RIVER VALLEY HIGH SCHOOL
2019 JC2 Preliminary Examination
Higher 2

NAME	

CLASS							INDEX NUMBER		

MATHEMATICS

9758/01

Paper 1

19 September 2019

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

For examiner's use only	
Question number	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

READ THESE INSTRUCTIONS FIRST

Write your class, index number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
You are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Calculator Model:

This document consists of 5 printed pages.

1 Find the range of values of x that satisfy

$$\frac{2x}{x+5} < \frac{1}{x-1}.$$ [4]

Hence deduce the range of values of x that satisfy

$$\frac{2|x|}{|x|+5} - \frac{1}{|x|-1} < 0.$$ [2]

2. A curve $y = \mathrm{f}(x)$ passes through the point (0 , 2) and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{2y}$.

(i) Obtain the Maclaurin series of y in ascending powers of x, up to and including the term in x^2.
Write down the equation of the tangent to the curve $y = \mathrm{f}(x)$ at $x = 0$. [4]

(ii) Show that $y = \sqrt{4 + \sin x}$. [2]

(iii) Using your results in part (ii), and assuming x to be sufficiently small for terms in x^3 and higher powers to be ignored, obtain the binomial series of y. [3]

3. The curve C has equation

$$y = \frac{2x-a}{2x^2-b},$$

where a and b are positive integers.

C passes through the point $\left(\frac{9}{2}, 0\right)$ and the equation of an asymptote of C is $x = 3$.

(i) Show that $a = 9$ and $b = 18$. [2]

(ii) Sketch C, stating clearly the equations of asymptotes, the x-coordinates of turning points and axial intercepts. [4]

(iii) Hence, giving your reasons, deduce the range of values of h such that the graph of $\frac{(x-8)^2}{h^2} + \frac{y^2}{100} = 1$, where h is a positive integer, intersects C at exactly 6 distinct points. [2]

4. A function is said to be self-inverse when $\mathrm{f} = \mathrm{f}^{-1}$ for all x in the domain of f.

Given that the function f is defined by

$$\mathrm{f} : x \mapsto \frac{ax+b}{cx-a}, \text{ for } x \in \mathbb{R}, x \neq \frac{a}{c},$$

where a, b and c are positive constants.

(i) Show that f is self-inverse. [2]

(ii) Using the result of part (i), deduce $\mathrm{f}^2(x)$ and state the range of f^2. [2]

(iii) Solve the equation $\mathrm{f}^{-1}(x) = x$, leaving your answers in the exact form. [3]

For the rest of the question, let $a = 2$, $b = 5$ and $c = 3$.
The function g is defined by $\mathrm{g} : x \mapsto \mathrm{e}^x + 2$ for $x \in \mathbb{R}$.

(iv) Show that the composite function fg exists, justifying your answer clearly. [1]

(v) By considering the graph of f, or otherwise, find the exact range of fg. [2]

5 Consider the following definitions:

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \quad \text{and} \quad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

They are known as *hyberbolic functions*. They are used in modeling suspended bridges and equations of motion related to skydiving.

(i) Find an expression for $\tanh x$ in terms of e^{-2x}. [1]

(ii) Given that

$$f(n+1) - f(n) = \sinh x\left[\operatorname{sech}\left(n+\frac{1}{2}\right)x\right]\left[\operatorname{sech}\left(n-\frac{1}{2}\right)x\right],$$

where $f(n) = \tanh\left(n - \frac{1}{2}\right)x$ and $x \neq 0$, find an expression for

$$S_N = \sum_{n=1}^{N}\left[\operatorname{sech}\left(n+\frac{1}{2}\right)x\right]\left[\operatorname{sech}\left(n-\frac{1}{2}\right)x\right]$$

in the form $\left(\operatorname{cosech} x\right)\left(\tanh Ax - \tanh\frac{1}{2}x\right)$ where A is a constant to be determined. [3]

(iii) Explain why S_∞ exists. Deduce an expression for S_∞ in the form $\left(\operatorname{cosech} x\right)\left(P - \tanh Qx\right)$, where P and Q are constants to be determined. [3]

6 (a) Find the roots of the equation $(1+i)z^2 - z + (2-2i) = 0$, giving your answers in the form $x + iy$, where x and y are exact real numbers. [4]

(b) Given that $f(z) = z^4 - 2z^3 + z^2 + az + b$, where a, $b \in \mathbb{R}$, and that $1 + 2i$ satisfies the equation $f(z) = 0$, find the values of a and b and the other roots. [5]
Hence solve the equation $w^4 - 2iw^3 - w^2 - 8iw - 20 = 0$, showing your workings clearly. [2]

7 The area A of the region bounded by $y = \sin^6 x$, the x-axis, $x = 0$ and $x = \dfrac{\pi}{2}$ is to be found.

(i) The area A can be approximated by dividing the region into 5 vertical strips of rectangles of equal width.
Show that $0.10625\pi < A < 0.20625\pi$. [3]

(ii) It is given that $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$, where $n \geq 0$.
By writing $\sin^{2n} x = (\sin x)(\sin^{2n-1} x)$, show by integration by parts that

$$I_n = \frac{2n-1}{2n} I_{n-1}, \text{ for } n \geq 1.$$

Deduce that $I_2 = \dfrac{3}{16}\pi$.
Using the value of I_2, obtain the exact value of A. [7]

8. Two planes have equations given by

$$p_1 : x + 5y + 4z = 4,$$
$$p_2 : x - y + z = 4.$$

(i) Explain why p_1 and p_2 intersect in a line ℓ and determine the cartesian equation of ℓ. [3]

(ii) The plane p_3 contains the line ℓ and the point $Q(5, 3, -6)$, find a cartesian equation of p_3.
Describe the geometrical relationship between these 3 planes and ℓ. [5]

(iii) The line m passes through the points $S(0,2,0)$ and $T(4,0,0)$. Find the position vector of the foot of perpendicular of S on p_2.
Hence find a vector equation of the line of reflection of m in p_2. [5]

9 On 1 January 2019, Mr Tan started a savings account with a bank with an initial deposit of \$5000. The bank offered compound interest of 1% computed based on the amount of money in this account at the end of each month. Due to personal financial needs, Mr Tan withdrew \$100 from the savings account at the beginning of each month starting from the second month, i.e. 1 February 2019.

(i) Find the amount of money in Mr Tan's account at the end of March 2019. [3]

(ii) Deduce that the amount of money in Mr Tan's account at the end of the n^{th} month is given by $\$100\left(101 - 50\left(1.01^n\right)\right)$. [3]

(iii) Determine the month and year Mr Tan will deplete the savings in his account if he continues to withdraw money from this account. [2]

On 1 January 2019, Mrs Tan also started a savings account with the same bank with an initial deposit of \$3000. The bank offered her interest of \$10 for the first month and increment of \$5 for each subsequent month. For example, in the first 3 months, her interest was \$10 for January, \$15 for February and \$20 for March. In addition, Mrs Tan further deposits \$50 into her savings account every mid-month starting from 15 January 2019.

(iv) Determine the month and year Mrs Tan will first have more money in her account than that of Mr Tan by the end of month. [5]

10 Since the 1990s, a group of scientists has started conservation work to prevent the extinction of a rare species of flying fox in the wild Western Australia forests. The number of such species, in thousands, observed in the forests at time t years after the start of the conservation is denoted by N. It is known that the death rate of the flying foxes is proportional to the number of flying foxes present and that the birth rate of the flying foxes has been maintained constant at 2000 per year. There were only 1000 flying foxes in the forests at the start of the conservation and the rate of increase of the number of flying foxes was 1500 per year when there were 3000 flying foxes.

(i) Form a differential equation relating N and t and show that it can be reduced to

$$6\frac{dN}{dt} = 12 - N.$$ [2]

(ii) Solve the differential equation to find N in terms of t. [5]

(iii) Deduce the minimum number of years in integer, needed for the number of flying foxes to exceed 6000. [2]

(iv) Sketch a graph to show how the number of flying foxes in the forests varies with time and suggest the long term behavior of the flying fox population. [3]

(v) Suggest a possible limitation of the model in part (i). [1]

END OF PAPER

RIVER VALLEY HIGH SCHOOL
2019 JC2 Preliminary Examination
Higher 2

NAME	

CLASS					

INDEX NUMBER		

MATHEMATICS 9758/02

Paper 2

23 September 2019

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

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For examiner's use only	
Question number	Mark
1	
2	
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Total	

Calculator Model:

This document consists of **7** printed pages.

Section A: Pure Mathematics [40 marks]

1 A sequence u_0, u_1, u_2, … is such that $u_{n+1} = -3u_n + An + B$, where A and B are constants and $n \geq 0$.

(i) Given that $u_0 = 4$, $u_1 = 2$ and $u_2 = 16$, find the values of A and B. [3]

It is known the nth term of the same sequence is given by $u_n = a(-3)^n + bn + c$, where a, b and c are constants.

