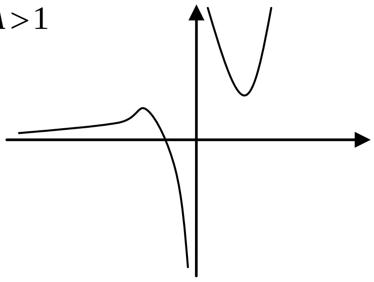
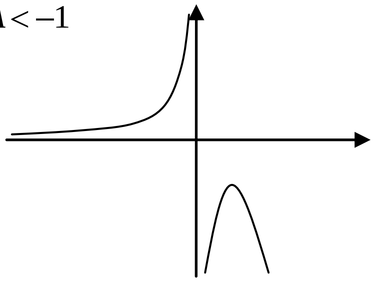
RV 2012 Yr 6 H2 MA Prelim Paper 2

Question 1 [7 Marks]		
	$x\frac{\mathrm{d}y}{\mathrm{d}x}+(1-x^3)y-x^2=0---(1)$ Let $u=xy \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$ Subs. into (1): $\frac{\mathrm{d}u}{\mathrm{d}x}-y+y-x^3y-x^2=0$ $\frac{\mathrm{d}u}{\mathrm{d}x}=ux^2+x^2=(u+1)x^2$ (shown) $\int\frac{1}{u+1}\mathrm{d}u=\int x^2\mathrm{d}x$ $\ln\lvert u+1\rvert=\frac{x^3}{3}+C$ $u+1=A\mathrm{e}^{x^3/3}$ where $A=\pm\mathrm{e}^C$ $u=A\mathrm{e}^{x^3/3}-1$ $\therefore y=\frac{A\mathrm{e}^{x^3/3}-1}{x}$ (shown) and $\mathrm{f}(x)=\frac{1}{3}x^3$	
(i)	$A>1$ 	
(ii)	$A<-1$ 	

Question 2 [10 Marks]

(a)

$\tan y = \mathrm{e}^x \quad \Rightarrow \quad \sec^2 y \dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$

$$\therefore \int \frac{1}{\mathrm{e}^x(1+\mathrm{e}^{2x})}\,\mathrm{d}x$$

$$= \int \cot y \cdot \frac{1}{1+\tan^2 y} \cdot \cot y \sec^2 y \,\mathrm{d}y$$

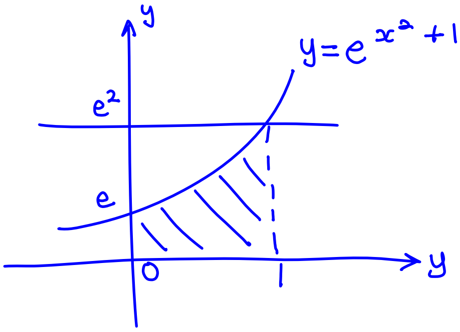
$$= \int \cot^2 y \,\mathrm{d}y$$

$$= \int (\operatorname{cosec}^2 y - 1)\,\mathrm{d}y$$

$$= -\cot y - y + C$$

$$= -\mathrm{e}^{-x} - \tan^{-1}(\mathrm{e}^x) + C$$

(b)



$y = \mathrm{e}^{x^2+1}$

When $x = 0$, $y = \mathrm{e}$ and when $x = 1$, $y = \mathrm{e}^2$.

$y = \mathrm{e}^{x^2+1} \Rightarrow x^2 = \ln y - 1$

Required volume

$= \pi(1)^2\mathrm{e}^2 - J$

where $J = \pi\int_{\mathrm{e}}^{\mathrm{e}^2} (\ln y - 1)\mathrm{d}y$

J

$$= \pi\int_{\mathrm{e}}^{\mathrm{e}^2} (\ln y - 1)\mathrm{d}y$$

$$= \pi\left\{\left[y(\ln y)\right]_{\mathrm{e}}^{\mathrm{e}^2} - \int_{\mathrm{e}}^{\mathrm{e}^2} 1\,\mathrm{d}y - \left[y\right]_{\mathrm{e}}^{\mathrm{e}^2}\right\}$$

