2,\ 0)$, and thus $r = 3$.

$$F_{min} = \frac{k}{3^2}$$

$$F_{min} = \frac{k}{9}$$

(v) By symmetry, B must be at $(0,1)$ or $(0,-1)$
with x-coordinate $= 0$

	With $x = 0$, $F = \dfrac{4k}{0+8} = \dfrac{k}{2}$	

12	Solution [13] Differential Equations	
(i)	$\frac{\mathrm{d}Q_{in}}{\mathrm{d}t} = k$ $\frac{\mathrm{d}Q_{out}}{\mathrm{d}t} \propto \sqrt{Q}$ $\frac{\mathrm{d}Q_{out}}{\mathrm{d}t} = c\sqrt{Q},\ c \in \mathbb{R}$ $\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}Q_{in}}{\mathrm{d}t} - \frac{\mathrm{d}Q_{out}}{\mathrm{d}t} = k - c\sqrt{Q}$ Starting with a new clean tank: When $t = 0$, $Q = 0$, $\frac{\mathrm{d}Q}{\mathrm{d}t} = 5$ ---(*) $\frac{\mathrm{d}Q}{\mathrm{d}t} = k - c\sqrt{Q}$ $5 = k - c\sqrt{0}$ $k = 5$ With filter in a new clean tank, level of pollution stabilizes at 75 units: As $t \to \infty$, $Q \to 75$, $\frac{\mathrm{d}Q}{\mathrm{d}t} \to 0$, ----(**) $\frac{\mathrm{d}Q}{\mathrm{d}t} = 5 - c\sqrt{Q}$ $0 = 5 - c\sqrt{75}$ $c = \frac{1}{\sqrt{3}}$ $\therefore\ \frac{\mathrm{d}Q}{\mathrm{d}t} = 5 - \sqrt{\frac{Q}{3}}$ (shown)	

(ii)	$\frac{dQ}{dt} = 5 - \sqrt{\frac{Q}{3}}$ ----(1) $x = \sqrt{\frac{Q}{3}}$ $Q = 3x^2$ ----(2) $\frac{dQ}{dt} = 6x\frac{dx}{dt}$ ----(3) Sub (2) and (3) into (1): $6x\frac{dx}{dt} = 5 - x$	

(iii)	$6x\frac{\mathrm{d}x}{\mathrm{d}t}=5-x$ $\int \frac{6x}{5-x}\,\mathrm{d}x=\int \mathrm{d}t$ $6\int -1+\frac{5}{5-x}\,\mathrm{d}x=\int \mathrm{d}t$ $-x-5\ln\|5-x\|=\frac{1}{6}t+c$ $x+\ln\|5-x\|^5=\frac{-t}{6}-c$ $\ln\|5-x\|^5=-x-\frac{t}{6}-c$ $\|5-x\|^5=\mathrm{e}^{-x-\frac{t}{6}-c}$ $(5-x)^5=\pm\mathrm{e}^{-c}\,\mathrm{e}^{-x}\mathrm{e}^{-\frac{t}{6}}$ $(5-x)^5\,\mathrm{e}^x=A\,\mathrm{e}^{\frac{-t}{6}}$ where $A=\pm\mathrm{e}^{-c}$ -----(*) $\left(5-\sqrt{\frac{Q}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{Q}{3}}}=A\,\mathrm{e}^{\frac{-t}{6}}$ When $t = 0$, $Q = 0$: $\left(5-\sqrt{\frac{Q}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{Q}{3}}}=A\,\mathrm{e}^{\frac{-t}{6}}$ $\left(5-\sqrt{\frac{0}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{0}{3}}}=A\,\mathrm{e}^{0}$ $A=5^5=3125$ $\left(5-\sqrt{\frac{Q}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{Q}{3}}}=3125\,\mathrm{e}^{\frac{-t}{6}}$ Therefore, $a = 5$, $b = 5$, $m = 3125$, and $p=\frac{-1}{6}$.	

(iv)	When $Q = 48$, $\left(5-\sqrt{\frac{48}{3}}\right)^5 e^{\sqrt{\frac{48}{3}}} = 3125\,e^{\frac{-t}{6}}$ $t = -6\ln\left(\frac{e^4}{3125}\right) = 24.3$ days (3 s.f.) Without a filter, the pollutant level would reach 48 units in $t = 48\ /\ 5 = 9.6$ days. Therefore the filter is effective.	