(ii) By considering a system of linear equations, find the values of a, b and c. [3]

2 Two planes p_1 and p_2 have equations

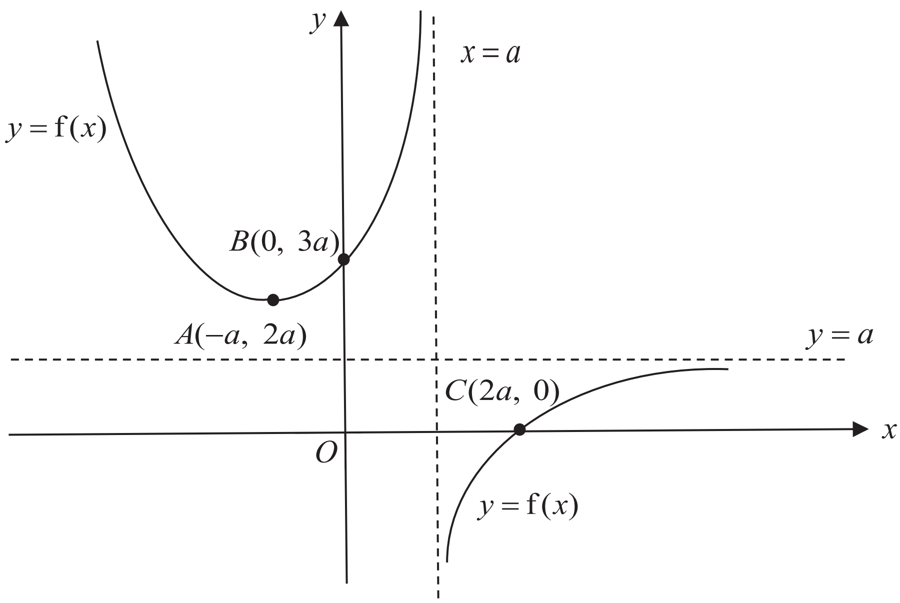
$$p_1 : \mathbf{r} = \mathbf{a} + \lambda_1\mathbf{m} + \mu_1\mathbf{n},$$
$$p_2 : \mathbf{r} = \mathbf{b} + \lambda_2\mathbf{m} + \mu_2\mathbf{n},$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

(a) The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The angle between $\mathbf{a}$ and $\mathbf{m}\times\mathbf{n}$ is 45° and that between $\mathbf{b}$ and $\mathbf{m}\times\mathbf{n}$ is 60°. Show that the distance between p_1 and p_2 is given by $\left|\dfrac{|\mathbf{b}|}{2} - \dfrac{|\mathbf{a}|}{\sqrt{2}}\right|$. [2]

(b) Q is a variable point between p_1 and p_2 such that the distance of Q from p_1 is twice that of the distance from p_2. Write down a possible position vector of Q, in terms of $\mathbf{a}$ and $\mathbf{b}$. Describe the locus of Q and write down its vector equation. [3]

3 (a) The diagram below shows the graph of $y = \mathrm{f}(x)$. The graph has a minimum point at $A(-a,\ 2a)$ where $a > 0$. It passes through the points $B(0,\ 3a)$ and $C(2a,\ 0)$. The equations of the asymptotes are $x = a$ and $y = a$. Each of the gradients of the tangents at points B and C is a.



Sketch on separate clearly labelled diagrams, the graphs of

(i) $y = \mathrm{f}(2x + a)$, [2]

(ii) $y = \mathrm{f}'(x)$, [2]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

Label in each case, the coordinates of points corresponding to the points *A*, *B* and *C* (if any) of $y = \mathrm{f}(x)$ and the equation of asymptote(s).

(b) Describe a sequence of transformations which transforms the graph of $2y^2 - x^2 = 1$ onto the graph of $y^2 - (x-1)^2 = 1$. [2]

4 The parametric equations of a curve are

$$x = \sin t, \quad y = \cos t, \quad \text{where } 0 < t < \frac{\pi}{2}.$$

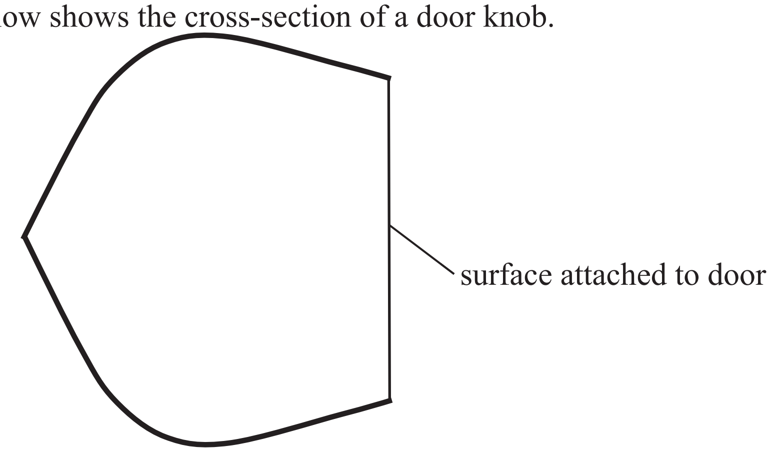
(i) Sketch the graph of the curve. [1]

(ii) Show that the equation of the tangent to the curve at the point where $t = p$ is given by the equation $y = -(\tan p)x + \sec p$. [3]

(iii) Given that the above tangent meets the *x*-axis at the point *P* and the *y*-axis at the point *Q*, find the coordinates of the mid-point *M* of *PQ* in terms of *p*.
Hence, determine the cartesian equation of the locus of *M*. [3]

(iv) Find the exact area of the region enclosed by the curve, the lines $x = \dfrac{1}{2}$, $x = \dfrac{1}{\sqrt{2}}$ and the *x*-axis. [3]

5 The diagram below shows the cross-section of a door knob.

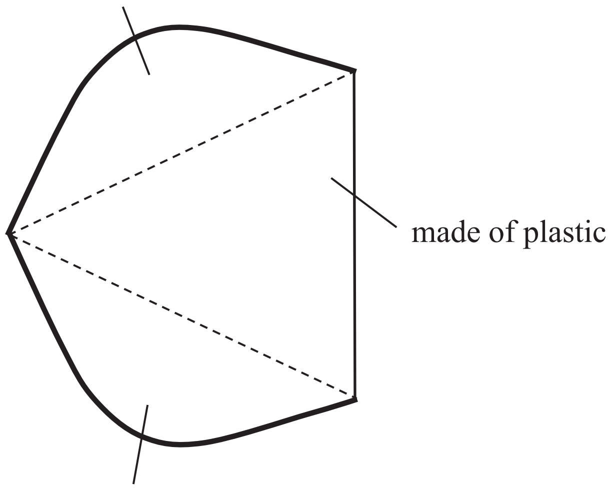


The curve part of the door knob can be modelled by the curve *C* with equation $y = \dfrac{2x}{1+x^2}$ between $x = 0$ and $x = 2$. The solid door knob is obtained by rotating *C* through four right angles about the *x*-axis.

(i) Sketch the graph of *C* for $0 \le x \le 2$. [1]

(ii) Using the substitution $x = \tan\theta$, find the exact volume of the door knob. [6]

The exterior and the interior of the door knob is made of wood and plastic respectively. The interior of the door knob consists of a solid cone.



The cost of wood and plastic used in manufacturing of the door knob is \$3 and \$1.20 per unit volume respectively. Find the cost of manufacturing one such door knob, giving your answer to the nearest dollar. [3]

Section B: Probability and Statistics [60 marks]

6 At a funfair game store, 3 red, 2 blue, 1 white, 1 yellow and 1 green beads are being arranged. The beads are identical apart from their colours.

(i) Find the number of ways to arrange all the beads in a row with all the red beads separated. [2]

(ii) Five beads, one of each colour, are string together to form a rigid ring. Find the number of distinct rings that can be formed. [2]

At another game store, the letters of the word INITIATE are being arranged in a row instead. Find the probability that all the letters are used and the letters A, E and N are arranged in alphabetical order. [2]

7 In a game, Alfred is given 6 keys of which n of the keys can unlock a box. He tries the keys randomly without replacement to unlock the box.
The random variable X denotes the number of tries Alfred takes to unlock the box.

(i) Write down an expression for $\text{P}(X = 3)$ in terms of n. [1]

Use $n = 3$ for the rest of this question.
For the game, Alfred wins \$2.00 if he takes less than 4 tries to open the box else he wins \$1.00. The random variable W denotes Alfred's winnings in a game.

(ii) Find the probability distribution table for W and hence find the value of $\text{E}(W)$. [2]

(iii) Find the probability that Alfred wins at least \$9.00 after playing the game 5 times. [3]

8 Two boxes, one white and one black, are used in a game. The boxes contains balls labelled '2', '5' and '10'. The balls are identical except for their numbers. The table below shows the number of balls in each box.

	Number of balls labelled		
	'2'	'5'	'10'
White box	5	2	1
Black box	5	1	2

In the first round of the game, a player draws 3 balls randomly from the white box. If the sum of the numbers on the 3 balls add up to 9 or more, the player enters the second round of the game to draw another 3 balls randomly from the black box.

(i) Find the probability that the player draws 3 different numbers in the first round. [1]

(ii) Show that the probability that sum of the numbers add up to 12 and 6, in the first and second round respectively, is $\frac{25}{1568}$.
Hence, find the probability that the sum of the numbers drawn from the two rounds adds up to 18. [4]

In the third round of the game, only the original black box with its 8 balls is used. The player draws 2 balls from it randomly. Find the expected sum of the numbers drawn. [2]

9 A multiple-choice test consists of 12 questions and each question has 5 options for a candidate to choose from. The candidate has to choose one option as the answer for each question. For every question that a candidature answered correctly, the candidate is awarded 1 mark. There are no marks awarded or deducted for wrong answers. Let X denotes the number of marks a candidate scores for the test.

(i) State an assumption on how a candidate decides which option to choose in order for X to be modelled by a binomial distribution. Explain why this assumption is necessary. [2]

Suppose the assumption stated in part (i) holds.