$$= \pi\left[y(\ln y) - 2y\right]_{\mathrm{e}}^{\mathrm{e}^2}$$

$$= \pi\left[\mathrm{e}^2 \ln \mathrm{e}^2 - \mathrm{e}\ln\mathrm{e} - 2\mathrm{e}^2 + 2\mathrm{e}\right]$$

$$= \pi\,\mathrm{e}$$

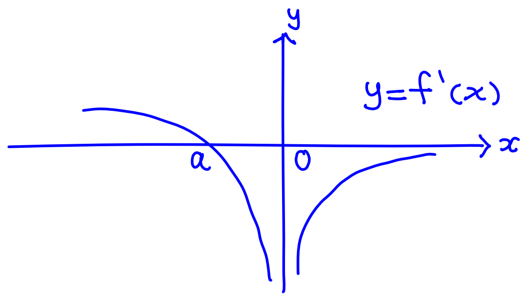
Required volume

$= \pi(1)^2\mathrm{e}^2 - J$

$= \pi\mathrm{e}^2 - \pi\mathrm{e}$

$= \pi\mathrm{e}(\mathrm{e} - 1)$

Note that $\int \ln y \,\mathrm{d}y = y\ln y - \int 1\,\mathrm{d}y = y\ln y - y + c$

<table>
<tr><td colspan="3">Question 3 [11 Marks]</td></tr>
<tr><td>ai</td><td>
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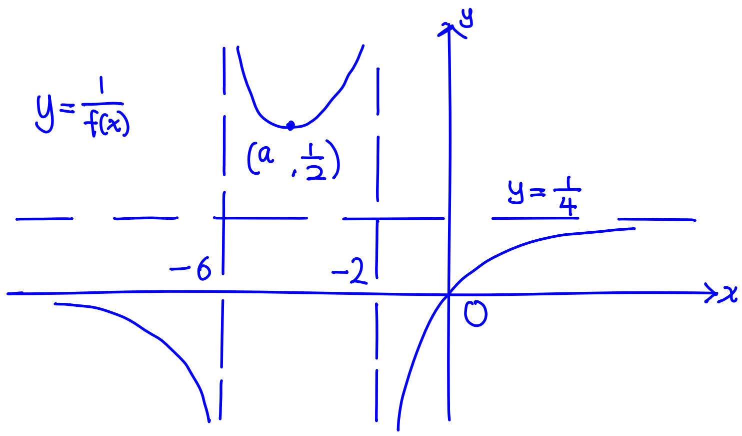
<tr><td>aii</td><td>
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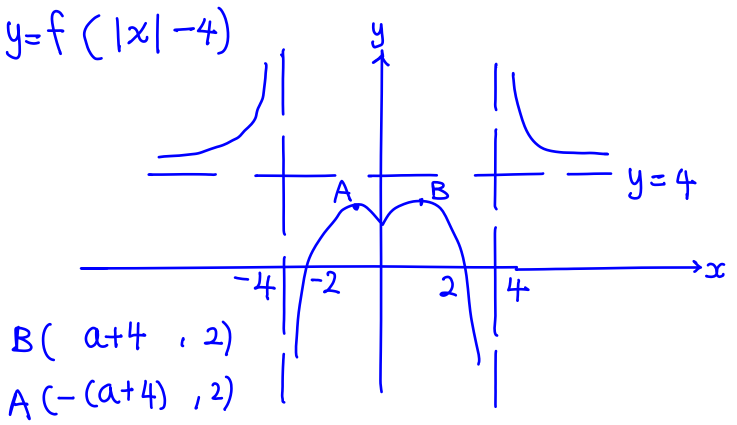
<tr><td>aiii</td><td>
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<tr><td>b</td><td>

$x^2 - \frac{y^2}{4} = 1 \xrightarrow{A} x^2 - y^2 = 1 \xrightarrow{B} x^2 - (y-1)^2 = 1$

A: Scaling parallel to the y-axis with a scale factor of $\frac{1}{2}$.
B: Translation of 1 unit in the positive direction of the y-axis.