(ii) Find the most probable number of marks that a candidate will score in the test. [2]

(iii) A candidate found that he will score at most 5 marks. Find the probability that he will score more than 4 marks. [2]

(iv) A candidate sat for n such tests (where n is large), with each test consisting of 12 questions and each question has 5 options. Determine the least value of n so that the probability of obtaining a mean test score of at most 2.7 marks is 0.95. [2]

10 Engineers claimed that a newly developed engine is efficient. The number of hours x the engine ran on one litre of fuel was measured on 70 occasions. The results are summarized by

$$\sum(x-46)=-27, \quad \sum(x-46)^2=30939.$$

(i) Calculate unbiased estimates for the population mean and variance of the number of hours the engine runs on one litre of fuel. [2]

(ii) Determine the greatest mean number of hours the engineers should claim so that there is sufficient evidence at the 5% level of significance that the engine is efficient. [3]

(iii) The engineers collected five more data points:

56, 34, 63, 50, 54.

The engineers found that when all 75 data points are considered, they can no longer claim that the engine is efficient at the α% level of significance.
Show that the unbiased estimate for the population variance, for the set of 75 data points is 426.378, correct to 3 decimal places. Find the greatest integer value of α. [4]

11 (a) The equation of the estimated least squares regression line of y on x for a set of bivariate data is $y=a+bx$. Explain what do you understand by the least squares regression line of y on x. [2]

(b) A student wishes to determine the relationship between the length of a metal wire l in millimetres, and the duration of time t in minutes for it to completely dissolve in a particular chemical solution. After conducting the experiment, he obtained the following set of data.

l	15	30	45	60	75	90	105	120	135	150
t	0.779	1.10	1.35	1.56	1.74	1.91	2.07	2.22	2.31	2.45

(i) Draw a scatter diagram to illustrate the data. [1]

(ii) State with a reason, which of the following model is appropriate for the data collected.

(A) $t=a+bl^2$, (B) $t=a+b\ln l$, (C) $t=ae^{bl}$,

where a and b are some constants. [2]

(iii) For the above chosen model, calculate the values of a and b and the product moment correlation coefficient. [2]

(iv) Using the model in part (iii), estimate the length of metal wire that took 1.00 minute to dissolve completely in the chemical solution.
Comment on the reliability of your answer. [3]

(v) Given that 1 millimetre = 0.03937 inch, find an equation that can be used to estimate the duration of time taken to completely dissolve a metal wire of length L, where L is measured in inches. [2]

12 An orchard produces apples and pears.

(i) The masses of apples produced by the orchard have mean 70 grams and standard deviation 40 grams.

Explain why the masses of apples in the orchard are unlikely to be normally distributed. [1]

(ii) The masses of pears produced by the orchard are normally distributed. It is known that one-third of the pears produced weigh less than 148 grams, and one-third weigh more than 230 grams. Find the mean and standard deviation of the masses of pears produced in the orchard. [4]

During each production period, the orchard produces 300 apples and 400 pears and the owner is able to sell all the apples at \$0.005 per gram and all the pears at \$0.008 per gram. It is further known that the owner needs to spend a fixed amount of \$$c$ as the operating cost for each production period and that the orchard makes a profit 90% of the time.

(iii) Explain why the total mass of the 300 apples is approximately normally distributed. [1]

(iv) By letting R be the amount of money collected from selling the 300 apples and 400 pears, find the distribution of R and hence, determine the value of c. [6]

END OF PAPER

2019 RVHS H2 Maths Prelim P1 Solutions

1 Solution [5] Inequality

$$\frac{2x}{x+5} < \frac{1}{x-1} \Rightarrow \frac{2x}{x+5} - \frac{1}{x-1} < 0 \text{ --------------- (1)}$$

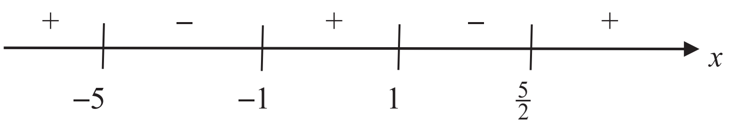
Then, we have

$$\frac{2x^2 - 2x - x - 5}{(x+5)(x-1)} < 0$$

$$\frac{2x^2 - 3x - 5}{(x+5)(x-1)} < 0$$

$$\frac{(2x-5)(x+1)}{(x+5)(x-1)} < 0$$

Applying number line test:



Therefore, the solution to inequality (1) is

$-5 < x < -1$ or $1 < x < \frac{5}{2}$.

Next, we replace 'x' by '$|x|$' in inequality (1)

to obtain $\frac{2|x|}{|x|+5} - \frac{1}{|x|-1} < 0$ -------------- (2)

Thus, the solution to inequality (2) correspondingly is

$$-5 < |x| < -1 \quad \text{or} \quad 1 < |x| < \frac{5}{2}$$

i.e. $-5 < |x| < -1$ (NA) or $1 < |x| < \frac{5}{2}$

Thus, the solution to inequality (2) is

$-\frac{5}{2} < x < -1$ or $1 < x < \frac{5}{2}$.

2	Solution [8] Maclaurin's Series	
	(i) $\frac{dy}{dx} = \frac{\cos x}{2y}$ $2y\frac{dy}{dx} = \cos x$ Diff Implicitly w.r.t x, $2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dy}{dx} = -\sin x$ $2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = -\sin x$ When $x = 0$, $y = 2$ (GIVEN) Hence $\frac{dy}{dx} = \frac{1}{4}$ $(2)(2)\frac{d^2y}{dx^2} + 2\left(\frac{1}{4}\right)^2 = 0$ $\frac{d^2y}{dx^2} = -\frac{1}{32}$ Using the Maclaurin's formula, $y = 2 + x(\frac{1}{4}) + \frac{x^2}{2!}(-\frac{1}{32}) + ...$ $\approx 2 + \frac{x}{4} - \frac{x^2}{64}$ (up to the x^2 term) Equation of tangent at $x = 0$: $y = 2 + \frac{1}{4}x$	
	(ii) $\frac{dy}{dx} = \frac{\cos x}{2y}$ $\int 2y \, dy = \int \cos x \, dx$ $y^2 = \sin x + C$ Subst $(0, 2)$, then $C = 4$ $y = \pm\sqrt{4 + \sin x}$ Since $x = 0$ and $y = 2$, $\therefore y = \sqrt{4 + \sin x}$	

	(iii) $y=\sqrt{4+\sin x}$ $=(4+\sin x)^{\frac{1}{2}}$ $\approx(4+x)^{\frac{1}{2}}$ (since x is small) $=2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$ $=2\left(1+\frac{1}{2}.\frac{x}{4}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)^2}{2!}+...\right)$ $\approx 2+\frac{x}{4}-\frac{x^2}{64}$	

3	Solution [8] Curve Sketching	

(i)

Since $x=3$ is an asymptote,

$2(3)^2 - b = 0$

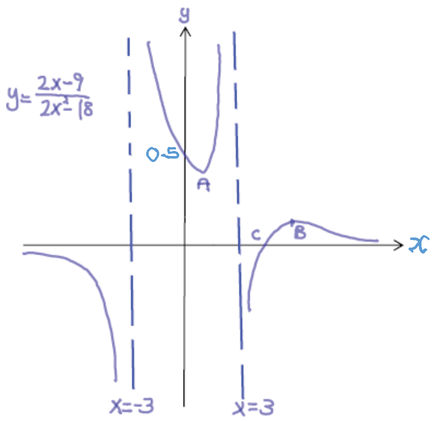
$b = 18$

C passes through $\left(\frac{9}{2}, 0\right)$ implies

$2\left(\frac{9}{2}\right) - a = 0$

$a = 9$

(ii)

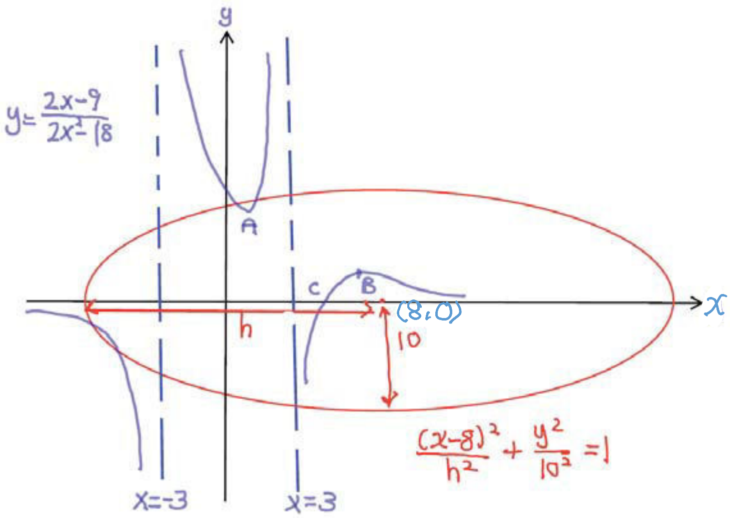


A(1.15, 0.436)
B(7.85, 0.0637)
C(4.50, 0.00)

TO find stationary points, set $\frac{dy}{dx} = 0$

And work through the maths to do it. There should be 2 stationary points. This is actually the most important part. The rest is the axes intercepts (let x=0 and then let y=0), identify the asymptotes. And the shape is important – especially the part on moving as close to the asymptote as possible.

(iii)



$\frac{(x-8)^2}{h^2}+\frac{y^2}{10^2}=1$ is the ellipse with centre at $(8,0)$, with axes of length h and 10.

Horizontal width need to be at least $8+4=12$ units.

Therefore $h\geq 12$, for 6 distinct points of intersections.

4	Solution [10] Functions	
	(i) Let $y=\dfrac{ax+b}{cx-a}, x\in\mathbb{R}, x\neq\dfrac{a}{c}$ $y(cx-a)=ax+b$ $x(cy-a)=ay+b$ $x=\dfrac{ay+b}{cy-a}$ Replacing y by x, $\therefore \mathrm{f}^{-1}(x)=\dfrac{ax+b}{cx-a}, x\in\mathbb{R}, x\neq\dfrac{a}{c}$ Since $\mathrm{f}(x)=\mathrm{f}^{-1}(x)=\dfrac{ax+b}{cx-a}$, f is self-inverse. (SHOWN)	
	(ii) $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ Composing function f on both sides, $\mathrm{ff}(x)=\mathrm{ff}^{-1}(x)$ $\mathrm{f}^2(x)=x$ $\mathrm{D}_{\mathrm{f}^2}=\mathbb{R}\setminus\left\{\dfrac{a}{c}\right\}, \quad \mathrm{R}_{\mathrm{f}^2}=\mathbb{R}\setminus\left\{\dfrac{a}{c}\right\}$ or present as $\mathrm{R}_{\mathrm{f}^2}=\left(-\infty,\dfrac{a}{c}\right)\cup\left(\dfrac{a}{c},\infty\right)$	
	(iii) $\mathrm{f}^{-1}(x)=x$ $\dfrac{ax+b}{cx-a}=x$ $ax+b=cx^2-ax$ $cx^2-2ax-b=0$ $x^2-\dfrac{2a}{c}x-\dfrac{b}{c}=0$ $\left(x-\dfrac{a}{c}\right)^2=\dfrac{b}{c}+\left(\dfrac{a}{c}\right)^2$	

	$x-\frac{a}{c}=\pm\sqrt{\frac{bc+a^2}{c^2}}$ $x=\frac{a}{c}+\sqrt{\frac{bc+a^2}{c^2}}$ or $\frac{a}{c}-\sqrt{\frac{bc+a^2}{c^2}}$ $=\frac{a+\sqrt{bc+a^2}}{c}$ or $\frac{a-\sqrt{bc+a^2}}{c}$	
	(iv) Now, $a=2$, $b=5$ and $c=3$ $\text{f}(x)=\frac{2x+5}{3x-2}, x\in\mathbb{R}, x\neq\frac{2}{3}$ $\text{g}(x)=\text{e}^x+2, x\in\mathbb{R}$ FACT: For fg to exist , need $\text{R}_\text{g}\subseteq \text{D}_\text{f}$ to hold , $\text{R}_\text{g}=(2,\infty)\subseteq \mathbb{R}\setminus\left\{\frac{2}{3}\right\}=\text{D}_\text{f}$ fg does exist.	
	(v) $D_g=\mathbb{R}\overset{g}{\mapsto}R_g=(2,\infty)\overset{\text{f}}{\mapsto}R_{\text{fg}}=\left(\frac{2}{3},\frac{9}{4}\right)$ Therefore Range of fg is $\left(\frac{2}{3},\frac{9}{4}\right)$	

5	Solution [7] MOD	
	(i) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $= \dfrac{e^x(1-e^{-2x})}{e^x(1+e^{-2x})}$ $= \dfrac{1-e^{-2x}}{1+e^{-2x}}$	
	(ii) $f(n+1)-f(n) = [\sinh x][\operatorname{sech}\left(n+\frac{1}{2}\right)x][\operatorname{sech}\left(n-\frac{1}{2}\right)x]$ $\sum_{n=1}^{N} f(n+1)-f(n) = (\sinh x)\sum_{n=1}^{N}[\operatorname{sech}\left(n+\frac{1}{2}\right)x][\operatorname{sech}\left(n-\frac{1}{2}\right)x]$ $\sum_{n=1}^{N}[\operatorname{sech}\left(n+\frac{1}{2}\right)x][\operatorname{sech}\left(n-\frac{1}{2}\right)x] = \frac{1}{\sinh x}\sum_{n=1}^{N} f(n+1)-f(n)$ $S_n = \dfrac{1}{\sinh x}\sum_{n=1}^{N} f(n+1)-f(n)$ $= \dfrac{1}{\sinh x}\begin{bmatrix} \cancel{f(2)} & - & f(1) \\ \cancel{f(3)} & - & \cancel{f(2)} \\ \cancel{f(4)} & - & \cancel{f(3)} \\ & \dots & \\ \cancel{f(N)} & - & \cancel{f(N-1)} \\ f(N+1) & - & \cancel{f(N)} \end{bmatrix}$ $= \dfrac{1}{\sinh x}\left(f(N+1)-f(1)\right)$ $= (\operatorname{cosech} x)\left(\tanh(N+\frac{1}{2})x - \tanh(\frac{1}{2})x\right)$ $\therefore A = N + \frac{1}{2}$	
	(iii) S_∞ $= \lim_{N\to\infty}\sum_{n=1}^{N}\left[\operatorname{sech}\left(n+\frac{1}{2}x\right)\right]\left[\operatorname{sech}\left(n-\frac{1}{2}x\right)\right]$ $= \lim_{N\to\infty}(\operatorname{cosech} x)\left[\tanh\left(N+\frac{1}{2}\right)x - \tanh\left(\frac{1}{2}\right)x\right]$	

	Consider $\lim_{N\to\infty} \tanh\left(N+\frac{1}{2}\right)x$ $= \lim_{N\to\infty} \frac{1-e^{-2\left(N+\frac{1}{2}\right)x}}{1+e^{-2\left(N+\frac{1}{2}\right)x}}$ $=1$ Since $\tanh\left(N+\frac{1}{2}\right)x \to 1$ as $N\to\infty$, therefore S_∞ exists. S_∞ $=(\text{cosech}\, x)\left[1-\tanh\left(\frac{1}{2}\right)x\right]$ $P=1,\ Q=\frac{1}{2}$	

6	Solution [11] Complex Numbers	
	(a) $(1+\text{i})z^2 - z + (2-2\text{i}) = 0$ $z = \frac{1 \pm \sqrt{1-4(1+\text{i})(2-2\text{i})}}{2+2\text{i}}$ $= \frac{1 \pm \sqrt{1-8(1+\text{i})(1-\text{i})}}{2+2\text{i}}$ $= \frac{1 \pm \sqrt{1-8(2)}}{2+2\text{i}}$ $= \frac{1 \pm \sqrt{15}\text{i}}{2+2\text{i}} \quad ----(*)$ $= \frac{1+\sqrt{15}\text{i}}{2+2\text{i}} \quad or \quad \frac{1-\sqrt{15}\text{i}}{2+2\text{i}}$ $= \frac{(1+\sqrt{15}\text{i})(2-2\text{i})}{(2+2\text{i})(2-2\text{i})} \quad or \quad \frac{(1-\sqrt{15}\text{i})(2-2\text{i})}{(2+2\text{i})(2-2\text{i})}$ $= \frac{(1+\sqrt{15}) + \text{i}(-1+\sqrt{15})}{4} \quad or \quad \frac{(1-\sqrt{15}) + \text{i}(-1-\sqrt{15})}{4}$	
	(ii) $z^4 - 2z^3 + z^2 + az + b = 0$ ---- (*) Sub $z = 1+2\text{i}$ into (*), $(1+2\text{i})^4 - 2(1+2\text{i})^3 + (1+2\text{i})^2 + a(1+2\text{i}) + b = 0$ $(-7-24\text{i}) - 2(-11-2\text{i}) + (-3+4\text{i}) + a(1+2\text{i}) + b = 0$ $(12+a+b) + (2a-16)\text{i} = 0$ Comparing the real and imaginary coefficients: $2a - 16 = 0$ ---- (1) $12 + a + b = 0$ ---- (2) Solving (1) & (2): $a = 8,\ b = -20$ Therefore $z^4 - 2z^3 + z^2 + 8z - 20 = 0$ Using GC: $z = 1+2\text{i},\ 1-2\text{i},\ 2,\ -2$	
	(ii) Alternative Method $\text{f}(z) = z^4 - 2z^3 + z^2 + 8z - 20$ Since 1 + 2i is a root of f(z)=0, and the polynomial have all	

	real coefficients, this imply 1 – 2i is also a root. Hence $z^4-2z^3+z^2+az+b=(z-(1+2i))(z-(1-2i))(z^2+pz+q)$ $=((z-1)-2i)((z-1)+2i)(z^2+pz+q)$ $=((z-1)^2+4)(z^2+pz+q)$ $=(z^2-2z+5)(z^2+pz+q)$ Equating coeff of z^3 in f(z): $-2=p-2$. Hence $p=0$ Equating coeff of z^2 in f(z): $1=q-2p+5$. Hence $q=-4$ Now $z^4-2z^3+z^2+az+b=(z^2-2z+5)(z^2-4)$ Equating coeff of z in f(z): $a=8$ Equating constant in f(z): $b=-20$ **Hence** $a=8$ and $b=-20$ $f(z)=0$ $(z^2-2z+5)(z^2-4)=0$ Hence the other 2 roots are ± 2	
	$z^4-2z^3+z^2+az+b=0$ ----(1) Let $z=-iw$ $w^4-2iw^3-w^2-8iw-20=0$ ----(2) Therefore $w=\frac{1}{-i}z \Rightarrow w=iz$ $w=i(1+2i),\ i(1-2i),\ i(2),\ i(-2)$ $w=i-2,\ i+2,\ 2i,\ -2i$	