Alternatively

$x^2 - \frac{y^2}{4} = 1 \xrightarrow{A} x^2 - \frac{(y-2)^2}{4} = 1 \xrightarrow{B} x^2 - \frac{(2y-2)^2}{4} = 1$

Note that $x^2 - \frac{(2y-2)^2}{4} = 1$ is equivalent to

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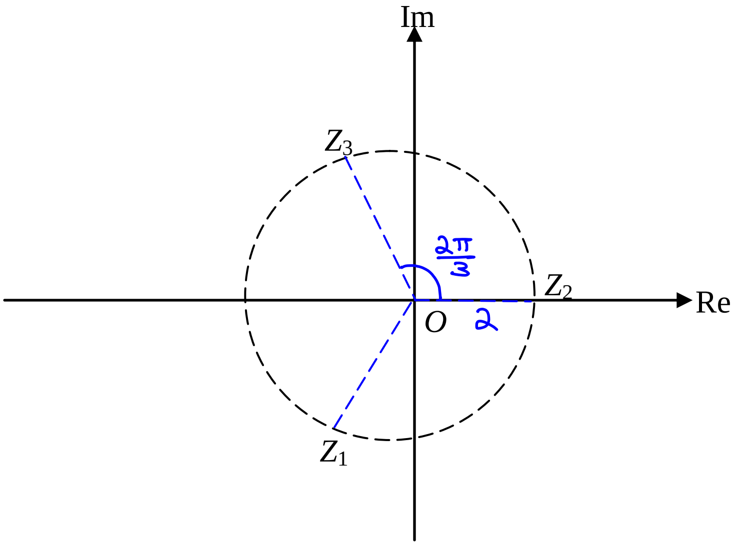
	$x^2-(y-1)^2=1$. A: Translation of 2 units in the positive direction of the y-axis. B: Scaling parallel to the y-axis with a scale factor of $\frac{1}{2}$. Note: Scale factors must be positive numbers.	

Question 4 [11 Marks]		
i	$z^3=8=8e^{i(0)}$ $z^3=8e^{i(0+2k\pi)}=8e^{i(2k\pi)}$ $z=2e^{i\left(\frac{2k}{3}\right)\pi}, k=0,\pm1$	
ii	$z_1=2e^{\left(-\frac{2\pi}{3}\right)i}$, $z_2=2$, $z_3=2e^{\left(\frac{2\pi}{3}\right)i}$  Note that the points Z_1, Z_2 and Z_3 lie on the circumference of a circle with centre O and radius 2. The points are equally spaced out separated by an angle of $\frac{2}{3}\pi$ radians. Z_1 and Z_3 represent complex numbers which are conjugates to each other. Thus Z_1 and Z_3 lie on the same vertical line.	
iii	Since $z=z_1$ satisfy the equation $\lvert z-z_2\rvert=\lvert z-z_3\rvert$ with $\lvert z_1-z_2\rvert=\lvert z_1-z_3\rvert=\sqrt{12}=2\sqrt{3}$, thus locus of P passes through the point Z_1. Note $\lvert z_1-z_2\rvert=\lvert z_1-z_3\rvert=\sqrt{12}$ can be easily found using GC.	

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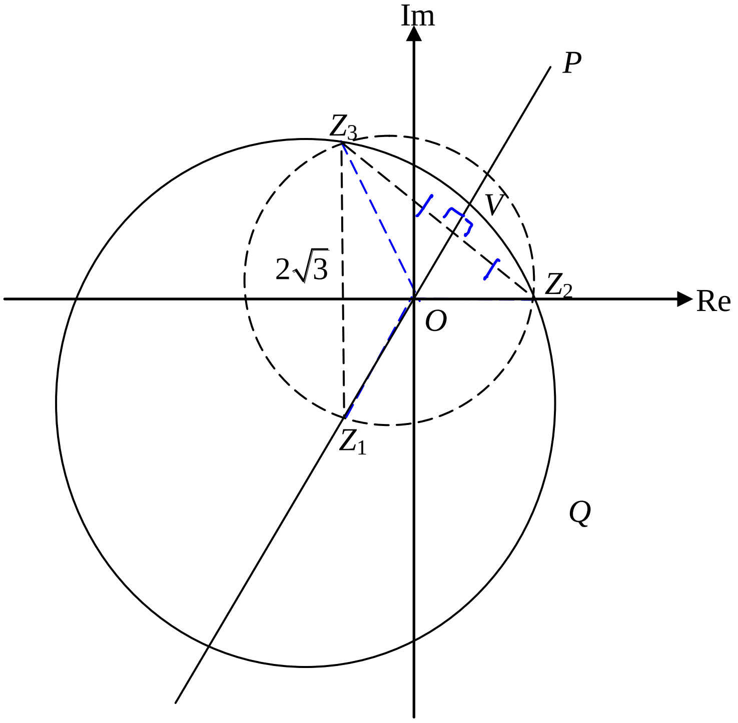
Locus of P: $|z-z_2|=|z-z_3|$, is the perpendicular bisector of the line joining Z_2 and Z_3. It passes through the origin and Z_1.
Locus of Q: $|w-z_1|=2\sqrt{3}$, is a circle with centre at Z_1 and radius $2\sqrt{3}$. It passes through Z_2 and Z_3.


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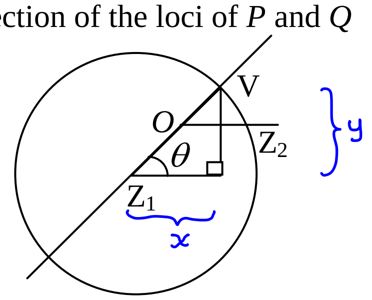
Let V be the point of intersection of the loci of P and Q in the first quadrant.


$\angle Z_1OZ_2 = \frac{2}{3}\pi$
$\angle VOZ_2 = \theta = \frac{1}{3}\pi$
$z_1 = 2e^{\left(-\frac{2\pi}{3}\right)i} = -1 - i\sqrt{3}$
$V \equiv v = -1 - i\sqrt{3} + (x + iy)$
where $x = 2\sqrt{3}\cos\theta = 2\sqrt{3}\left(\frac{1}{2}\right) = \sqrt{3}$,
$y = 2\sqrt{3}\sin\theta = 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 3$.
$\therefore v = -1 - i\sqrt{3} + (\sqrt{3} + 3i) = (-1+\sqrt{3}) + i(-\sqrt{3}+3)$

Alternatively
$|v| = Z_1V - Z_1O = 2\sqrt{3} - 2$
$\arg(v) = \angle VOZ_2 = \theta = \frac{1}{3}\pi$ (Refer to above working.)
$v = \left(2\sqrt{3}-2\right)e^{\left(\frac{\pi}{3}\right)i}$
$v = \left(2\sqrt{3}-2\right)\cos\left(\frac{\pi}{3}\right) + i\left(2\sqrt{3}-2\right)\sin\left(\frac{\pi}{3}\right)$
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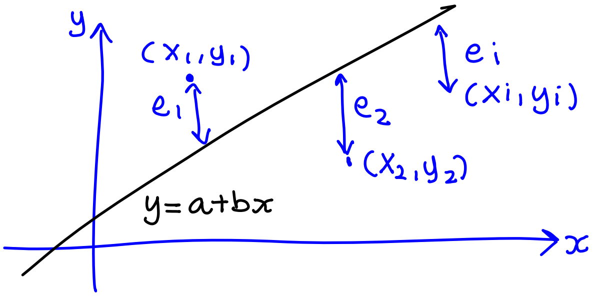
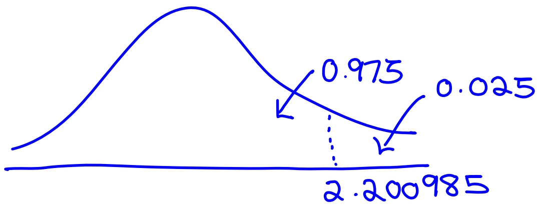
	$v=\left(2\sqrt{3}-2\right)\left(\frac{1}{2}\right)+i\left(2\sqrt{3}-2\right)\left(\frac{\sqrt{3}}{2}\right)$ $\therefore v=(-1+\sqrt{3})+\mathrm{i}(-\sqrt{3}+3)$	