Question 7 [10] Integration		
(i)	When the left height of a rectangle is used, area of region $=\frac{\pi}{10}\left(0+\sin^6\frac{\pi}{10}+\sin^6\frac{2\pi}{10}+\sin^6\frac{3\pi}{10}+\sin^6\frac{4\pi}{10}\right)$ $=\frac{\pi}{10}(1.0625)=0.10625\pi$ When the right height of a rectangle is used, area of region $=\frac{\pi}{10}\left(\sin^6\frac{\pi}{10}+\sin^6\frac{2\pi}{10}+\sin^6\frac{3\pi}{10}+\sin^6\frac{4\pi}{10}+\sin^6\frac{5\pi}{10}\right)$ $=\frac{\pi}{10}(2.0625)=0.20625\pi$ Thus, $0.10625\pi < A < 0.20625\pi$.	
(ii)	$I_n=\int_0^{\frac{\pi}{2}}\sin^{2n}x\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{2}}\sin x\sin^{2n-1}x\,\mathrm{d}x$ $=\left[-\cos x\sin^{2n-1}x\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x(2n-1)\sin^{2n-2}x\cos x\,\mathrm{d}x$ $=(2n-1)\int_0^{\frac{\pi}{2}}\cos^2 x\sin^{2n-2}x\,\mathrm{d}x$ $=(2n-1)\int_0^{\frac{\pi}{2}}(1-\sin^2 x)\sin^{2n-2}x\,\mathrm{d}x$ $=(2n-1)\int_0^{\frac{\pi}{2}}\sin^{2n-2}x-\sin^{2n}x\,\mathrm{d}x$ $=(2n-1)(I_{n-1}-I_n)$ $\Rightarrow I_n+(2n-1)I_n=(2n-1)I_{n-1}$ $\Rightarrow I_n=\frac{2n-1}{2n}I_{n-1}$	

	I_2 $=\frac{3}{4}I_1$ $=\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)I_0$ $=\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{2}}1\,\mathrm{d}x,$ $=\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$ $=\frac{3}{16}\pi$ (Shown) Area of region I_3 $=\frac{5}{6}I_2$ $=\left(\frac{5}{6}\right)\left(\frac{3}{16}\pi\right)$ $=\frac{5}{32}\pi$	

8	Solution [13] Vectors	
	(i) $p_1: \mathbf{r} \bullet \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = 4, \quad p_2: \mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 4$ Since the two normal vectors are not parallel to each other, the 2 planes are not parallel and hence intersecting. From GC, $x = 4 - \frac{3}{2}\lambda \quad ------(1)$ $y = -\frac{1}{2}\lambda \quad ------(2)$ $z = \lambda \quad ------(3)$ Hence $\ell: \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ where $\lambda \in \mathbb{R}$ $\frac{x-4}{3} = y = -\frac{z}{2}$	
	(ii) p_3 contains $\ell: \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and point $Q(5,3,-6)$. Let T denote the point $(4,0,0)$. $\overrightarrow{QT} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ are 2 direction vectors parallel to p_3. Consider $\begin{pmatrix} -1 \\ -3 \\ 6 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 16 \\ 8 \end{pmatrix} = 8 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ Therefore $\mathbf{n} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is a normal vector to p_3.	

	$\mathbf{r}\cdot\mathbf{n}=\begin{pmatrix}4\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\2\\1\end{pmatrix}=0$ $p_3\ :\ \mathbf{r}\cdot\begin{pmatrix}0\\2\\1\end{pmatrix}=0$ $p_3\ :\ 2y+z=0$ *Geometrical Relationship:* 3 planes/intersect at the common line l.	
	(iii) $p_2:\mathbf{r}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=4$ ---- (1) Let N be the foot of perpendicular of $S(0,2,0)$ on p_2. Let l_{NS} be the line passing through points N and S. $l_{NS}:\mathbf{r}=\begin{pmatrix}0\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ ---- (2) Sub (2) into (1): $\left[\begin{pmatrix}0\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\right].\begin{pmatrix}1\\-1\\1\end{pmatrix}=4$ ---- (*) $-2+3\lambda=4$ $\lambda=2$ $\overrightarrow{ON}=\begin{pmatrix}0\\2\\0\end{pmatrix}+2\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\0\\2\end{pmatrix}$ ---- (2) Let S' be the point of reflection of S in p_2. N is the midpoint of SS'. $\overrightarrow{ON}=\frac{1}{2}\left(\overrightarrow{OS}+\overrightarrow{OS'}\right)$ $\overrightarrow{OS'}=2\overrightarrow{ON}-\overrightarrow{OS}$	

	$\overrightarrow{OS'} = 2\begin{pmatrix}2\\0\\2\end{pmatrix} - \begin{pmatrix}0\\2\\0\end{pmatrix} = \begin{pmatrix}4\\-2\\4\end{pmatrix}$ Let m' be the line of reflection of m in p_2. The point $T(4,0,0)$ lying on m also lies on p_2. Therefore $T(4,0,0)$ also lies on m'. m' passes through $T(4,0,0)$ and $S'(4,-2,4)$ $m': \mathbf{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda\left[\begin{pmatrix}4\\0\\0\end{pmatrix} - \begin{pmatrix}4\\-2\\4\end{pmatrix}\right], \lambda \in \mathbb{R}$ $m': \mathbf{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\-4\end{pmatrix}, \lambda \in \mathbb{R}$ $m': \mathbf{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\-2\end{pmatrix}, \lambda \in \mathbb{R}$	

9 Solution [13] APGP

(i)

We first observe the following pattern:

Month	Beginning ($)	End ($)
Jan' 2019	5000	5000×1.01
Feb'2019	$5000\times1.01-100$	$(5000\times1.01-100)\times1.01$
Mar'2019	$5000\times1.01^2-100\times1.01-100$	$(5000\times1.01^2-100\times1.01-100)\times1.01$

Thus, by the end of March 2019, John's account is left with $\$(5000\times1.01^3-100\times1.01^2-100\times1.01)$
$= \$4948.50$

(ii)

From (i), we can deduce that by the end of the n^{th} month, the money left in John's saving account is

$$5000\times1.01^n-\left(100\times1.01^{n-1}+100\times1.01^{n-2}+....+100\times1.01\right)$$

$$=5000\times1.01^n-100\times1.01\left(1+1.01+1.01^2+...+1.01^{n-2}\right)$$

$$=5000\times1.01^n-100\times1.01\times\frac{1\left(1.01^{n-1}-1\right)}{1.01-1}$$

$$=5000\times1.01^n-10000\times1.01\times\left(1.01^{n-1}-1\right)$$

$$=10100+5000\times1.01^n-10000\times1.01^n$$

$$=100\left(101-50\times1.01^n\right) \text{ (shown)}$$

(iii)

Consider $100\left(101-50\times1.01^n\right)\le 0$

$$\Rightarrow 50\times1.01^n\ge101$$

$$\Rightarrow 1.01^n\ge\frac{101}{50}$$

$$\Rightarrow n\ge70.66$$

Thus, it will take Mr Tan 71 months to deplete his saving account and it will be by November of 2024.

Alternative solution using table:

n	Amt left in account
70	$66.18>0$
71	$-34.16<0$

Account depleted in the 71st month.

(iv)
The amount of interest earned by Mrs Tan's for each subsequent month forms an AP: 10, 10+5, 10+2×5,…..
i.e. an AP with first term 10 and common difference 5
Thus, by the end of n^{th} month, the total amount of money in Mrs Tan's saving account

$$= 3000 + 50n + \frac{n}{2}\left[2(10) + 5(n-1)\right]$$

$$= \; = 3000 + 50n + 10n + 2.5n(n-1)$$

$$= 3000 + 57.5n + 2.5n^2$$

For Mrs Tan's account to be more than Mr Tan's, we let

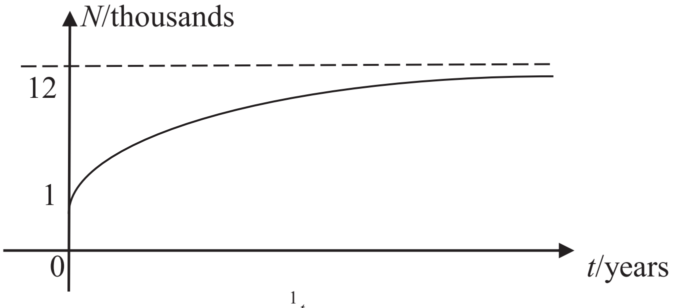
$$3000 + 57.5n + 2.5n^2 > 100\left(101 - 50 \times 1.01^n\right) \text{ ---- (*)}$$

Then using GC: we have

n	LHS of (*)	RHS of (*)
14	4295	4352.6
15	4425	4295.2
16	4560	4327.1

It takes 15 months from Jan 2019 for Mrs' Tan's account to exceed that of Mr Tan.
Thus, it is by end of March 2020 that Mrs Tan's account will first be more than that of Mr Tan's.

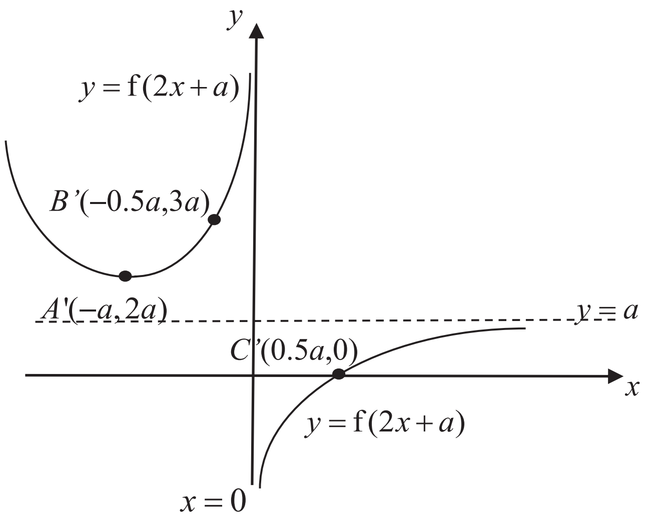
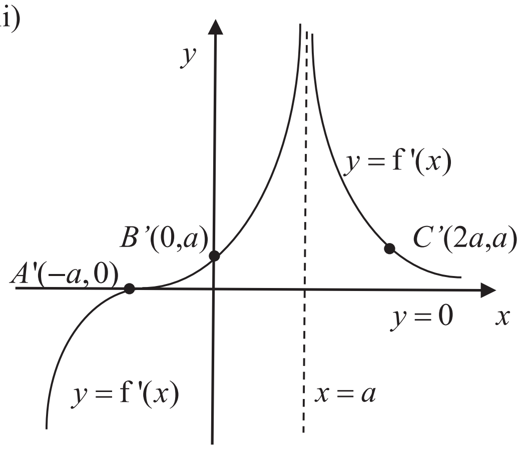
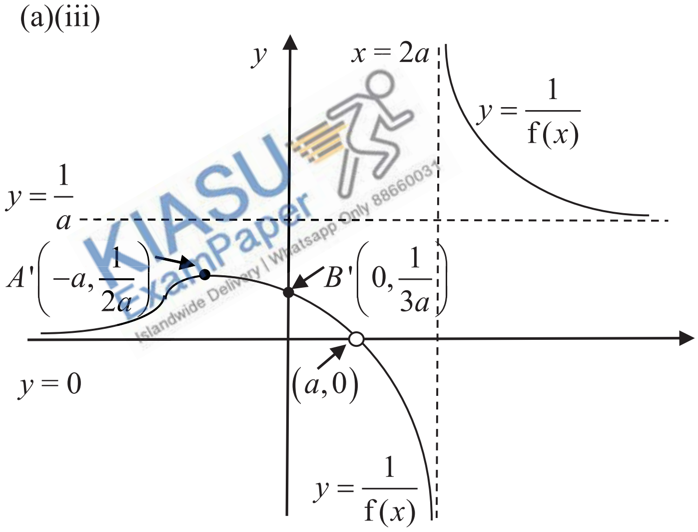
10	Solution [13] DE	
	(i) Based on the given information, we have $\frac{dN}{dt} = 2 - kN, k \in R$ Since it is given that $\frac{dN}{dt} = 1.5$ when $N = 3$, we have $1.5 = 2 - 3k \Rightarrow k = \frac{1}{6}$. Thus, $\frac{dN}{dt} = 2 - \frac{1}{6}N \Rightarrow 6\frac{dN}{dt} = 12 - N$ (shown)	
	(ii) Now, $6\frac{dN}{dt} = 12 - N$ $\Rightarrow \int \frac{dN}{12-N} = \int \frac{1}{6}\ dt$ $\Rightarrow \frac{\ln\|12-N\|}{-1} = \frac{1}{6}t + c$ $\Rightarrow \ln\|12-N\| = -\frac{1}{6}t + C$ Then, we have $\|12-N\| = e^{-\frac{1}{6}t+C}$ $\Rightarrow 12 - N = \pm\ e^{-\frac{1}{6}t+C}$ $\Rightarrow 12 - N = Ae^{-\frac{1}{6}t}$ where $A = \pm e^{C}$ Next, given that when $t = 0$, $N = 1$, we have $A = 12 - 1 = 11$. Hence, the required equation connecting N and t is $N = 12 - 11e^{-\frac{1}{6}t}$	
	(iii) Let $12 - 11e^{-\frac{1}{6}t} \geq 6$	

	$11e^{-\frac{1}{6}t} \le 6$ $\Rightarrow e^{-\frac{1}{6}t} \le \frac{6}{11}$ $\Rightarrow -\frac{1}{6}t \le \ln\left(\frac{6}{11}\right)$ $\Rightarrow t \ge 3.64$ Thus, a minimum of 4 years are needed for the number of flying fox to first exceed 6000.	
	(iv)  We note that as $t \to \infty$, $e^{-\frac{1}{6}t} \to 0$. Thus, $N = 12 - 11e^{-\frac{1}{6}t} \to 12$ So, in the long run, the number of flying fox approaches 12 thousands.	
	(v) One possible limitation may be that the model fails to take into account external factors that affect the population. For example (i) outburst of sudden natural disasters which will affect population of the flying fox; (ii) the increase in the hunting of flying fox over the years; (iii) adverse climate change which affect their living habitat	

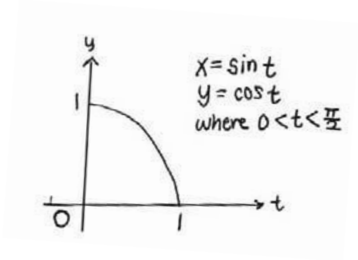
2019 RVHS H2 Maths Prelim P2 Solutions

1	Solution [6] System of linear Eqns	
	(i) $u_{n+1} = -3u_n + An + B$ When $n = 0$, $u_1 = -3u_0 + B$ ---- (*) $2 = -3(4) + B$ $B = 14$ When $n = 1$, $u_2 = -3u_1 + A + B$ ---- (**) $16 = -3(2) + A + 14$ $A = 8$	
	(ii) $u_0 = 4 \Rightarrow a \quad + c = 4$ ---- (1) $u_1 = 2 \Rightarrow -3a + b + c = 2$ ---- (2) $u_2 = 16 \Rightarrow 9a + 2b + c = 16$ ---- (3) Using GC, $a = 1$, $b = 2$, $c = 3$	

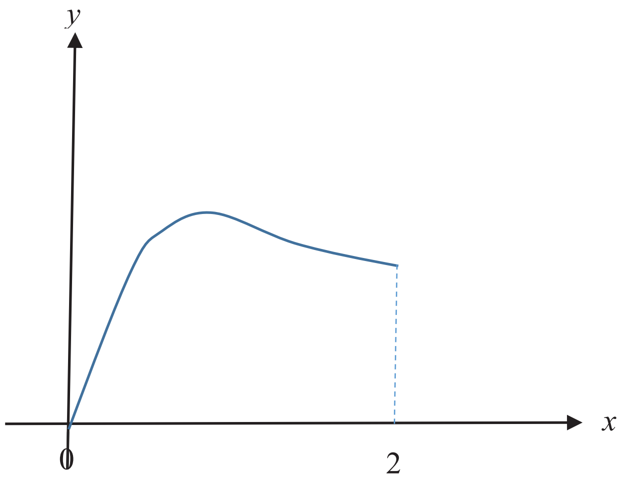
Question 2 [5] vectors		
(a)	Distance between p_1 and p_2 $=\dfrac{\left\lvert \overrightarrow{AB}\cdot(\mathbf{m}\times\mathbf{n})\right\rvert}{\lvert\mathbf{m}\times\mathbf{n}\rvert}$ $=\dfrac{\left\lvert \mathbf{b}\cdot(\mathbf{m}\times\mathbf{n})-\mathbf{a}\cdot(\mathbf{m}\times\mathbf{n})\right\rvert}{\lvert\mathbf{m}\times\mathbf{n}\rvert}$ $=\dfrac{\left\lvert \lvert\mathbf{b}\rvert\lvert(\mathbf{m}\times\mathbf{n})\rvert\cos 60^\circ-\lvert\mathbf{a}\rvert\lvert(\mathbf{m}\times\mathbf{n})\rvert\cos 45^\circ\right\rvert}{\lvert\mathbf{m}\times\mathbf{n}\rvert}$ $=\left\lvert\dfrac{\lvert\mathbf{b}\rvert}{2}-\dfrac{\lvert\mathbf{a}\rvert}{\sqrt{2}}\right\rvert$	
(b)	By ratio theorem, $\overrightarrow{OQ}=\dfrac{\mathbf{a}+2\mathbf{b}}{3}$. The locus of Q is a plane containing fixed point with position vector $\dfrac{\mathbf{a}+2\mathbf{b}}{3}$ and parallel to **m** and **n**. Equation of locus of Q: $\mathbf{r}=\dfrac{\mathbf{a}+2\mathbf{b}}{3}+\lambda\mathbf{m}+\mu\mathbf{n},\quad \lambda,\mu\in\mathbb{R}$	

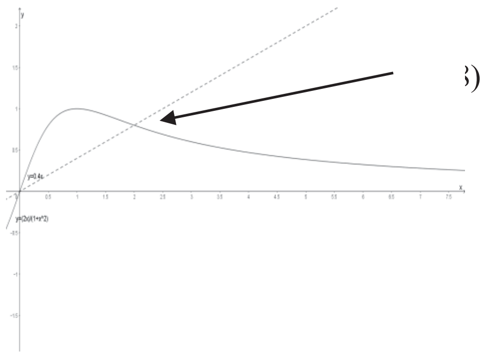
3	Solution [9] Transformation	
	(a)(i) $\mathrm{f}(x) \to \mathrm{f}(x+a) \to \mathrm{f}(2x+a)$ 	
	(a)(ii) 	
	(a)(iii) 	

	(b) We first note the following transformation steps: $2y^2 - x^2 = 1$ $\downarrow$ Step 1: Replace 'x' by '$x-1$' $2y^2 - (x-1)^2 = 1$ $\downarrow$ Step 2: Replace 'y' by '$\frac{y}{\sqrt{2}}$' $2\left(\frac{y}{\sqrt{2}}\right)^2 - (x-1)^2 = 1$ $y^2 - (x-1)^2 = 1$ Thus, the sequence of transformations needed are as follow 1. A translation of 1 unit in the positive x axis direction; 2. A scaling parallel to the y axis of factor $\sqrt{2}$.	