Question 5 [4 Marks]		
i	The age group may be classified as 21 – 30, 31 – 40, 41 – 50 and last group with age more than 50. For each age group, interview 25 people which add up to 100.	
ii	The sample selected is biased and not representative of the district.	
iii	Stratified random sampling	

Question 6 [7 Marks]		
	No. of combinations = $2\cdot 3\cdot 3\cdot 4=72$	
	No. of ways $=[72\times(1\cdot 2\cdot 2\cdot 3)]\div 2^4$ $=54$ Alternatively No. of ways $=\binom{2}{2}\times\binom{3}{2}\times\binom{3}{2}\times\binom{4}{2}=54$	
	No. of ways = $2^8-1=255$ Alternatively, number. of ways = $\sum_{r=1}^{8}\binom{8}{r}=255$ Note: $\binom{8}{0}$ is not counted as the customer must choose at least 1 ingradient.	

Question 7 [8 Marks]		
a	Let the sample of bivariate data be (x_i, y_i) where $i = 1, 2, \ldots n$. Let $e_i = y_i - (a + bx_i)$ be the vertical deviation between the point (x_i, y_i) and the line $y = a + bx$.  The line $y = a + bx$ is the least square regression line for the sample of bivariate data if the sum of the squares of the vertical deviation $\sum_{i=1}^{n}(e_i)^2$ is the minimum.	
b i	From GC, r = 0.96953 $\approx$ 0.970 Although the value of r is close to 1, it does not indicate that the linear model is suitable as the scatter diagram may show a **curvilinear relationship** between the points.	
b ii	Calculate the r values for both models and the model with a higher absolute value of r closer to 1 is a better model. For $y = a + b\ln x$, value of r = 0.995> 0.970 $y = a + b\ln x$ = 1.4595375 + 4.2820506 ln$x$ Thus y = 1.46 + 4.30 ln$x$ is a better model.	
b iii	From (ii), $y =$ 1.4595375 + 4.2820506 ln$x$ Sub $y = 6.4$, then $x = 3.17$. Since $r = 0.995 \approx 1$, the regression line of $\ln x$ on y and the regression line of y on $\ln x$, it is reasonable to use the regression line of y on $\ln x$ for the estimate. –(1) In addition, $y = 6.4$ is within the input data range of $2.2 \le y \le 9.9$, the estimate is an interpolation. –(2) Hence, the estimate is reliable. Note: Both (1) and (2) should be stated to justify the reliability of the estimate. Simply quoting $r = 0.995 \approx 1$ in (1) is not acceptable.	