4	Solution [10] Parametric Eqn + Integration Application (Area)	
	(i)  	
	(ii) Given $x=\sin t,\quad y=\cos t$ where $0<t<\frac{\pi}{2}$, we have $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\sin t}{\cos t}=-\tan t$ Thus, at the point where $t=p$, ie $(\sin p,\cos p)$, the equation of the tangent is $y-\cos p=-\tan p(x-\sin p)$ ---- (*) So, $y=-(\tan p)x+\frac{\sin^2 p}{\cos p}+\cos p$ $=-(\tan p)x+\frac{\sin^2 p+\cos^2 p}{\cos p}$ $=-(\tan p)x+\frac{1}{\cos p}$ $=-(\tan p)x+\sec p$ (shown)	
	(iii) The equation of the tangent is $y=-(\tan p)x+\sec p$, Let $x=0$, $y=\sec p$. So, $Q=\left(0,\frac{1}{\cos p}\right)$. Let $y=0$, $x=\frac{\sec p}{\tan p}=\frac{1}{\cos p}\times\frac{\cos p}{\sin p}=\frac{1}{\sin p}$. So, $P=\left(\frac{1}{\sin p},0\right)$ Hence mid point of PQ = $M=\left(\frac{1}{2}\left(\frac{1}{\sin p}+0\right),\frac{1}{2}\left(\frac{1}{\cos p}+0\right)\right)=\left(\frac{1}{2\sin p},\frac{1}{2\cos p}\right)$ To find the Cartesian equation of the locus of the point M,	

	We let $x=\dfrac{1}{2\sin p}$ and $y=\dfrac{1}{2\cos p}$ Then $\sin p=\dfrac{1}{2x}$ and $\cos p=\dfrac{1}{2y}$ And thus, $\left(\dfrac{1}{2x}\right)^2+\left(\dfrac{1}{2y}\right)^2=1$ Hence the Cartesian equation of the locus of M is $\dfrac{1}{x^2}+\dfrac{1}{y^2}=4$ or $x^2+y^2=4x^2y^2$	
	(iv) Exact area = $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} y \, \mathrm{d}x$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} y \dfrac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cos t)(\cos t) \, \mathrm{d}t$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 t \, \mathrm{d}t$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \dfrac{1+\cos 2t}{2} \, \mathrm{d}t$ $=\left[\dfrac{1}{2}t+\dfrac{\sin 2t}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$ $=\left(\dfrac{\pi}{8}+\dfrac{1}{4}\right)-\left(\dfrac{\pi}{12}+\dfrac{\sqrt{3}}{8}\right)$ $=\dfrac{\pi}{24}+\dfrac{1}{4}-\dfrac{\sqrt{3}}{8}$ units^2	

Question 5 [10]		
(i)		
(ii)	$x=\tan\theta \Rightarrow \frac{dx}{d\theta}=\sec^2\theta$ Volume $=\pi\int_0^2\left(\frac{2x}{1+x^2}\right)^2 dx$ $=4\pi\int_0^2\frac{x^2}{\left(1+x^2\right)^2}dx$ $=4\pi\int_0^{\tan^{-1}2}\frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\,d\theta$ $=4\pi\int_0^{\tan^{-1}2}\frac{\tan^2\theta}{\sec^2\theta}d\theta$ $=4\pi\int_0^{\tan^{-1}2}\sin^2\theta\,d\theta$ $=4\pi\int_0^{\tan^{-1}2}\frac{1-\cos 2\theta}{2}d\theta$ $=2\pi\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\tan^{-1}2}$ $=2\pi\left[\theta-\sin\theta\cos\theta\right]_0^{\tan^{-1}2}$ $=2\pi\left[\tan^{-1}2-\frac{2}{\sqrt5}\frac{1}{\sqrt5}\right]$ $=2\pi\left[\tan^{-1}2-\frac{2}{5}\right]$ units^3	



Volume of plastic

$$= \frac{1}{3}\pi\left(\frac{4}{5}\right)^2(2) = \frac{32}{75}\pi \text{ units}^3$$

Cost of one paperweight

$$= \left(2\pi\left[\tan^{-1}2 - \tfrac{2}{5}\right] - \frac{32}{75}\pi\right)(3) + \left(\frac{32}{75}\pi\right)(1.2)$$

$= \$11$ (to the nearest dollar)

<table>
<tr><td colspan="3">Question 6 [6]</td></tr>
<tr><td>(i)</td><td>RRR BB W Y G
Step 1 : Arrange BB W Y G in $\frac{5!}{2!}$ ways
Step 2: Slot the RRR into the 6 appropriate slots _ to the left/ right of the arranged letters in Step:1 generally denoted by X

_X_X_X_X_X_

No. of ways $=\frac{5!}{2!}\times\binom{6}{3}=1200$

Note:
Number of ways in which RRR are separated
$\neq$ Total number of ways $-$ All R separated ----(*)

Why is this so?</td><td></td></tr>
<tr><td>(ii)</td><td>No. of ways for the letters to form a circle = $\frac{5!}{5}$
Since clockwise and anti-clockwise are indistinguishable in a ring,
No. of ways = $\frac{5!}{5}\div 2=12$

Note:
For a physical 3-dimensional ring made up of beads, what happen when you flip it the other side?</td><td></td></tr>
<tr><td>(iii)</td><td><u>Method 1</u>
III N TT A E
1. There is only 1 way to arrange A, E and N in alphabetical order.
2. Without restriction there are $(3!)$ ways to arrange A, E and N.

Without restriction, total number of ways to arrange all the letters $=\frac{8!}{3!2!}$ ---- (**)
To get number of ways to arrange all letters and A, E and N in alphabetical order, REMOVE all the UNWANTED arrangements in (**), using the ideas in (1) and (2).</td><td></td></tr>
</table>

No. of ways to arrange all letters and A, E and N in alphabetical order $=\left[\dfrac{8!}{3!2!}\right]\div(3!)=\dfrac{8!}{3!3!2!}$

(Note: A, E and N need not be together.)

Required prob $=\dfrac{8!}{3!3!2!}\Big/\dfrac{8!}{3!2!}=\dfrac{1}{3!}=\dfrac{1}{6}$

Method 2

Step 1: Other than A, E, N, the other letters are III TT,

number of ways to arrange III TT $=\dfrac{5!}{3!2!}$

Step 2: After arranging III TT, we create slots _ to the left / right of each letter.

_X_X_X_X_X_

Step3: We now slot in A, E, N into the slots so that A, E, N are in alphabetical order.

We could group A, E, N as
- 1 item: [AEN]
- 2 items: [AE], [N]
- 2 items: [A], [NE]
- 3 items: [A], [E], [N]

For example to slot 2 items: [AE], [N] into _X_X_X_X_X_,

Number of ways $=\dfrac{5!}{3!2!}\times\dbinom{6}{2}$

Number of ways to arrange all letters and A, E and N in alphabetical order

$=\dfrac{5!}{3!2!}\times\left[\dbinom{6}{1}+\dbinom{6}{2}+\dbinom{6}{2}+\dbinom{6}{3}\right]$

7	Solution [6] DRV	
	(i) $\mathrm{P}(X=3)$ $=\frac{6-n}{6}\cdot\frac{5-n}{5}\cdot\frac{n}{4}$ $=\frac{(6-n)(5-n)n}{120}$	
	(ii) When $n=3$, $\mathrm{P}(W=1)$ $=\mathrm{P}(X\geq 4)$ = P(keys from first 3 trials can't unlock box) $=\left(\frac{3}{6}\right)\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)=\frac{1}{20}=0.05$ w | 2 | 1 $\mathrm{P}(W=w)$ | 0.95 | 0.05 $\mathrm{E}(W)=2(0.95)+(0.05)=\$1.95$ Alternatively $\mathrm{P}(W=2)$ $=\mathrm{P}(X=1)+\mathrm{P}(X=2)+\mathrm{P}(X=3)$ $=\left(\frac{3}{6}\right)+\left(\frac{3}{6}\right)\left(\frac{3}{5}\right)+\left(\frac{3}{6}\right)\left(\frac{2}{5}\right)\left(\frac{3}{4}\right)$ $=\frac{19}{20}=0.95$	
	(iii) Let T denote the r.v the number of times that Alfred wins \$2 from a game, out of 5 games played. $T\sim \mathrm{B}(5,0.95)$ P(Alfred wins more at least \$9 out of 5 games) $=\mathrm{P}(T\geq 4)$ $=1-\mathrm{P}(T\leq 3)$ $=0.977$	

Question 8 [7] Probability

(i) P(Player draws 3 different numbers in the first round)

$$=\frac{\binom{5}{1}\binom{2}{1}}{\binom{8}{3}}=\frac{5}{28}$$

(ii) Probability of sum of numbers from first round is 12 and sum of numbers from second round is 6)

$$=\frac{\binom{5}{1}\binom{2}{2}}{\binom{8}{3}}\frac{\binom{5}{3}}{\binom{8}{3}}$$

$$=\frac{25}{1568}$$

P(Sum of the numbers drawn adds up to 18)
= P(Sum of each of first and second round is 9)
+ P(Sum of first round is 12 and second round is 6)

$$=\frac{\binom{5}{2}\binom{2}{1}}{\binom{8}{3}}\frac{\binom{5}{2}}{\binom{8}{3}}+\frac{25}{1568}$$

$$=\frac{125}{1568}$$

Let X be the sum of the 2 numbers drawn.

x	4	7	12	15	20
$P(X = x)$	$\frac{\binom{5}{2}}{\binom{8}{2}}=\frac{10}{28}$	$\frac{\binom{5}{1}}{\binom{8}{2}}=\frac{5}{28}$	$\frac{\binom{5}{1}\binom{2}{1}}{\binom{8}{2}}=\frac{10}{28}$	$\frac{\binom{2}{1}}{\binom{8}{2}}=\frac{2}{28}$	$\frac{\binom{2}{2}}{\binom{8}{2}}=\frac{1}{28}$