Question 8 [8 Marks]		
i	Let S and L be the r.v. denoting the amount of oil in a Small and Regular bottle respectively. $S \sim \mathrm{N}\left(545,\ 20^2\right)$ and $R \sim \mathrm{N}\left(1020,\ 50^2\right)$ $\mathrm{P}\left(R \geq \alpha\right) \geq 0.97$ $1-\mathrm{P}\left(R < \alpha\right) \geq 0.97$ $\mathrm{P}\left(R < \alpha\right) \leq 0.03$ $\alpha \leq 925.96$ Greatest $\alpha = 925$	
ii	$S_1 + S_2 + S_3 \sim \mathrm{N}\left(3 \times 545,\ 3 \times 20^2\right)$ $R_1 + R_2 \sim \mathrm{N}\left(2 \times 1020,\ 2 \times 50^2\right)$ Let $T = \frac{1}{5}\left(S_1 + S_2 + S_3 + R_1 + R_2\right)$ $S_1 + S_2 + S_3 + R_1 + R_2$ $\sim \mathrm{N}\left(3 \times 545 + 2 \times 1020,\ 3 \times 20^2 + 2 \times 50^2\right)$ $T \sim N\left(\frac{1}{5} \times 3675,\ \frac{1}{5^2} \times 6200\right)$ $T \sim N\left(735,\ 248\right)$ $\mathrm{P}\left(T < 730\right) = 0.375$	
iii	$S \sim \mathrm{N}\left(545,\ 20^2\right)$ P(bottle with the least content holds more than 560 ml) = P(all 3 bottles holds more than 548 ml of oil) $= \left[\mathrm{P}\left(S > 547\right)\right]^3$ $= 0.0974$ (3.s.f)	
iv	Suppose $R \sim \mathrm{N}\left(1020,\ 50^2\right)$ $P\left(0 < R < 1080\right) = 0.885$ OR $P\left(R \geq 1080\right) = 0.115$ There is a probability of 0.115 that the Regular bottle of cooking oil will not fall within the bottle, hence using a normal distribution is not appropriate. Alternatively, Suppose $R \sim \mathrm{N}\left(1020,\ 50^2\right)$, then $\mu = 1020$ and $\sigma = 50$ $P\left(\mu - 3\sigma < R < \mu - 3\sigma\right) = P\left(930 < R < 1230\right) = 0.997$ There is a probability of 0.997 that the amount of cooking oil in a Regular bottle falls within the range 930 ml to 1230 ml. A wide range of this set of values exceeds the capacity of the bottle i.e. 1080 ml, hence using a normal distribution is not appropriate.	

Question 9 [11 Marks]		
i	$\mathrm{H}_0 : \mu = \mu_0$ $\mathrm{H}_1 : \mu > \mu_0$	
ii	It means that there is a probability of 0.025 that the test concludes that the mean time spent by a technician of Open Network to install a fibre optic cable unit in a household unit is more than μ_0 minutes when in fact it is not true.	
iii	$s^2 = \dfrac{n}{n-1} \times \text{sample variance} \Rightarrow s^2 = \dfrac{12}{11} \times 4.5^2$ Under H_0, $T = \dfrac{\overline{\mathrm{X}} - \mu_o}{s/\sqrt{n}} \sim t(11)$  Using GC, invT(0.975,11) gives 2.200985 H_0 is not rejected if $t_{cal} \le 2.200985$ $\dfrac{21.3 - \mu_0}{\frac{\sqrt{12}}{\sqrt{11}}(4.5) \big/ \sqrt{12}} \le 2.200985$ $21.3 - \mu_0 \le \dfrac{(2.200985)(4.5)}{\sqrt{11}}$ $\mu_0 \ge 18.31370$ $\mu_0 \ge 18.4$ (3s.f.)	
	The time spent by a technician of Open Network to install a fibre optic cable unit in a household unit, is assumed to be normally distributed. Note: In this case, *X* is defined in the question. Therefore the statement "*X* is normally distributed" is acceptable. Candidates are to ensure that *X* is properly defined if they are to use this statement	
iv	Under H_0, $Z = \dfrac{\overline{\mathrm{X}} - \mu_o}{\sigma/\sqrt{n}} \sim N(0,1)$ Taking $\mu_0 = 19$, $n = 12$, $\bar{x} = 21.3$ and $\sigma = 5$ Using GC – Z Test, p-value $= 0.0555255$ Since H_0 is rejected, p-value $< \dfrac{\alpha}{100}$ $0.0555255 < \dfrac{\alpha}{100} \Rightarrow \alpha > 5.552$ The least significance level for which H_0 is rejected is 5.56%.	

Question 10 [11 Marks]		
a	Given that $\mathrm{P}(B)=0.4$ and $\mathrm{P}(A\mid B')=0.15$, $\mathrm{P}(A\mid B')=0.15$ $\dfrac{\mathrm{P}(A\cap B')}{\mathrm{P}(B')}=0.15$ $\dfrac{\mathrm{P}(A\cup B)-\mathrm{P}(B)}{1-\mathrm{P}(B)}=0.15$ $\dfrac{\mathrm{P}(A\cup B)-0.4}{1-0.4}=0.15$ $\mathrm{P}(A\cup B)=0.49$	
b	P(wins grand prize in 2^{nd} round) $=\dfrac{2}{6}\cdot\dfrac{1}{5}=\dfrac{1}{15}$ (Shown)	
bi	P(correct key in 1st round $\cap$ wins grand prize) = P(1^{st} correct, 2^{nd} correct) + P(1^{st} correct, 2^{nd} wrong, 3^{rd} correct) $=\left(\dfrac{2}{6}\cdot\dfrac{1}{5}\right)+\left(\dfrac{2}{6}\cdot\dfrac{4}{5}\cdot\dfrac{1}{4}\right)=\dfrac{2}{15}$	
bii	P(correct key selected in first round | wins grand prize) $=\dfrac{\text{P(correct key in 1st round } \cap \text{ wins grand prize)}}{\text{P(wins grand prize)}}$ $=\dfrac{\frac{2}{15}}{\left(\frac{2}{6}\cdot\frac{1}{5}\right)+\left(\frac{2}{6}\cdot\frac{4}{5}\cdot\frac{1}{4}\right)+\left(\frac{4}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}\right)}$ $=\dfrac{2}{15}\Big/\dfrac{3}{15}$ $=\dfrac{2}{3}$	
	Let X be the random variable for the no. of shows in a week (out of 5 shows) for which the grand prize is won in the 2^{nd} round. Then $X\sim \mathrm{B}\left(5,\dfrac{1}{15}\right)$ Let $p=\frac{1}{15}$, then $\mathrm{E}(X)=5p$, $\mathrm{Var}(X)=5p(1-p)$ For $n=60$ weeks, since n is large, $\overline{X}\sim \mathrm{N}\left(5p,\dfrac{5p(1-p)}{60}\right)$ <u>approximately by **Central Limit Theorem.**</u> $\overline{X}\sim \mathrm{N}\left(\dfrac{1}{3},\dfrac{7}{1350}\right)$ Required probability = $\mathrm{P}(\overline{X}>0.4)$ = 0.177 (correct to 3 s.f. Note: Application of Central Limit Theorem must be clearly stated.	

Question 11 [11 Marks]		
i	Let X and Y denote the number of male and female customers entering the cafe in a randomly chosen 30 minute period. $X \sim Po(2)$, $Y \sim Po(4)$ $X+Y \sim Po(6)$ Required probability $= \mathrm{P}\left(X+Y \geq 6\right)$ $= 1-\mathrm{P}\left(X+Y \leq 5\right)$ $= 0.5543203586 \approx 0.554$ (3.s.f)	
ii	Let W denote the number of days on which more than 5 customers enter the café between 9:00 am and 9:30 am, out of 60 days. $W \sim B(60, 0.5543203586)$ Since $n = 60 \geq 50$ is large, $np = 33.25 > 5$ and $nq = 26.74 > 5$ Therefore $W \sim N(33.2592, 14.8229)$ approximately $\mathrm{P}\left(W \geq 30\right) \overset{cc}{=} \mathrm{P}\left(W \geq 29.5\right)$ $= 0.836$ (3.s.f)	
iii	Let A and B denote the number of male and female customers entering the cafe in a randomly chosen 15 minute period. $A \sim Po(1)$, $B \sim Po(2)$ $A+B \sim Po(3)$ $\mathrm{P}\left(A \leq 2 \middle\vert A+B \leq 4\right)$ $= \dfrac{\mathrm{P}\left(A \leq 2 \cap A+B \leq 4\right)}{\mathrm{P}\left(A+B \leq 4\right)}$ $= \dfrac{\begin{bmatrix} \mathrm{P}\left(A=0\right)\cdot\mathrm{P}\left(B \leq 4\right)+\mathrm{P}\left(A=1\right)\cdot\mathrm{P}\left(B \leq 3\right) \\ +\mathrm{P}\left(A=2\right)\cdot\mathrm{P}\left(B \leq 2\right) \end{bmatrix}}{\mathrm{P}\left(A+B \leq 4\right)}$ $= 0.967$ (3.s.f)	
iv	The Poisson distribution may not be a good model for a day as the **mean** number of customers entering the café may fluctuate, with more customers during the lunch and dinner period.	