E(X)

$$=4\left(\frac{10}{28}\right)+7\left(\frac{5}{28}\right)+12\left(\frac{10}{28}\right)+15\left(\frac{2}{28}\right)+20\left(\frac{1}{28}\right)$$

$$=8.75$$

Question 9 [8] Binomial Dist		
(i)	Assumption: The candidature has no knowledge of which option is the correct answer and chooses one of the 5 options **randomly/by guessing**. This assumption is important to ensure that the probability of choosing a particular option is constant at 0.2.	
(ii)	Let X denotes the number of marks a candidate scores for the test, out of 12. $X \sim B(12, 0.2)$ $P(X=1) = 0.206$ $P(X=2) = 0.283$ $P(X=3) = 0.236$ Mode of X is 2. Majority of the candidates will score 2 marks.	
(iii)	$P(X > 4 \mid X \leq 5)$ $= \frac{P(X=5)}{P(X \leq 5)}$ $= 0.0542 \ \ (3\text{ s.f})$	
(iv)	Assuming n is large, by CLT, $\overline{X} \sim N\left(2.4, \frac{1.92}{n}\right)$ approx. $P\left(\overline{X} \leq 2.7\right) \geq 0.95$ $n = 57, P\left(\overline{X} \leq 2.7\right) = 0.94893 < 0.95$ $n = 58, P\left(\overline{X} \leq 2.7\right) = 0.95041 > 0.95$ Thus, least value of n is 58.	

Question 10 [9] Hypo Test		
(i)	$\bar{x} = -\frac{27}{70} + 46 = 45.61428 = 45.6 \text{ (to 3 s.f.)}$ $s^2 = \frac{1}{69}\left(30939 - \frac{(-27)^2}{70}\right) = 448.2403727 = 448 \text{ (to 3 s.f.)}$	
(ii)	Let μ_0 be the mean number of hours the engineer claimed. Test $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ at 5% significance level Test statistic: Under H_0, $Z = \frac{\bar{X} - \mu_0}{s / \sqrt{70}} \sim N(0,1)$ To reject H_0, $\frac{\bar{x} - \mu_0}{s/\sqrt{70}} > 1.644853632$ $\mu_0 < \bar{x} - 1.644853632 \frac{s}{\sqrt{70}}$ $\mu_0 < 41.45197671$ $\mu_0 < 41.5 \text{ (to 3 s.f.)}$ Thus, the greatest mean number of hours the engineers should claim is 41.5.	
(iii)	Based on $n = 70$, $\sum x = \left(-\frac{27}{70} + 46\right) \times 70 = 3193$ Based on $n = 75$, $\bar{x} = \frac{3193 + 56 + 34 + 63 + 50 + 54}{75} = 46$ $\sum (x - 46)^2$ $= \sum (x - \bar{x})^2$ $= 30939 + (56-46)^2 + (34-46)^2 + (63-46)^2$ $+ (50-46)^2 + (54-46)^2$ $= 31552$ $\therefore s^2 = \frac{\sum (x - \bar{x})^2}{74} = 426.3783783784 = 426.378 \text{ (3.s.f)}$	

Test $H_0 : \mu = 41.5$
against $H_1 : \mu > 41.5$ at α% significance level

Test statistic: Under H_0, $Z = \dfrac{\bar{X} - 41.5}{s / \sqrt{75}} \sim N(0,1)$

By GC, p-value = 0.029559

Do not reject H_0,

$$p - value \geq \frac{\alpha}{100}$$

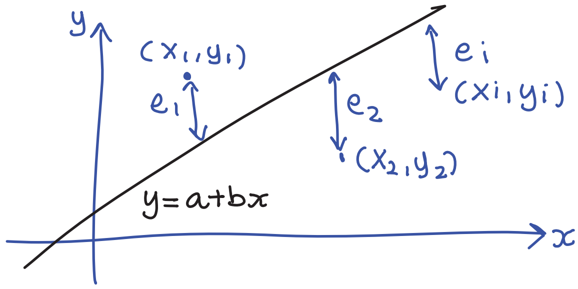
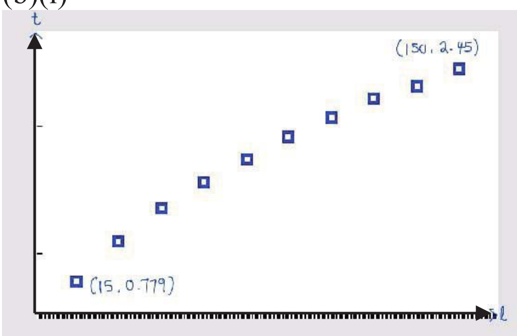
$$\Rightarrow \frac{\alpha}{100} \leq 0.029559$$

$$\Rightarrow \alpha \leq 2.9559$$

The greatest integer value of α is 2.

[If $\mu_0 = 41.4$, p-value = 0.026649, then the greatest integer value of α is 2.

If $\mu_0 = 41.45197671$, p-value = 0.028230374, then the greatest integer value of α is 2.]

11	Solution [12] Correlation & Regression	
	(a) Let the sample of bivariate data be (x_i, y_i) where $i = 1, 2, \ldots n$. Let $e_i = y_i - (a + bx_i)$ be the vertical deviation between the point (x_i, y_i) and the line $y = a + bx$.  The line $y = a + bx$ is the least square regression line for the sample of bivariate data if the sum of the squares of the vertical deviation $\sum_{i=1}^{n}(e_i)^2$ is the minimum.	
	(b)(i)  	
	(b)(ii) Model (B). As l increases, t increases at a decreasing rate, therefore model (B) is most appropriate. Note that for Model (A), the graph of $t = a + bl^2$ has a turning point located on the vertical axis, therefore Model A is not suitable.	
	(b)(iii) $t = a + b\ln l$ Using GC, $a = -1.370621901 \approx -1.37$ (3.s.f) $b = 0.7394875471 \approx 0.740$ (3 s.f) $r = 0.9871042012 \approx 0.987$ (3 s.f)	

	(b)(iv) $t = -1.370621901 + +0.7394875471\ln l$ ---- (1) To estimate l when t = 1.00, use the regression line of t on ln l, since t is the dependent variable. Sub $t = 1.00$ into (1), $l = 24.6743 \approx 24.7$ mm. The appropriate regression line is used since t is the dependent variable. Since $\lvert r \rvert \approx 0.987$ is close to 1, the model based on $t = a + b\ln l$, is a good fit for the data. $t = 1.00$ is within the input data range $0.779 \le t \le 2.45$, the estimate is an interpolation. Therefore, the estimate is reliable.	
	(b)(v) 1 millimeter = 0.03937 inch 1 inch = $\dfrac{1}{0.03937}$ millimeters L inches = $\dfrac{L}{0.03937}$ millimeters $(L = 0.03937l)$ $t = a + b\ln\dfrac{L}{0.03937}$ $t = a + b(\ln L - \ln 0.03937)$ $t = (a - b\ln 0.03937) + b\ln L$ $t = 1.02143631 + 0.7394875471\ln L$ $t = 1.02 + 0.739\ln L$ (3 s.f)	

12	Solution [12 marks] Normal Dist	
(i)	Let A be the mass of a randomly chosen apple. Suppose $A \sim N\left(70, 40^2\right)$ Then $\mathrm{P}(A<0)=0.0401$ (3 s.f.), but it's impossible for apples to have negative mass, so the probability should be much closer to 0. Hence, the normal distribution is not a suitable distribution for the masses of apples.	
(ii)	Let Y be the mass of a pear. Let μ be the mean mass of the pears, and σ^2 be the variance in the mass of the pears. $Y \sim N\left(\mu, \sigma^2\right)$ Given, $\mathrm{P}(Y<148)=\mathrm{P}(Y>230)=\frac{1}{3}$ By symmetry, $\mu=\frac{148+230}{2}=189$ Let $Z=\frac{Y-189}{\sigma} \sim N(0,1)$ $\mathrm{P}(Y<148)=\frac{1}{3}$ $\mathrm{P}\left(Z<\frac{-41}{\sigma}\right)=\frac{1}{3}$ From GC, $\frac{-41}{\sigma}=-0.430727$ $\sigma=95.18783606$ $\sigma=95.2\,\mathrm{g}$ (3 s.f.)	
(iii)	Let T_A be the total mass of the 300 apples. Then $T_A=A_1+A_2+\ldots+A_{300}$ and $T_A \sim \mathrm{N}\left(300\times 70, 300\times 40^2\right)$ approx. by the Central Limit Theorem (CLT) as $n=300$ is large.	
(iv)	Similarly, by letting T_P be the total mass of the 400 pears, we have $T_P \sim N(400\times 189, 400\times 9060.724134)$	

	So, we have $T_A \sim \mathrm{N}(21000, 480000)$ and $T_P \sim N(75600, 3624289.654)$ Thus, we have $R = 0.005T_A + 0.008T_P$ and $R \sim \mathrm{N}(0.005\times 21000+0.008\times 75600,$ $0.005^2 \times 480000 + 0.008^2 \times 3624289.654)$ or $R \sim \mathrm{N}(709.8, 243.9545379)$ Since c is the running cost of the orchard, $\mathrm{P}(R > c) = 0.9$ From GC, $c = 689.7833896$ Therefore, the cost of running the orchard is \$689.78 (2 d.p) Note: In accounting terms, the amount of money collected from sales in a business, is referred to as the revenue.	